Solving Linear Equations and Inequalities: Linear Inequalities in One Variable*

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Abstract

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. In this chapter, the emphasis is on the mechanics of equation solving, which clearly explains how to isolate a variable. The goal is to help the student feel more comfortable with solving applied problems. Ample opportunity is provided for the student to practice translating words to symbols, which is an important part of the "Five-Step Method" of solving applied problems (discussed in modules (<link document="m21980"/>) and (<link document="m21979"/>)). Objectives of this module: understand the meaning of inequalities, be able to recognize linear inequalities, know, and be able to work with, the algebra of linear inequalities and with compound inequalities.

1 Overview

- Inequalities
- Linear Inequalities
- The Algebra of Linear Inequalities
- Compound Inequalities

2 Inequalities

Relationships of Inequality

We have discovered that an equation is a mathematical way of expressing the relationship of equality between quantities. Not all relationships need be relationships of equality, however. Certainly the number of human beings on earth is greater than 20. Also, the average American consumes less than 10 grams of vitamin C every day. These types of relationships are not relationships of equality, but rather, relationships of **inequality**.

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3 Linear Inequalities

Linear Inequality
A **linear inequality** is a mathematical statement that one linear expression is greater than or less than another linear expression.
Inequality Notation
The following notation is used to express relationships of inequality:

$>$	Strictly greater than
$<$	Strictly less than
$\geq$	Greater than or equal to
$\leq$	Less than or equal to

Note that the expression $x > 12$ has infinitely many solutions. Any number strictly greater than 12 will satisfy the statement. Some solutions are 13, 15, 90, 12.1, 16.3 and 102.51.

4 Sample Set A

The following **are** linear inequalities in one variable.

Example 1

1. $x \leq 12$
2. $x + 7 > 4$
3. $y + 3 \geq 2y - 7$
4. $P + 26 < 10\,(4P - 6)$
5. $\frac{2r-9}{5} > 15$

The following **are not** linear inequalities in one variable.

Example 2

1. $x^2 < 4$.
   The term x^2 is quadratic, not linear.
2. $x \leq 5y + 3$.
   There are two variables. This is a linear inequality in two variables.
3. $y + 1 \neq 5$.
   Although the symbol $\neq$ certainly expresses an inequality, it is customary to use only the symbols $<$, $>$, $\leq$, $\geq$.

5 Practice Set A

A linear equation, we know, may have exactly one solution, infinitely many solutions, or no solution. Speculate on the number of solutions of a linear inequality. (**Hint:** Consider the inequalities $x < x - 6$ and $x \geq 9$.)

A linear inequality may have infinitely many solutions, or no solutions.

6 The Algebra of Linear Inequalities

Inequalities can be solved by basically the same methods as linear equations. There is one important exception that we will discuss in item 3 of the algebra of linear inequalities.

The Algebra of Linear Inequalities

Let a, b, and c represent real numbers and assume that

$a < b \qquad (\text{or } a > b)$

Then, if $a < b$,

1. $a + c < b + c \quad \text{and} \quad a - c < b - c$.
   If any real number is added to or subtracted from both sides of an inequality, the sense of the inequality remains unchanged.
2. If c is a **positive** real number, then if $a < b$,
   $ac < bc \quad \text{and} \quad \frac{a}{c} < \frac{b}{c}$.
   If both sides of an inequality are multiplied or divided by the same positive number the sense of the inequality remains unchanged.
3. If c is a **negative** real number, then if $a < b$,
   $ac > bc \quad \text{and} \quad \frac{a}{c} > \frac{b}{c}$.
   If both sides of an inequality are multiplied or divided by the same **negative** number, **the inequality sign must be reversed** (change direction) in order for the resulting inequality to be equivalent to the original inequality. (See problem 4 in the next set of examples.)

For example, consider the inequality $3 < 7$.

Example 3

For $3 < 7$, if 8 is added to both sides, we get

$3 + 8 < 7 + 8.$

$11 < 15 \qquad \text{True}$

Example 4

For $3 < 7$, if 8 is subtracted from both sides, we get

$3 - 8 < 7 - 8.$

$-5 < -1 \qquad \text{True}$

Example 5

For $3 < 7$, if both sides are multiplied by 8 (a positive number), we get

$8(3) < 8(7)$

$24 < 56 \qquad \text{True}$

Example 6

For $3 < 7$, if both sides are multiplied by -8 (a negative number), we get

$(-8)3 > (-8)7$

Notice the change in direction of the inequality sign.

