Roots, Radicals, and Square Root Equations: Division of Square Root Expressions*

Wade Ellis
Denny Burzynski

This work is produced by The Connexions Project and licensed under the
Creative Commons Attribution License †

Abstract

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The distinction between the principal square root of the number x and the secondary square root of the number x is made by explanation and by example. The simplification of the radical expressions that both involve and do not involve fractions is shown in many detailed examples; this is followed by an explanation of how and why radicals are eliminated from the denominator of a radical expression. Real-life applications of radical equations have been included, such as problems involving daily output, daily sales, electronic resonance frequency, and kinetic energy. Objectives of this module: be able to use the division property of square roots, the method of rationalizing the denominator, and conjugates to divide square roots.

1 Overview

- The Division Property of Square Roots
- Rationalizing the Denominator
- Conjugates and Rationalizing the Denominator

2 The Division Property of Square Roots

In our work with simplifying square root expressions, we noted that

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

Since this is an equation, we may write it as

*Version 1.4: Jun 1, 2009 11:43 am -0500
†http://creativecommons.org/licenses/by/3.0/

$$\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$$

To divide two square root expressions, we use the division property of square roots.
The Division Property $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$

$$\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$$

The quotient of the square roots is the square root of the quotient.

3 Rationalizing the Denominator

As we can see by observing the right side of the equation governing the division of square roots, the process may produce a fraction in the radicand. This means, of course, that the square root expression is not in simplified form. It is sometimes more useful to rationalize the denominator of a square root expression before actually performing the division.

4 Sample Set A

Simplify the square root expressions.

Example 1
$\sqrt{\frac{3}{7}}$.

This radical expression is not in simplified form since there is a fraction under the radical sign. We can eliminate this problem using the division property of square roots.

$$\sqrt{\frac{3}{7}} = \frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{3}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{3}\sqrt{7}}{7} = \frac{\sqrt{21}}{7}$$

Example 2
$\frac{\sqrt{5}}{\sqrt{3}}$.

A direct application of the rule produces $\sqrt{\frac{5}{3}}$, which must be simplified. Let us rationalize the denominator before we perform the division.

$$\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{5}\sqrt{3}}{3} = \frac{\sqrt{15}}{3}$$

Example 3
$\frac{\sqrt{21}}{\sqrt{7}} = \sqrt{\frac{21}{7}} = \sqrt{3}$.

The rule produces the quotient quickly. We could also rationalize the denominator first and produce the same result.

$$\frac{\sqrt{21}}{\sqrt{7}} = \frac{\sqrt{21}}{7} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{21 \cdot 7}}{7} = \frac{\sqrt{3 \cdot 7 \cdot 7}}{7} = \frac{\sqrt{3 \cdot 7^2}}{7} = \frac{7\sqrt{3}}{7} = \sqrt{3}$$

Example 4

$$\frac{\sqrt{80x^9}}{\sqrt{5x^4}} = \sqrt{\frac{80x^9}{5x^4}} = \sqrt{16x^5} = \sqrt{16}\sqrt{x^4x} = 4x^2\sqrt{x}$$

Example 5

$$\frac{\sqrt{50a^3b^7}}{\sqrt{5ab^5}} = \sqrt{\frac{50a^3b^7}{5ab^5}} = \sqrt{10a^2b^2} = ab\sqrt{10}$$

Example 6
$\frac{\sqrt{5a}}{\sqrt{b}}$.

Some observation shows that a direct division of the radicands will produce a fraction. This suggests that we rationalize the denominator first.

$$\frac{\sqrt{5a}}{\sqrt{b}} = \frac{\sqrt{5a}}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{5a}\sqrt{b}}{b} = \frac{\sqrt{5ab}}{b}$$

Example 7

$$\frac{\sqrt{m-6}}{\sqrt{m+2}} = \frac{\sqrt{m-6}}{\sqrt{m+2}} \cdot \frac{\sqrt{m+2}}{\sqrt{m+2}} = \frac{\sqrt{m^2-4m-12}}{m+2}$$

Example 8

$$\frac{\sqrt{y^2-y-12}}{\sqrt{y+3}} = \sqrt{\frac{y^2-y-12}{y+3}} = \sqrt{\frac{(y+3)(y-4)}{(y+3)}} = \sqrt{\frac{\cancel{)(y+3)}(y-4)}{\cancel{)(y+3)}}} = \sqrt{y-4}$$

5 Practice Set A

Simplify the square root expressions.

