Lecture 5:The Laplace Transform Method of Solution*

Truc Pham-Dinh
Tam Huynh-Ngoc
Anh Tuan Hoang
Thanh Dinh Vu

Abstract

Continue to describe methods for representing signals as superpositions of complex exponential functions. Develop efficient methods for analyzing LTI systems.

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Lecture #5: The Laplace transform method of solution
Motivation:

- Continue to describe methods for representing signals as superpositions of complex exponential functions
- Develop efficient methods for analyzing LTI systems

Outline:

- Review of last lecture
- Laplace transform of the family of singularity functions
- More on the region of convergence
- Analysis of networks with the Laplace transform — the impedance method
- Finding inverse transforms — partial fraction expansion
- Conclusion

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- Historical perspective — Oliver Heaviside

Review

- The Laplace transform represents a time function as a superposition of complex exponentials.
- A time function is related uniquely to a Laplace transform if the ROC is specified.
- If the Laplace transform of a sum of causal and anti-causal exponential time functions exists, its ROC is a strip in the s-plane parallel to the jω-axis.

Laplace transforms of singularity functions

Unit impulse function

$L\{\delta(t)\} = \sum_{-\infty}^{\infty} \delta(t) . e^{-\mathrm{st}} \mathrm{dt}$

Recall the definition of the unit impulse

$\sum_{=\infty}^{\infty} \delta(t) f(t) \mathrm{dt} = f(0)$

Hence,

$L\{\delta(t)\} = 1$

for all values of s. The region of convergence is the entire s plane.

Unit impulse function delayed — use of properties

The Laplace transform of an impulse located at t = 0 is

$L\{\delta(t)\} = 1$

Using the delay property, $x(t) \overset{L}{\Leftrightarrow} X(s)$

$x(t-T) \overset{L}{\Leftrightarrow} X(s) e^{-s/T}$

the Laplace transform of the delayed impulse is

$L\{\delta(t-T)\} = e^{-\mathrm{sT}}$

and the region of convergence is the whole s plane.

Two-minute miniquiz problem

Problem 5-1

Find the Laplace transform including the ROC for

$x(t) = e^{-2(t-4)} u(t-4)$

Two-minute miniquiz solution

Problem 5-1

We use the Laplace transform of the causal exponential time function and time delay property to solve this problem.

$e^{-2t} u(t) \overset{L}{\Leftrightarrow} \frac{1}{s+2} \quad \text{for} \quad \sigma > -2$

$e^{-2(t-4)} u(t-4) \overset{L}{\Leftrightarrow} \frac{1}{s+2} e^{-4s} \quad \text{for} \quad \sigma > -2$

Singularity functions and their relatives

The Laplace transform of a unit impulse is

$\delta(t) \overset{L}{\Leftrightarrow} 1 \quad \text{for} \quad \text{all} \quad s$

and from the Laplace transform of a causal exponential with $\alpha = 0$ we have the Laplace transform of a causal step function

$u(t) \overset{L}{\Leftrightarrow} \frac{1}{s} \quad \text{for} \quad \sigma > 0$

Note this fits together with the time differentiation property

$\frac{\mathrm{dx}(t)}{\mathrm{dt}} \overset{L}{\Leftrightarrow} \mathrm{sX}(s)$

since in a generalized function sense

$\delta(t) = \frac{\mathrm{du}(t)}{\mathrm{dt}} \overset{L}{\Leftrightarrow} L\{\delta(t)\} = s\left(\frac{1}{s}\right) = 1$