$-24 > -56 \quad \text{True}$

If we had forgotten to reverse the direction of the inequality sign we would have obtained the incorrect statement$-24 < -56$.

Example 7

For $3 < 7$, if both sides are divided by 8 (a positive number), we get

$\frac{3}{8} < \frac{7}{8} \quad \text{True}$

Example 8

For $3 < 7$, if both sides are divided by -8 (a negative number), we get

$\frac{3}{-8} > \frac{7}{-8} \quad \text{True} \quad (\text{since } -.375 - .875)$

7 Sample Set B

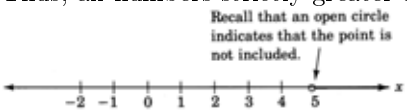
Solve the following linear inequalities. Draw a number line and place a point at each solution.

Example 9

$3x > 15$ Divide both sides by 3. The 3 is a positive number, so we need not reverse the sense of the inequality.

$x > 5$

Thus, all numbers strictly greater than 5 are solutions to the inequality $3x > 15$.

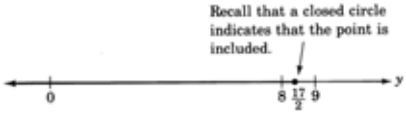


Example 10

$2y - 1 \leq 16$ Add 1 to both sides.

$2y \leq 17$ Divide both sides by 2.

$y \leq \frac{17}{2}$

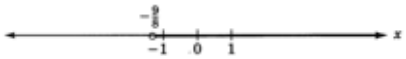


Example 11

$-8x + 5 < 14$ Subtract 5 from both sides.

$-8x < 9$ Divide both sides by -8. We must reverse the sense of the inequality since we are dividing by a negative number.

$x > -\frac{9}{8}$



Example 12

$5 - 3(y + 2) < 6y - 10$

$5 - 3y - 6 < 6y - 10$

$-3y - 1 < 6y - 10$

$-9y < -9$

$y > 1$

Example 13

$\frac{2z+7}{-4} \geq -6$ Multiply by -4

$2z + 7 \leq 24$ Notice the change in the sense of the inequality.

$2z \leq 17$

$z \leq \frac{17}{2}$

8 Practice Set B

Solve the following linear inequalities.

Exercise 1 *(Solution on p. 11.)*

$y - 6 \leq 5$

Exercise 2 *(Solution on p. 11.)*

$x + 4 > 9$

Exercise 3 *(Solution on p. 11.)*

$4x - 1 \geq 15$

Exercise 4 *(Solution on p. 11.)*

$-5y + 16 \leq 7$

Exercise 5 *(Solution on p. 11.)*

$7(4s - 3) < 2s + 8$

Exercise 6 *(Solution on p. 11.)*

$5(1 - 4h) + 4 < (1 - h)2 + 6$

Exercise 7 *(Solution on p. 11.)*

$18 \geq 4(2x - 3) - 9x$

Exercise 8 *(Solution on p. 11.)*

$-\frac{3b}{16} \leq 4$

Exercise 9 *(Solution on p. 11.)*

$\frac{-7z+10}{-12} < -1$

Exercise 10 *(Solution on p. 11.)*

$-x - \frac{2}{3} \leq \frac{5}{6}$

9 Compound Inequalities

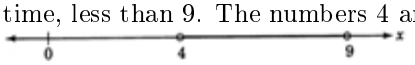
Compound Inequality

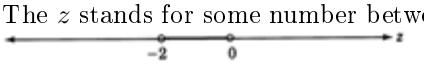
Another type of inequality is the **compound inequality**. A compound inequality is of the form:

$a < x < b$

There are actually two statements here. The first statement is $a < x$. The next statement is $x < b$. When we read this statement we say "a is less than x," then continue saying "and x is less than b."

