Systems of Linear Equations: Elimination by Substitution*

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Abstract

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. Beginning with the graphical solution of systems, this chapter includes an interpretation of independent, inconsistent, and dependent systems and examples to illustrate the applications for these systems. The substitution method and the addition method of solving a system by elimination are explained, noting when to use each method. The five-step method is again used to illustrate the solutions of value and rate problems (coin and mixture problems), using drawings that correspond to the actual situation. Objectives of this module: know when the substitution method works best, be able to use the substitution method to solve a system of linear equations, know what to expect when using substitution with a system that consists of parallel lines.

1 Overview

- When Substitution Works Best
- The Substitution Method
- Substitution and Parallel Lines
- Substitution and Coincident Lines

2 When Substitution Works Best

We know how to solve a linear equation in one variable. We shall now study a method for solving a system of two linear equations in two variables by transforming the two equations in two variables into one equation in one variable.

To make this transformation, we need to eliminate one equation and one variable. We can make this **elimination by substitution**.

When Substitution Works Best

The substitution method works best when **either** of these conditions exists:

1. One of the variables has a coefficient of 1, *or*
2. One of the variables can be made to have a coefficient of 1 without introducing fractions.

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3 The Substitution Method

The Substitution Method
To solve a system of two linear equations in two variables,

1. Solve one of the equations for one of the variables.
2. Substitute the expression for the variable chosen in step 1 into the other equation.
3. Solve the resulting equation in one variable.
4. Substitute the value obtained in step 3 into the equation obtained in step 1 and solve to obtain the value of the other variable.
5. Check the solution in both equations.
6. Write the solution as an ordered pair.

4 Sample Set A

Example 1

Solve the system $\begin{cases} 2x+3y=14 & (1) \\ 3x+y=7 & (2) \end{cases}$

Step 1: Since the coefficient of y in equation 2 is 1, we will solve equation 2 for y.

$y=-3x+7$

Step 2: Substitute the expression $-3x+7$ for y in equation 1.

$2x+3(-3x+7)=14$

Step 3: Solve the equation obtained in step 2.

$$\begin{aligned} 2x+3(-3x+7) &= 14 \\ 2x-9x+21 &= 14 \\ -7x+21 &= 14 \\ -7x &= -7 \\ x &= 1 \end{aligned}$$

Step 4: Substitute $x=1$ into the equation obtained in step 1, $y=-3x+7$.

$$\begin{aligned} y &= -3(1)+7 \\ y &= -3+7 \\ y &= 4 \end{aligned}$$

We now have $x=1$ and $y=4$.

Step 5: Substitute $x=1, y=4$ into each of the original equations for a check.

(1)
$$\begin{aligned} 2x+3y &= 14 & \\ 2(1)+3(4) &= 14 & \text{Is this correct?} \\ 2+12 &= 14 & \text{Is this correct?} \\ 14 &= 14 & \text{Yes, this is correct.} \end{aligned}$$

(2)
$$\begin{aligned} 3x+y &= 7 & \\ 3(1)+(4) &= 7 & \text{Is this correct?} \\ 3+4 &= 7 & \text{Is this correct?} \\ 7 &= 7 & \text{Yes, this is correct.} \end{aligned}$$

Step 6: The solution is $(1,4)$. The point $(1,4)$ is the point of intersection of the two lines of the system.

5 Practice Set A

Exercise 1 ***(Solution on p. 10.)***

Slove the system $\begin{cases} 5x-8y=18 \\ 4x+\ y=\ \ 7 \end{cases}$

6 Substitution And Parallel Lines

The following rule alerts us to the fact that the two lines of a system are parallel.
Substitution and Parallel Lines
If computations eliminate all the variables and produce a contradiction, the two lines of a system are parallel, and the system is called inconsistent.

7 Sample Set B

Example 2

Solve the system $\begin{cases} 2x-y=1 & (1) \\ 4x-2y=4 & (2) \end{cases}$

Step 1: Solve equation 1 for y.