Singularity functions and their relatives, cont'd

We use the multiplication by t property
$\mathrm{tx}\left(t\right)\overset{L}{\Leftrightarrow}-\frac{\mathrm{dX}(s)}{\mathrm{ds}}$
to obtain
$\mathrm{tu}\left(t\right)\overset{L}{\Leftrightarrow}-\frac{d}{\mathrm{ds}}\left(\frac{1}{s}\right)=\frac{1}{s^2}\mathrm{for}\sigma>0$
and use it again to obtain
$t^2u\left(t\right)\overset{L}{\Leftrightarrow}-\frac{d}{\mathrm{ds}}\left(\frac{1}{s^2}\right)=\frac{1}{s^3}\mathrm{for}\sigma>0$
which implies that by induction
or
$\mathrm{t}^nu\left(t\right)\overset{L}{\Leftrightarrow}\frac{n!}{s^{n+1}}\mathrm{for}\sigma>0$
$\frac{t^{n-1}}{(n-1)!}u\left(t\right)\overset{L}{\Leftrightarrow}\frac{1}{s^n}\mathrm{for}\sigma>0$
Summary of singularity functions and their relatives

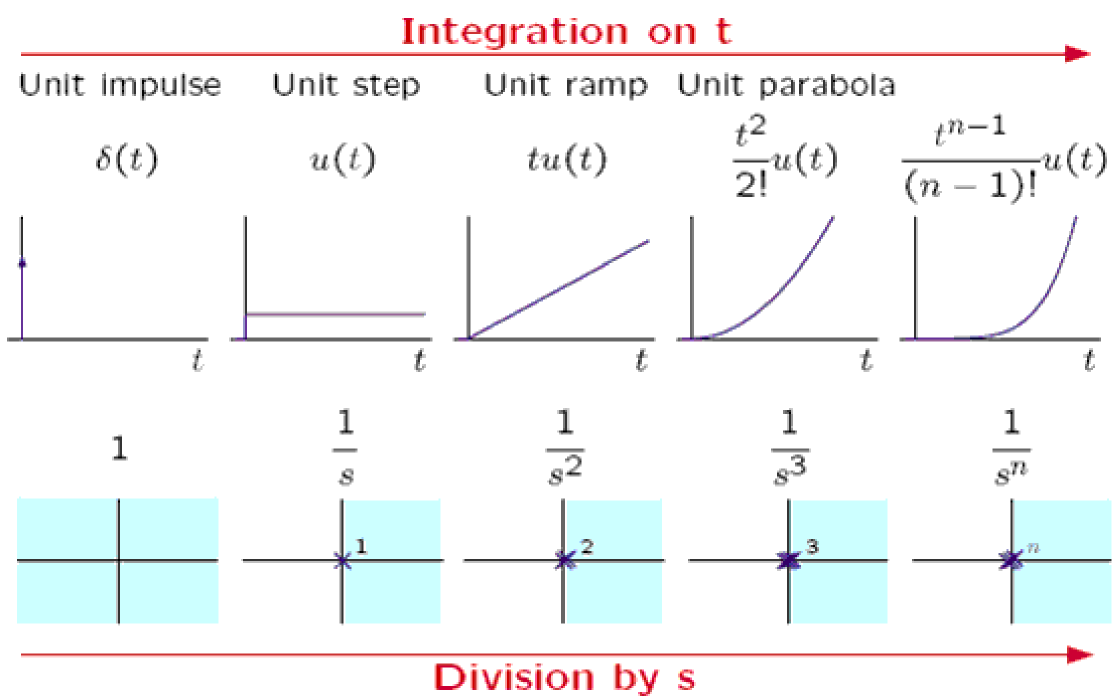

Figure 1

Wild and crazy singularity functions

Since taking the derivative of a time function corresponds to multiplying the Laplace transform by s we can contemplate the derivative of the unit impulse called the unit doublet.

$\frac{\mathrm{d}\delta(t)}{\mathrm{dt}}=\delta\left(t\right)\overset{L}{\Leftrightarrow}s$

This process can be continued by taking successive derivatives of the impulse to form the unit triplet which has Laplace transform s^2, unit quadruplet, etc. In general, the nth derivative of the unit impulse has a Laplace transform s^n. We shall consider the usefulness of these higher order singularity functions later!

General comments on the ROC

- Unit impulse ⇒ ROC is the whole s plane.
- Finite duration, absolutely integrable time function ⇒ ROC is the whole s plane.
- Time shifting a time function does not change its ROC.
- Right-sided time function ⇒ ROC is to the right of the rightmost pole.
- Left-sided time function ⇒ ROC is to left of the left-most pole.