Exercise 1 *(Solution on p. 10.)*
$\frac{\sqrt{26}}{\sqrt{13}}$

Exercise 2 *(Solution on p. 10.)*
$\frac{\sqrt{7}}{\sqrt{3}}$

Exercise 3 *(Solution on p. 10.)*
$\frac{\sqrt{80m^5n^8}}{\sqrt{5m^2n}}$

Exercise 4 *(Solution on p. 10.)*
$\frac{\sqrt{196(x+7)^8}}{\sqrt{2(x+7)^3}}$

Exercise 5 *(Solution on p. 10.)*
$\frac{\sqrt{n+4}}{\sqrt{n-5}}$

Exercise 6 *(Solution on p. 10.)*

$\frac{\sqrt{a^2-6a+8}}{\sqrt{a-2}}$

Exercise 7 *(Solution on p. 10.)*

$\frac{\sqrt{x^{3n}}}{\sqrt{x^n}}$

Exercise 8 *(Solution on p. 10.)*

$\frac{\sqrt{a^{3m-5}}}{\sqrt{a^{m-1}}}$

6 Conjugates and Rationalizing the Denominator

To perform a division that contains a binomial in the denominator, such as $\frac{3}{4+\sqrt{6}}$, we multiply the numerator and denominator by a **conjugate** of the denominator.

Conjugate

A conjugate of the binomial $a+b$ is $a-b$. Similarly, a conjugate of $a-b$ is $a+b$.

Notice that when the conjugates $a+b$ and $a-b$ are multiplied together, they produce a difference of two squares.

$$(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

This principle helps us eliminate square root radicals, as shown in these examples that illustrate finding the product of conjugates.

Example 9

$$\begin{aligned}\left(5+\sqrt{2}\right)\left(5-\sqrt{2}\right) &= 5^2 - \left(\sqrt{2}\right)^2 \\ &= 25-2 \\ &= 23\end{aligned}$$

Example 10

$$\begin{aligned}\left(\sqrt{6}-\sqrt{7}\right)\left(\sqrt{6}+\sqrt{7}\right) &= \left(\sqrt{6}\right)^2 - \left(\sqrt{7}\right)^2 \\ &= 6-7 \\ &= -1\end{aligned}$$

7 Sample Set B

Simplify the following expressions.

Example 11

$\frac{3}{4+\sqrt{6}}$.

The conjugate of the denominator is $4-\sqrt{6}$. Multiply the fraction by 1 in the form of $\frac{4-\sqrt{6}}{4-\sqrt{6}}$.

$$\begin{aligned}\frac{3}{4+\sqrt{6}} \cdot \frac{4-\sqrt{6}}{4-\sqrt{6}} &= \frac{3\left(4-\sqrt{6}\right)}{4^2-\left(\sqrt{6}\right)^2} \\ &= \frac{12-3\sqrt{6}}{16-6} \\ &= \frac{12-3\sqrt{6}}{10}\end{aligned}$$

Example 12
$\frac{\sqrt{2x}}{\sqrt{3}-\sqrt{5x}}$.

The conjugate of the denominator is $\sqrt{3}+\sqrt{5x}$. Multiply the fraction by 1 in the form of $\frac{\sqrt{3}+\sqrt{5x}}{\sqrt{3}+\sqrt{5x}}$.

$$\begin{aligned}\frac{\sqrt{2x}}{\sqrt{3}-\sqrt{5x}}\cdot\frac{\sqrt{3}+\sqrt{5x}}{\sqrt{3}+\sqrt{5x}} &= \frac{\sqrt{2x}\left(\sqrt{3}+\sqrt{5x}\right)}{\left(\sqrt{3}\right)^2-\left(\sqrt{5x}\right)^2}\\ &= \frac{\sqrt{2x}\sqrt{3}+\sqrt{2x}\sqrt{5x}}{3-5x}\\ &= \frac{\sqrt{6x}+\sqrt{10x^2}}{3-5x}\\ &= \frac{\sqrt{6x}+x\sqrt{10}}{3-5x}\end{aligned}$$

8 Practice Set B

Simplify the following expressions.