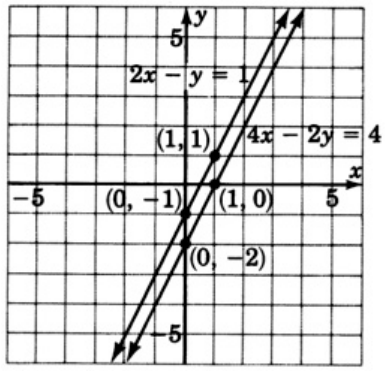
$$\begin{aligned} 2x-y &= 1 \\ -y &= -2x+1 \\ y &= 2x-1 \end{aligned}$$

Step 2: Substitute the expression $2x-1$ for y into equation 2.
$4x-2(2x-1)=4$

Step 3: Solve the equation obtained in step 2.

$$\begin{aligned} 4x-2(2x-1) &= 4 \\ 4x-4x+2 &= 4 \\ 2 &\neq 4 \end{aligned}$$

Computations have eliminated all the variables and produce a contradiction. These lines are parallel.



This system is inconsistent.

8 Practice Set B

Exercise 2 ***(Solution on p. 10.)***

Slove the system $\begin{cases} 7x - 3y = 2 \\ 14x - 6y = 1 \end{cases}$

9 Substitution And Coincident Lines

The following rule alerts us to the fact that the two lines of a system are coincident.
Substitution and Coincident Lines
If computations eliminate all the variables and produce an identity, the two lines of a system are coincident and the system is called dependent.

10 Sample Set C

Example 3

Solve the system $\begin{cases} 4x + 8y = 8 & (1) \\ 3x + 6y = 6 & (2) \end{cases}$

Step 1: Divide equation 1 by 4 and solve for x.

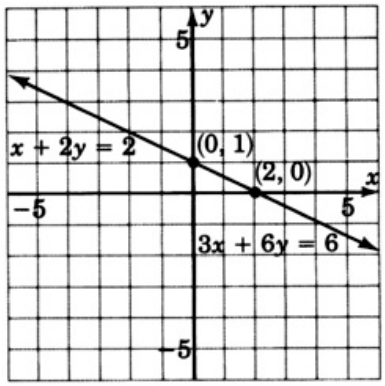
$$\begin{aligned} 4x + 8y &= 8 \\ x + 2y &= 2 \\ x &= -2y + 2 \end{aligned}$$

Step 2: Substitute the expression $-2y + 2$ for x in equation 2.
$3(-2y + 2) + 6y = 6$

Step 3: Solve the equation obtained in step 2.

$$\begin{aligned} 3(-2y + 2) + 6y &= 6 \\ -6y + 6 + 6y &= 6 \\ 6 &= 6 \end{aligned}$$

Computations have eliminated all the variables and produced an identity. These lines are coincident.



This system is dependent.

11 Practice Set C

Exercise 3 ***(Solution on p. 10.)***

Solve the system $\begin{cases} 4x + 3y = 1 \\ -8x - 6y = -2 \end{cases}$

Systems in which a coefficient of one of the variables is not 1 or cannot be made to be 1 without introducing fractions are not well suited for the substitution method. The problem in Sample Set D illustrates this "messy" situation.

12 Sample Set D

Example 4

Solve the system $\begin{cases} 3x + 2y = 1 & (1) \\ 4x - 3y = 3 & (2) \end{cases}$

Step 1: We will solve equation (1) for y.

$$\begin{aligned} 3x + 2y &= 1 \\ 2y &= -3x + 1 \\ y &= \tfrac{-3}{2}x + \tfrac{1}{2} \end{aligned}$$

Step 2: Substitute the expression $\frac{-3}{2}x + \frac{1}{2}$ for y in equation (2).
$4x - 3\left(\frac{-3}{2}x + \frac{1}{2}\right) = 3$

Step 3: Solve the equation obtained in step 2.

$$\begin{aligned} 4x - 3\left(\tfrac{-3}{2}x + \tfrac{1}{2}\right) &= 3 \qquad \text{Multiply both sides by the LCD, 2.} \\ 4x + \tfrac{9}{2}x - \tfrac{3}{2} &= 3 \\ 8x + 9x - 3 &= 6 \\ 17x - 3 &= 6 \\ 17x &= 9 \\ x &= \tfrac{9}{17} \end{aligned}$$

Step 4: Substitute $x = \frac{9}{17}$ into the equation obtained in step 1, $y = \frac{-3}{2}x + \frac{1}{2}$.
$y = \frac{-3}{2}\left(\frac{9}{17}\right) + \frac{1}{2}$
$y = \frac{-27}{34} + \frac{17}{34} = \frac{-10}{34} = \frac{-5}{17}$
We now have $x = \frac{9}{17}$ and $y = \frac{-5}{17}$.