Just by looking at the inequality we can see that the number x is between the numbers a and b. The compound inequality $a < x < b$ indicates "betweenness." Without changing the meaning, the statement $a < x$ can be read $x > a$. (Surely, if the number a is less than the number x, the number x must be greater than the number a.) Thus, we can read $a < x < b$ as "x is greater than a and at the same time is less than b." For example:

1. $4 < x < 9$.
   The letter x is some number strictly between 4 and 9. Hence, x is greater than 4 and, at the same time, less than 9. The numbers 4 and 9 are not included so we use open circles at these points.
   

2. $-2 < z < 0$.
   The z stands for some number between -2 and 0. Hence, z is greater than -2 but also less than 0.
   

3. $1 < x + 6 < 8$.
   The expression $x + 6$ represents some number strictly between 1 and 8. Hence, $x + 6$ represents some number strictly greater than 1, but less than 8.

4. $\frac{1}{4} \leq \frac{5x-2}{6} \leq \frac{7}{9}$.
The term $\frac{5x-2}{6}$ represents some number between and including $\frac{1}{4}$ and $\frac{7}{9}$. Hence, $\frac{5x-2}{6}$ represents some number greater than or equal to $\frac{1}{4}$ to but less than or equal to $\frac{7}{9}$.

Consider problem 3 above, $1 < x + 6 < 8$. The statement says that the quantity $x + 6$ is between 1 and 8. This statement will be true for only certain values of x. For example, if $x = 1$, the statement is true since $1 < 1 + 6 < 8$. However, if $x = 4.9$, the statement is false since $1 < 4.9 + 6 < 8$ is clearly not true. The first of the inequalities is satisfied since 1 is less than 10.9, but the second inequality is not satisfied since 10.9 is not less than 8.

We would like to know for exactly which values of x the statement $1 < x + 6 < 8$ is true. We proceed by using the properties discussed earlier in this section, but now we must apply the rules to all **three** parts rather than just the two parts in a regular inequality.

10 Sample Set C

Example 14
Solve $1 < x + 6 < 8$.

$1 - 6 < x + 6 - 6 < 8 - 6$ Subtract 6 from all three parts.

$-5 < x < 2$

Thus, if x is any number strictly between -5 and 2, the statement $1 < x + 6 < 8$ will be true.

Example 15
Solve $-3 < \frac{-2x-7}{5} < 8$.

$-3(5) < \frac{-2x-7}{5}(5) < 8(5)$	Multiply each part by 5.
$-15 < -2x - 7 < 40$	Add 7 to all three parts.
$-8 < -2x < 47$	Divide all three parts by -2.
$4 > x > -\frac{47}{2}$	Remember to reverse the direction of the inequality signs.
$-\frac{47}{2} < x < 4$	It is customary (but not necessary) to write the inequality so that inequality arrows point to the left.

Thus, if x is any number between $-\frac{47}{2}$ and 4, the original inequality will be satisfied.

11 Practice Set C

Find the values of x that satisfy the given continued inequality.

Exercise 11 *(Solution on p. 11.)*
$4 < x - 5 < 12$

Exercise 12 *(Solution on p. 11.)*
$-3 < 7y + 1 < 18$

Exercise 13 *(Solution on p. 11.)*
$0 \leq 1 - 6x \leq 7$

Exercise 14 *(Solution on p. 11.)*
$-5 \leq \frac{2x+1}{3} \leq 10$

Exercise 15 *(Solution on p. 11.)*

$9 < \frac{-4x+5}{-2} < 14$

Exercise 16 *(Solution on p. 11.)*

Does $4 < x < -1$ have a solution?

12 Exercises

For the following problems, solve the inequalities.