General comments on the ROC, cont'd

- A sum of causal and anti-causal exponential time functions that has a Laplace transform ⇒ ROC is a strip in the s plane.
- There are no poles in the ROC.
- Some time functions do not have Laplace transforms, e.g., $x(t) = e^{-t}$ for all t.

Analysis of networks with the Laplace transform — the impedance method Kirchhoff's laws

Kirchhoff's current and voltage laws are algebraic equations that
link the branch variables in a network,

$\sum_{\text{node}} i_k(t) = 0 \quad \text{and} \quad \sum_{\text{loop}} v_k(t) = 0$

If we take the Laplace transform of these equations then we obtain

$\sum_{\text{node}} I_k(s) = 0 \quad \text{and} \quad \sum_{\text{loop}} V_k(s) = 0$

Hence, the Laplace transforms of the branch variables satisfy KCL and KVL.

Constitutive relations

Resistance, capacitance, and inductance

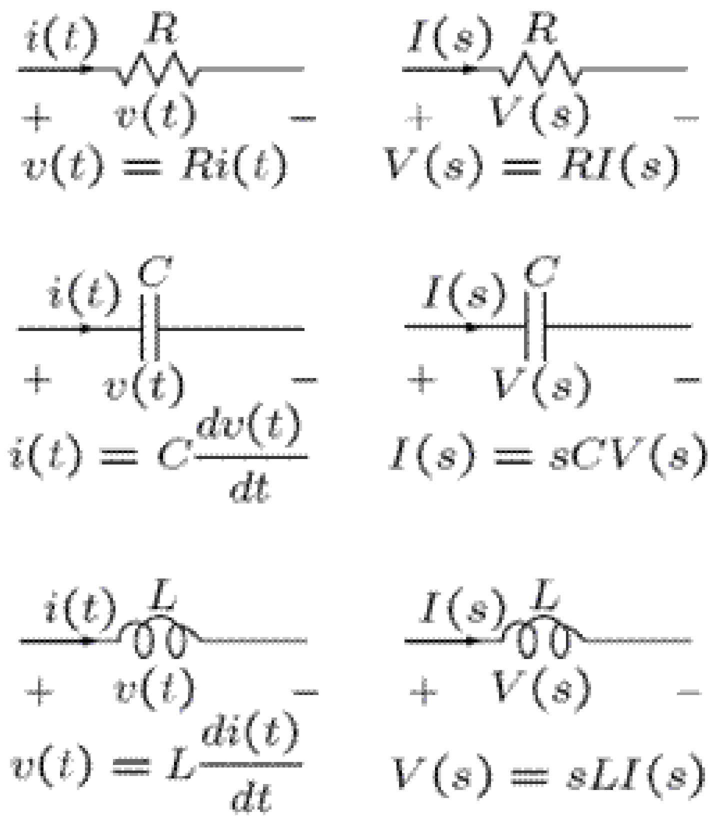

Figure 2

Impedance and admittance

The ratios of voltage to current and current to voltage are system functions with special names. The impedance is defined as

$Z(s) = \frac{V(s)}{I(s)}$

and the admittance is defined as

$Y(s) = \frac{I(s)}{V(s)}$

The impedance and admittance of the resistance, capacitance, and inductance are

Element	Z(s)	Y(s)
Resistance	R	G=1/R
Capacitance	$\frac{1}{sC}$	sC
Inductance	SL	$\frac{1}{sL}$

Figure 3

Significance

- The equilibrium equations of a network involve KCL, KVL, and the constitutive relations.
- KCL and KVL are algebraic equations for both time functions and their Laplace transforms.
- In terms of time functions, the constitutive relations involve derivatives.
- In terms of Laplace transforms, the constitutive relations are algebraic.

Conclusions

- Analysis of a network in the time domain leads to differential equations.
- Analysis of a network in the Laplace transform domain leads to algebraic equations.