Step 5: Substitution will show that these values of x and y check.

Step 6: The solution is $\left(\frac{9}{17}, \frac{-5}{17}\right)$.

13 Practice Set D

Exercise 4 *(Solution on p. 10.)*

Solve the system $\begin{cases} 9x - 5y = -4 \\ 2x + 7y = -9 \end{cases}$

14 Exercises

For the following problems, solve the systems by substitution.

Exercise 5 *(Solution on p. 10.)*

$$\begin{cases} 3x + 2y &= 9 \\ y &= -3x + 6 \end{cases}$$

Exercise 6

$$\begin{cases} 5x - 3y &= -6 \\ y &= -4x + 12 \end{cases}$$

Exercise 7 *(Solution on p. 10.)*

$$\begin{cases} 2x + 2y &= 0 \\ x &= 3y - 4 \end{cases}$$

Exercise 8

$$\begin{cases} 3x + 5y &= 9 \\ x &= 4y - 14 \end{cases}$$

Exercise 9 *(Solution on p. 10.)*

$$\begin{cases} -3x + y &= -4 \\ 2x + 3y &= 10 \end{cases}$$

Exercise 10

$$\begin{cases} -4x + y &= -7 \\ 2x + 5y &= 9 \end{cases}$$

Exercise 11 *(Solution on p. 10.)*

$$\begin{cases} 6x - 6 &= 18 \\ x + 3y &= 3 \end{cases}$$

Exercise 12

$$\begin{cases} -x-y &= 5 \\ 2x+y &= 5 \end{cases}$$

Exercise 13 *(Solution on p. 10.)*

$$\begin{cases} -5x+y &= 4 \\ 10x-2y &= -8 \end{cases}$$

Exercise 14

$$\begin{cases} x+4y &= 1 \\ -3x-12y &= -1 \end{cases}$$

Exercise 15 *(Solution on p. 10.)*

$$\begin{cases} 4x-2y &= 8 \\ 6x+3y &= 0 \end{cases}$$

Exercise 16

$$\begin{cases} 2x+3y &= 12 \\ 2x+4y &= 18 \end{cases}$$

Exercise 17 *(Solution on p. 10.)*

$$\begin{cases} 3x-9y &= 6 \\ 6x-18y &= 5 \end{cases}$$

Exercise 18

$$\begin{cases} -x+4y &= 8 \\ 3x-12y &= 10 \end{cases}$$

Exercise 19 *(Solution on p. 10.)*

$$\begin{cases} x+y &= -6 \\ x-y &= 4 \end{cases}$$

Exercise 20

$$\begin{cases} 2x+y &= 0 \\ x-3y &= 0 \end{cases}$$

Exercise 21 *(Solution on p. 10.)*

$$\begin{cases} 4x-2y &= 7 \\ y &= 4 \end{cases}$$

Exercise 22

$$\begin{cases} x+6y &= 11 \\ x &= -1 \end{cases}$$

Exercise 23 *(Solution on p. 10.)*

$$\begin{cases} 2x-4y &= 10 \\ 3x=5y &+ 12 \end{cases}$$

Exercise 24

$$\begin{cases} y+7x+4 &= 0 \\ x=-7y &+ 28 \end{cases}$$

Exercise 25 *(Solution on p. 10.)*

$$\begin{cases} x+4y=0 \\ x+\frac{2}{3}y=\frac{10}{3} \end{cases}$$

Exercise 26

$$\begin{cases} x=24-5y \\ x-\frac{5}{4}y=\frac{3}{2} \end{cases}$$

Exercise 27 *(Solution on p. 10.)*

$$\begin{cases} x=11-6y \\ 3x+18y=-33 \end{cases}$$

Exercise 28

$$\begin{cases} 2x+\frac{1}{3}y=4 \\ 3x+6y=39 \end{cases}$$

Exercise 29 *(Solution on p. 10.)*

$$\begin{cases} \frac{4}{5}x+\frac{1}{2}y=\frac{3}{10} \\ \frac{1}{3}x+\frac{1}{2}y=\frac{-1}{6} \end{cases}$$