Exercise 17 *(Solution on p. 11.)*

$x + 7 < 12$

Exercise 18

$y - 5 \leq 8$

Exercise 19 *(Solution on p. 11.)*

$y + 19 \geq 2$

Exercise 20

$x - 5 > 16$

Exercise 21 *(Solution on p. 11.)*

$3x - 7 \leq 8$

Exercise 22

$9y - 12 \leq 6$

Exercise 23 *(Solution on p. 11.)*

$2z + 8 < 7$

Exercise 24

$4x - 14 > 21$

Exercise 25 *(Solution on p. 11.)*

$-5x \leq 20$

Exercise 26

$-8x < 40$

Exercise 27 *(Solution on p. 11.)*

$-7z < 77$

Exercise 28

$-3y > 39$

Exercise 29 *(Solution on p. 11.)*

$\frac{x}{4} \geq 12$

Exercise 30

$\frac{y}{7} > 3$

Exercise 31 *(Solution on p. 11.)*

$\frac{2x}{9} \geq 4$

Exercise 32

$\frac{5y}{2} \geq 15$

Exercise 33 *(Solution on p. 11.)*

$\frac{10x}{3} \leq 4$

Exercise 34

$\frac{-5y}{4} < 8$

Exercise 35 *(Solution on p. 12.)*
$\frac{-12b}{5} < 24$

Exercise 36
$\frac{-6a}{7} \leq -24$

Exercise 37 *(Solution on p. 12.)*
$\frac{8x}{-5} > 6$

Exercise 38
$\frac{14y}{-3} \geq -18$

Exercise 39 *(Solution on p. 12.)*
$\frac{21y}{-8} < -2$

Exercise 40
$-3x + 7 \leq -5$

Exercise 41 *(Solution on p. 12.)*
$-7y + 10 \leq -4$

Exercise 42
$6x - 11 < 31$

Exercise 43 *(Solution on p. 12.)*
$3x - 15 \leq 30$

Exercise 44
$-2y + \frac{4}{3} \leq -\frac{2}{3}$

Exercise 45 *(Solution on p. 12.)*
$5(2x - 5) \geq 15$

Exercise 46
$4(x + 1) > -12$

Exercise 47 *(Solution on p. 12.)*
$6(3x - 7) \geq 48$

Exercise 48
$3(-x + 3) > -27$

Exercise 49 *(Solution on p. 12.)*
$-4(y + 3) > 0$

Exercise 50
$-7(x - 77) \leq 0$

Exercise 51 *(Solution on p. 12.)*
$2x - 1 < x + 5$

Exercise 52
$6y + 12 \leq 5y - 1$

Exercise 53 *(Solution on p. 12.)*
$3x + 2 \leq 2x - 5$

Exercise 54
$4x + 5 > 5x - 11$

Exercise 55 *(Solution on p. 12.)*
$3x - 12 \geq 7x + 4$

Exercise 56
$-2x - 7 > 5x$

Exercise 57 *(Solution on p. 12.)*

$-x-4>-3x+12$

Exercise 58

$3-x\geq 4$

Exercise 59 *(Solution on p. 12.)*

$5-y\leq 14$

Exercise 60

$2-4x\leq -3+x$

Exercise 61 *(Solution on p. 12.)*

$3\left[4+5\left(x+1\right)\right]<-3$

Exercise 62

$2\left[6+2\left(3x-7\right)\right]\geq 4$

Exercise 63 *(Solution on p. 12.)*

$7\left[-3-4\left(x-1\right)\right]\leq 91$

Exercise 64

$-2\left(4x-1\right)<3\left(5x+8\right)$

Exercise 65 *(Solution on p. 12.)*

$-5\left(3x-2\right)>-3\left(-x-15\right)+1$

Exercise 66

$-.0091x\geq 2.885x-12.014$

Exercise 67 *(Solution on p. 12.)*

What numbers satisfy the condition: twice a number plus one is greater than negative three?

Exercise 68

What numbers satisfy the condition: eight more than three times a number is less than or equal to fourteen?

Exercise 69 *(Solution on p. 12.)*

One number is five times larger than another number. The difference between these two numbers is less than twenty-four. What are the largest possible values for the two numbers? Is there a smallest possible value for either number?

Exercise 70

The area of a rectangle is found by multiplying the length of the rectangle by the width of the rectangle. If the length of a rectangle is 8 feet, what is the largest possible measure for the width if it must be an integer (positive whole number) and the area must be less than 48 square feet?

13 Exercises for Review

Exercise 71 *(Solution on p. 12.)*

(here[1]) Simplify $\left(x^2y^3z^2\right)^5$.

Exercise 72

(here[2]) Simplify $-\left[-\left(-\left|-8\right|\right)\right]$.