Exercise 9 *(Solution on p. 10.)*
$\frac{5}{9+\sqrt{7}}$

Exercise 10 *(Solution on p. 10.)*
$\frac{-2}{1-\sqrt{3x}}$

Exercise 11 *(Solution on p. 10.)*
$\frac{\sqrt{8}}{\sqrt{3x}+\sqrt{2x}}$

Exercise 12 *(Solution on p. 10.)*
$\frac{\sqrt{2m}}{m-\sqrt{3m}}$

9 Exercises

For the following problems, simplify each expressions.

Exercise 13 *(Solution on p. 10.)*
$\frac{\sqrt{28}}{\sqrt{2}}$

Exercise 14
$\frac{\sqrt{200}}{\sqrt{10}}$

Exercise 15 *(Solution on p. 10.)*
$\frac{\sqrt{28}}{\sqrt{7}}$

Exercise 16
$\frac{\sqrt{96}}{\sqrt{24}}$

Exercise 17 *(Solution on p. 10.)*
$\frac{\sqrt{180}}{\sqrt{5}}$

Exercise 18
$\frac{\sqrt{336}}{\sqrt{21}}$

Exercise 19 *(Solution on p. 10.)*
$\frac{\sqrt{162}}{\sqrt{18}}$

Exercise 20
$\sqrt{\frac{25}{9}}$

Exercise 21 *(Solution on p. 10.)*

$\sqrt{\frac{36}{35}}$

Exercise 22

$\sqrt{\frac{225}{16}}$

Exercise 23 *(Solution on p. 10.)*

$\sqrt{\frac{49}{225}}$

Exercise 24

$\sqrt{\frac{3}{5}}$

Exercise 25 *(Solution on p. 10.)*

$\sqrt{\frac{3}{7}}$

Exercise 26

$\sqrt{\frac{1}{2}}$

Exercise 27 *(Solution on p. 10.)*

$\sqrt{\frac{5}{2}}$

Exercise 28

$\sqrt{\frac{11}{25}}$

Exercise 29 *(Solution on p. 10.)*

$\sqrt{\frac{15}{36}}$

Exercise 30

$\sqrt{\frac{5}{16}}$

Exercise 31 *(Solution on p. 10.)*

$\sqrt{\frac{7}{25}}$

Exercise 32

$\sqrt{\frac{32}{49}}$

Exercise 33 *(Solution on p. 10.)*

$\sqrt{\frac{50}{81}}$

Exercise 34

$\frac{\sqrt{125x^5}}{\sqrt{5x^3}}$

Exercise 35 *(Solution on p. 10.)*

$\frac{\sqrt{72m^7}}{\sqrt{2m^3}}$

Exercise 36

$\frac{\sqrt{162a^{11}}}{\sqrt{2a^5}}$

Exercise 37 *(Solution on p. 11.)*

$\frac{\sqrt{75y^{10}}}{\sqrt{3y^4}}$

Exercise 38

$\frac{\sqrt{48x^9}}{\sqrt{3x^2}}$

Exercise 39 *(Solution on p. 11.)*

$\frac{\sqrt{125a^{14}}}{\sqrt{5a^5}}$

Exercise 40

$\frac{\sqrt{27a^{10}}}{\sqrt{3a^5}}$

Exercise 41 *(Solution on p. 11.)*

$\frac{\sqrt{108x^{21}}}{\sqrt{3x^4}}$

Exercise 42

$\frac{\sqrt{48x^6y^7}}{\sqrt{3xy}}$

Exercise 43 *(Solution on p. 11.)*

$\frac{\sqrt{45a^3b^8c^2}}{\sqrt{5ab^2c}}$

Exercise 44

$\frac{\sqrt{66m^{12}n^{15}}}{\sqrt{11mn^8}}$

Exercise 45 *(Solution on p. 11.)*

$\frac{\sqrt{30p^5q^{14}}}{\sqrt{5q^7}}$

Exercise 46

$\frac{\sqrt{b}}{\sqrt{5}}$

Exercise 47 *(Solution on p. 11.)*

$\frac{\sqrt{5x}}{\sqrt{2}}$

Exercise 48

$\frac{\sqrt{2a^3b}}{\sqrt{14a}}$

Exercise 49 *(Solution on p. 11.)*