Thus, R, L, and C networks can be analyzed using methods developed for resistive networks. These include: use of series and parallel combinations, voltage and current dividers, as well as Thévenin's and Norton's equivalents. Methods work just as well for any system (e.g., mechanical, acoustic, chemical, etc.) that is analogous to an electric circuit.

Example — impulse response of an RLC network

We wish to find the output voltage $v_0(t)$ for the RLC network to an impulse of input voltage,

$v_i(t) = \delta(t)$

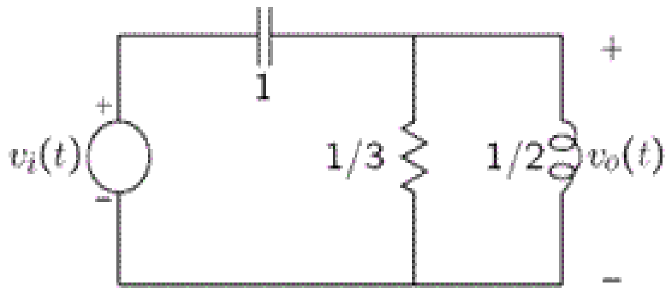

Figure 4

The numbers show the values of the capacitance, resistance, and inductance in farads, ohms, and henries, respectively.

The response of an LTI system to an impulse is important and is called the impulse response and is usually designated by h(t).

Solution to network by impedance method

The first step is to redraw the network in terms of Laplace transforms of variables and the impedances of the elements.

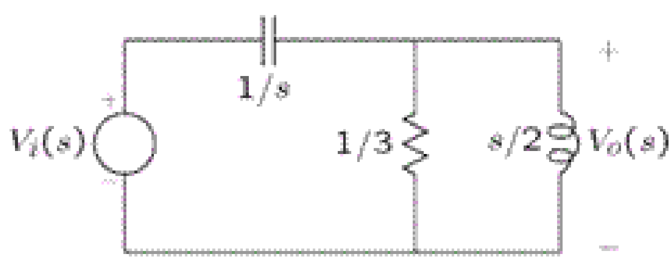

Figure 5

The impedance is shown next to each network element.

Vi(s) is divided between the voltage on the capacitance and that on the parallel resistance and inductance combination. The impedance of the parallel resistance and inductance is

$Z_{\text{RL}} = \frac{1}{\left(\frac{2}{s}\right)+3} = \frac{s}{3s+2}$

and therefore the output voltage is

$V_o\left(s\right) = \frac{\frac{s}{3s+2}}{\frac{1}{s}+\frac{s}{3s+2}} V_i\left(s\right) = \frac{s^2}{s^2+3s+2} V_i\left(s\right) = \frac{s^2}{(s+1)(s+2)} V_i\left(s\right)$

Laplace transform of output voltage

The next step is to find the Laplace transform of the input voltage. Since

$v_i\left(t\right) = \delta\left(t\right)$

$V_i(s) = 1 \quad \text{for} \quad \text{all} \quad s$

Therefore, since

$V_0(s) = H(s).V_i(s)$

Where

$V_0(s) = H(s) = \frac{s^2}{(s+1)(s+2)}$

This shows that the Laplace transform of the impulse response of a system equals the system function,

$h(t) \overset{L}{\Leftrightarrow} H(s)$

Since H(s) characterizes the system, so does h(t).

Region of convergence of system function

What is the ROC of this system function? Because the network is a passive RLC network, the system is causal, i.e., the impulse response cannot precede the occurrence of the impulse. Thus, the ROC is to the right of the rightmost pole, i.e., $\sigma > -1$. So we have the following pole-zero diagram and ROC for