[1] "Basic Properties of Real Numbers: The Power Rules for Exponents" <http://cnx.org/content/m21897/latest/>

[2] "Basic Operations with Real Numbers: Absolute Value" <http://cnx.org/content/m21876/latest/>

Exercise 73 *(Solution on p. 12.)*

(here[3]) Find the product. $(2x-7)\ (x+4)$.

Exercise 74

(here[4]) Twenty-five percent of a number is 12.32. What is the number?

Exercise 75 *(Solution on p. 12.)*

(here[5]) The perimeter of a triangle is 40 inches. If the length of each of the two legs is exactly twice the length of the base, how long is each leg?

[3] "Algebraic Expressions and Equations: Combining Polynomials Using Multiplication" <http://cnx.org/content/m21852/latest/>

[4] "Solving Linear Equations and Inequalities: Application II - Solving Problems" <http://cnx.org/content/m21980/latest/>

[5] "Solving Linear Equations and Inequalities: Application II - Solving Problems" <http://cnx.org/content/m21980/latest/>

Solutions to Exercises in this Module

Solution to Exercise (p. 5)
$y \leq 11$
Solution to Exercise (p. 5)
$x > 5$
Solution to Exercise (p. 5)
$x \geq 4$
Solution to Exercise (p. 5)
$y \geq \frac{9}{5}$
Solution to Exercise (p. 5)
$s < \frac{29}{2}$
Solution to Exercise (p. 5)
$h > \frac{1}{18}$
Solution to Exercise (p. 5)
$x \geq -30$
Solution to Exercise (p. 5)
$b \geq \frac{-64}{3}$
Solution to Exercise (p. 5)
$z < -\frac{2}{7}$
Solution to Exercise (p. 5)
$x \geq \frac{-3}{2}$
Solution to Exercise (p. 6)
$9 < x < 17$
Solution to Exercise (p. 6)
$-\frac{4}{7} < y < \frac{17}{7}$
Solution to Exercise (p. 6)
$-1 \leq x \leq \frac{1}{6}$
Solution to Exercise (p. 6)
$-8 \leq x \leq \frac{29}{2}$
Solution to Exercise (p. 6)
$\frac{23}{4} < x < \frac{33}{4}$
Solution to Exercise (p. 7)
no
Solution to Exercise (p. 7)
$x < 5$
Solution to Exercise (p. 7)
$y \geq -17$
Solution to Exercise (p. 7)
$x \leq 3$
Solution to Exercise (p. 7)
$z < -\frac{1}{2}$
Solution to Exercise (p. 7)
$x \geq -4$
Solution to Exercise (p. 7)
$z > -11$
Solution to Exercise (p. 7)
$x \geq 48$
Solution to Exercise (p. 7)
$x \geq 18$

Solution to Exercise (p. 7)
$x \leq \frac{6}{5}$
Solution to Exercise (p. 7)
$b > -10$
Solution to Exercise (p. 8)
$x < -\frac{15}{4}$
Solution to Exercise (p. 8)
$y > \frac{16}{21}$
Solution to Exercise (p. 8)
$y \geq 2$
Solution to Exercise (p. 8)
$x \leq 15$
Solution to Exercise (p. 8)
$x \geq 4$
Solution to Exercise (p. 8)
$x \geq 5$
Solution to Exercise (p. 8)
$y < -3$
Solution to Exercise (p. 8)
$x < 6$
Solution to Exercise (p. 8)
$x \leq -7$
Solution to Exercise (p. 8)
$x \leq -4$
Solution to Exercise (p. 8)
$x > 8$
Solution to Exercise (p. 9)
$y \geq -9$
Solution to Exercise (p. 9)
$x < -2$
Solution to Exercise (p. 9)
$x \geq -3$
Solution to Exercise (p. 9)
$x < -2$
Solution to Exercise (p. 9)
$x > -2$
Solution to Exercise (p. 9)
First number: any number strictly smaller that 6.
Second number: any number strictly smaller than 30.
No smallest possible value for either number.
No largest possible value for either number.
Solution to Exercise (p. 9)
$x^{10}y^{15}z^{10}$
Solution to Exercise (p. 10)
$2x^2 + x - 28$
Solution to Exercise (p. 10)
16 inches