$\frac{\sqrt{3m^4n^3}}{\sqrt{6mn^5}}$

Exercise 50

$\frac{\sqrt{5(p-q)^6(r+s)^4}}{\sqrt{25(r+s)^3}}$

Exercise 51 *(Solution on p. 11.)*

$\frac{\sqrt{m(m-6)-m^2+6m}}{\sqrt{3m-7}}$

Exercise 52

$\frac{\sqrt{r+1}}{\sqrt{r-1}}$

Exercise 53 *(Solution on p. 11.)*

$\frac{\sqrt{s+3}}{\sqrt{s-3}}$

Exercise 54

$\frac{\sqrt{a^2+3a+2}}{\sqrt{a+1}}$

Exercise 55 *(Solution on p. 11.)*

$\frac{\sqrt{x^2-10x+24}}{\sqrt{x-4}}$

Exercise 56

$\frac{\sqrt{x^2-2x-8}}{\sqrt{x+2}}$

Exercise 57 *(Solution on p. 11.)*

$\frac{\sqrt{x^2-4x+3}}{\sqrt{x-3}}$

Exercise 58

$\frac{\sqrt{2x^2-x-1}}{\sqrt{x-1}}$

Exercise 59 *(Solution on p. 11.)*

$\frac{-5}{4+\sqrt{5}}$

Exercise 60

$\frac{1}{1+\sqrt{x}}$

Exercise 61 *(Solution on p. 11.)*

$\frac{2}{1-\sqrt{a}}$

Exercise 62

$\frac{-6}{\sqrt{5}-1}$

Exercise 63 *(Solution on p. 11.)*

$\frac{-6}{\sqrt{7}+2}$

Exercise 64

$\frac{3}{\sqrt{3}-\sqrt{2}}$

Exercise 65 *(Solution on p. 11.)*

$\frac{4}{\sqrt{6}+\sqrt{2}}$

Exercise 66

$\frac{\sqrt{5}}{\sqrt{8}-\sqrt{6}}$

Exercise 67 *(Solution on p. 11.)*

$\frac{\sqrt{12}}{\sqrt{12}-\sqrt{8}}$

Exercise 68

$\frac{\sqrt{7x}}{2-\sqrt{5x}}$

Exercise 69 *(Solution on p. 11.)*

$\frac{\sqrt{6y}}{1+\sqrt{3y}}$

Exercise 70

$\frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

Exercise 71 *(Solution on p. 11.)*

$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$

Exercise 72

$\frac{\sqrt{8^3b^5}}{4-\sqrt{2ab}}$

Exercise 73 *(Solution on p. 11.)*

$\frac{\sqrt{7x}}{\sqrt{5x}+\sqrt{x}}$

Exercise 74

$\frac{\sqrt{3y}}{\sqrt{2y}-\sqrt{y}}$

10 Exercises for Review

Exercise 75 *(Solution on p. 11.)*

(**here**[1]) Simplify $x^8y^7\left(\frac{x^4y^8}{x^3y^4}\right)$.

Exercise 76

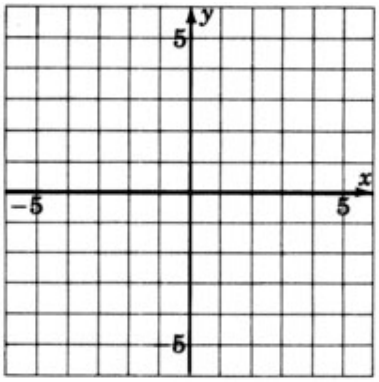
(**here**[2]) Solve the compound inequality $-8 \leq 7-5x \leq -23$.

[1] "Basic Properties of Real Numbers: Rules of Exponents" <http://cnx.org/content/m21900/latest/>

[2] "Solving Linear Equations and Inequalities: Linear Inequalities in One Variable" <http://cnx.org/content/m21979/latest/>

Exercise 77 *(Solution on p. 11.)*

(**here**[3]) Construct the graph of $y = \frac{2}{3}x - 4$.



Exercise 78

(**here**[4]) The symbol $\sqrt{x}$ represents which square root of the number x, $x \geq 0$?

Exercise 79 *(Solution on p. 12.)*

(**here**[5]) Simplify $\sqrt{a^2 + 8a + 16}$.