$V_0(s) = H(s) = \frac{s^2}{(s+1)(s+2)} \quad \text{for} \quad \sigma > 1$

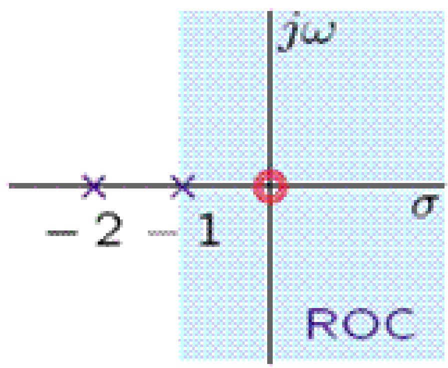

Figure 6

Partial fraction expansion of the Laplace transform of the output voltage

The Laplace transform of the output voltage is

$V_0(s) = H(s) = \frac{s^2}{(s+1)(s+2)} \quad \text{for} \quad \sigma > -1$

Note that H(s) is an improper rational function. A rational function is a ratio of polynomials. A proper rational function has a denominator polynomial whose order exceeds that of the numerator. The first step in finding the voltage as a function of time is to expand H(s) into a polynomial and a proper rational function.

$H(s) = P(s) + H_P(s)$

where P(s) is a polynomial and $H_P(s)$ is a proper rational function.

Synthetic division

We can synthetically divide the denominator into the numerator of $V_0(s) = H(s) = \frac{s^2}{(s+1)(s+2)} = \frac{s^2}{s^2+3s+2}$ as follows

$\frac{s^2}{(s^2+3s+2)} = \frac{(s^2+3s+2)-(3s+2)}{s^2+3s+2} = 1 - \frac{3s+2}{(s+1)(s+2)}$
Partial fraction expansion
We can expand the proper rational function in a partial fraction expansion of the form
$H_p(s) = -\frac{3s+2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$
The coefficient A is found as follows
$[(s+1)\,H_p(s)]\,.|_{s=-1} = \left[(s+1)\frac{A}{s+1} + (s+1)\frac{B}{s+2}\right].|_{s=-1} = A$
Therefore,
$A = \frac{3-2}{-1+2} = 1$
By a similar argument
$B = \frac{6-2}{-2+1} = -4$
so that
$V_0(s) = H(s) = 1 + \frac{1}{s+1} - \frac{4}{s+2}$
Inverse Laplace transform of output voltage
The partial fraction expansion shows that
$V_0(s) = H(s) = 1 + \frac{1}{s+1} - \frac{4}{s+2}\text{for}\sigma > -1$
Therefore,
$v_0(t) = h(t) = \delta(t) + e^{-t}u(t) - 4e^{-2t}u(t)$