Exercise 30

$$\begin{cases} x-\frac{1}{3}y &= \frac{-8}{3} \\ -3x+y &= 1 \end{cases}$$

15 Exercises For Review

Exercise 31 *(Solution on p. 10.)*

() Find the quotient: $\frac{x^2-x-12}{x^2-2x-15} \div \frac{x^2-3x-10}{x^2-2x-8}$.

Exercise 32

() Find the difference: $\frac{x+2}{x^2+5x+6} - \frac{x+1}{x^2+4x+3}$.

Exercise 33 *(Solution on p. 10.)*

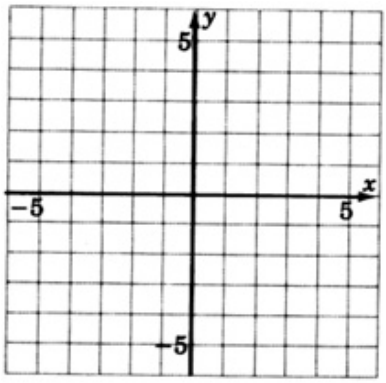
() Simplify $-\sqrt{81x^8y^5z^4}$.

Exercise 34

() Use the quadratic formula to solve $2x^2+2x-3=0$.

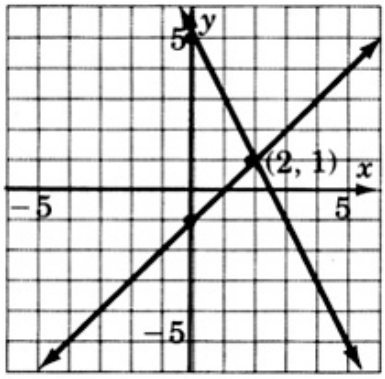
Exercise 35 *(Solution on p. 10.)*

() Solve by graphing $\begin{cases} x-y=1 \\ 2x+y=5 \end{cases}$


y
5
x
−5
5
−5

Solutions to Exercises in this Module

Solution to Exercise (p. 3)
The point $(2, -1)$ is the point of intersection of the two lines.
Solution to Exercise (p. 4)
Substitution produces $4 \neq 1$, or $\frac{1}{2} \neq 2$, a contradiction. These lines are parallel and the system is inconsistent.
Solution to Exercise (p. 5)
Computations produce $-2 = -2$, an identity. These lines are coincident and the system is dependent.
Solution to Exercise (p. 6)
These lines intersect at the point $(-1, -1)$.
Solution to Exercise (p. 6)
$(1, 3)$
Solution to Exercise (p. 6)
$(-1, 1)$
Solution to Exercise (p. 6)
$(2, 2)$
Solution to Exercise (p. 6)
$\left(4, -\frac{1}{3}\right)$
Solution to Exercise (p. 7)
Dependent (same line)
Solution to Exercise (p. 7)
$(1, -2)$
Solution to Exercise (p. 7)
inconsistent (parallel lines)
Solution to Exercise (p. 7)
$(-1, -5)$
Solution to Exercise (p. 7)
$\left(\frac{15}{4}, 4\right)$
Solution to Exercise (p. 7)
$(-1, -3)$
Solution to Exercise (p. 7)
$(4, -1)$
Solution to Exercise (p. 8)
inconsistent (parallel lines)
Solution to Exercise (p. 8)
$(1, -1)$
Solution to Exercise (p. 8)
$\frac{(x-4)^2}{(x-5)^2}$
Solution to Exercise (p. 8)
$-9x^4y^2z^2\sqrt{y}$
Solution to Exercise (p. 8)
$(2, 1)$


y
5
(2, 1)
x
−5
5
−5