[3] "Graphing Linear Equations and Inequalities: Graphing Equations in Slope-Intercept Form" <http://cnx.org/content/m22000/latest/>

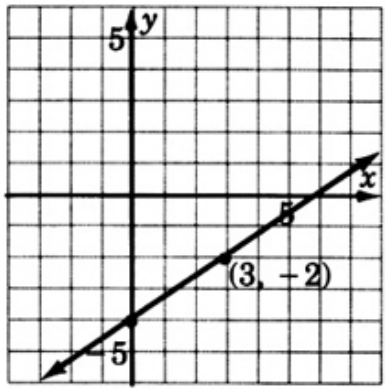
[4] "Roots, Radicals, and Square Root Equations: Square Root Expressions" <http://cnx.org/content/m18879/latest/>

[5] "Roots, Radicals, and Square Root Equations: Multiplication of Square Root Expressions" <http://cnx.org/content/m21975/latest/>

Solutions to Exercises in this Module

Solution to Exercise (p. 3)
$\sqrt{2}$
Solution to Exercise (p. 3)
$\frac{\sqrt{21}}{3}$
Solution to Exercise (p. 3)
$4mn^3\sqrt{mn}$
Solution to Exercise (p. 3)
$7(x+7)^2\sqrt{2\left(x+7\right)}$
Solution to Exercise (p. 3)
$\frac{\sqrt{n^2-n-20}}{n-5}$
Solution to Exercise (p. 4)
$\sqrt{a-4}$
Solution to Exercise (p. 4)
x^n
Solution to Exercise (p. 4)
a^{m-2}
Solution to Exercise (p. 5)
$\frac{45-5\sqrt{7}}{74}$
Solution to Exercise (p. 5)
$\frac{-2-2\sqrt{3x}}{1-3x}$
Solution to Exercise (p. 5)
$\frac{2\sqrt{6x}-4\sqrt{x}}{x}$
Solution to Exercise (p. 5)
$\frac{\sqrt{2m}+\sqrt{6}}{m-3}$
Solution to Exercise (p. 5)
$\sqrt{14}$
Solution to Exercise (p. 5)
2
Solution to Exercise (p. 5)
6
Solution to Exercise (p. 5)
3
Solution to Exercise (p. 6)
$\frac{6\sqrt{35}}{35}$
Solution to Exercise (p. 6)
$\frac{7}{15}$
Solution to Exercise (p. 6)
$\frac{\sqrt{21}}{7}$
Solution to Exercise (p. 6)
$\frac{\sqrt{10}}{2}$
Solution to Exercise (p. 6)
$\frac{\sqrt{15}}{6}$
Solution to Exercise (p. 6)
$\frac{\sqrt{7}}{5}$
Solution to Exercise (p. 6)
$\frac{5\sqrt{2}}{9}$

Solution to Exercise (p. 6)
$6m^2$
Solution to Exercise (p. 6)
$5y^3$
Solution to Exercise (p. 6)
$5a^4\sqrt{a}$
Solution to Exercise (p. 7)
$6x^8\sqrt{x}$
Solution to Exercise (p. 7)
$3ab^3\sqrt{c}$
Solution to Exercise (p. 7)
$p^2q^3\sqrt{6pq}$
Solution to Exercise (p. 7)
$\frac{\sqrt{10x}}{2}$
Solution to Exercise (p. 7)
$\frac{m\sqrt{2m}}{2n}$
Solution to Exercise (p. 7)
0
Solution to Exercise (p. 7)
$\frac{\sqrt{s^2-9}}{s-3}$
Solution to Exercise (p. 7)
$\sqrt{x-6}$
Solution to Exercise (p. 7)
$\sqrt{x-1}$
Solution to Exercise (p. 7)
$\frac{-20+5\sqrt{5}}{11}$
Solution to Exercise (p. 8)
$\frac{2\left(1+\sqrt{a}\right)}{1-a}$
Solution to Exercise (p. 8)
$-2\left(\sqrt{7}-2\right)$
Solution to Exercise (p. 8)
$\sqrt{6}-\sqrt{2}$
Solution to Exercise (p. 8)
$3+\sqrt{6}$
Solution to Exercise (p. 8)
$\frac{\sqrt{6y}-3y\sqrt{2}}{1-3y}$
Solution to Exercise (p. 8)
$\frac{a-\sqrt{ab}}{a-b}$
Solution to Exercise (p. 8)
$\frac{\sqrt{35}-\sqrt{7}}{4}$
Solution to Exercise (p. 8)
x^9y^{11}
Solution to Exercise (p. 8)



Solution to Exercise (p. 9)

$a + 4$