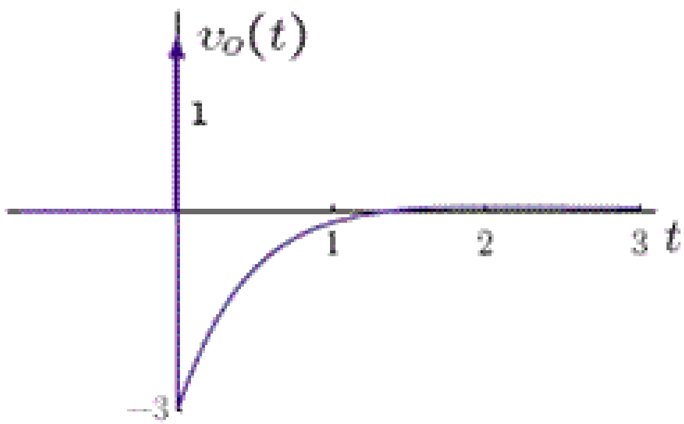

Figure 7

Physical interpretation of result

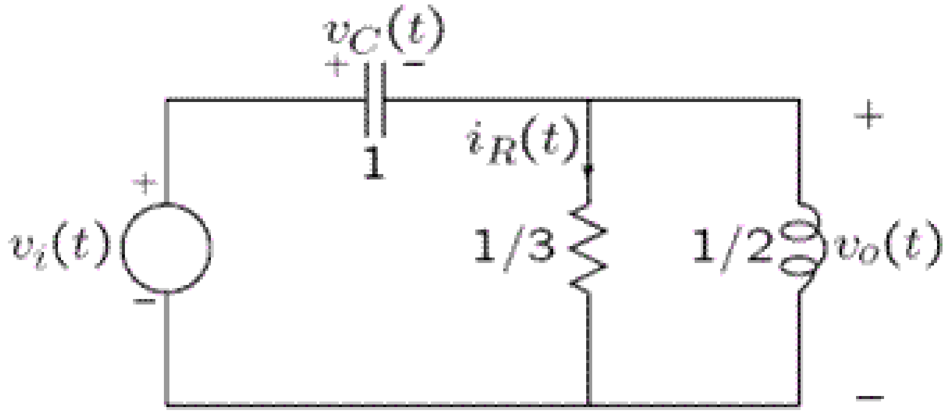

Figure 8

$v_0(t) = h(t) = \delta(t) + e^{-t}u(t) - 4e^{-2t}u(t)$

How can we explain the impulse response of this circuit in physical terms. There are three critical times:

(1) at t = 0, $v_0(t)$ has a unit impulse and a discontinuity of value −3;

(2) for t>0, $v_0(t)$ consists of complexex ponentials at the frequencies −1 and −2;

(3) as $t \to \infty$, $v_0(t) \to 0$

The voltages and currents in the network must satisfy KVL and KCL plus the constitutive relations of the elements.

- The reasoning at t = 0 is tricky. If the impulse in $v_i(t)$ appeared in $v_C(t)$ that would cause a doublet in current that cannot be matched to satisfy KCL. Therefore, the impulse appears in $v_0(t)$ which causes an impulse in $i_R(t) = 3\delta(t)$ which flows through the capacitance to cause a step $v_C(t) = 3u(t)$ which appears as an initial step in $v_0(t)$.

- After the impulse occurs, the capacitance has an initial voltage and the inductance has an initial current, i.e., the network is energized. All voltages and currents now relaxex ponentially at the natural frequencies of −1 and −2.

- Since the network is lossy, the natural frequencies are in the left-half of the s plane all voltages and current decay to zero.

Two-minute miniquiz problem

Problem 5-2

Consider the network shown below.

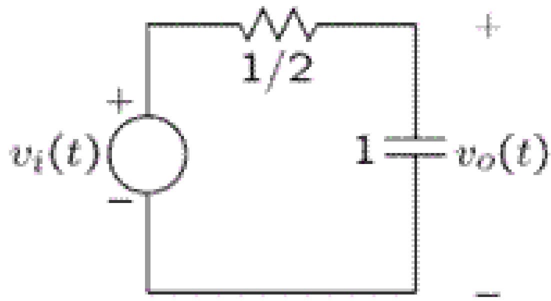

Figure 9

The input voltage $v_i\left(t\right)$is
$v_i\left(t\right) = e^{-t} u\left(t\right)$
Determine $v_0\left(t\right)$.
Two-minute miniquiz solution
Problem 5-2
The system function is
$H\left(s\right) = \frac{V_0(s)}{V_i(s)} = \frac{\frac{1}{s}}{\frac{1}{2}+\frac{1}{s}} = \frac{2}{s+2}$
The Laplace transform of the input voltage is
$V_i\left(s\right) = \frac{1}{s+1}$
Therefore,
$V_0\left(s\right) = V_i\left(s\right) H\left(s\right) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$
And
$v_0\left(t\right) = 2e^{-t} u\left(t\right) - 2e^{-2t} u\left(t\right)$
Conclusion — Laplace transform method for finding the response of an LTI system

Figure 10

- Find Laplace transform of input $x\left(t\right) \overset{L}{\Leftrightarrow} X\left(s\right)$

Conclusion cont'd

- Determine system function H(s) from

- impulse response of system $h\left(t\right) \overset{L}{\Leftrightarrow} H\left(s\right)$

- structural model of system using impedance method PLUS knowledge about causality, stability. etc.;
- differential equation PLUS knowledge about causality, stability, etc.

- Determine Laplace transform of output Y (s) = H(s)X(s).

- Determine output time function $y(t) \overset{L}{\Leftrightarrow} Y(s)$

Conclusion cont'd

This method can be summarized as follows

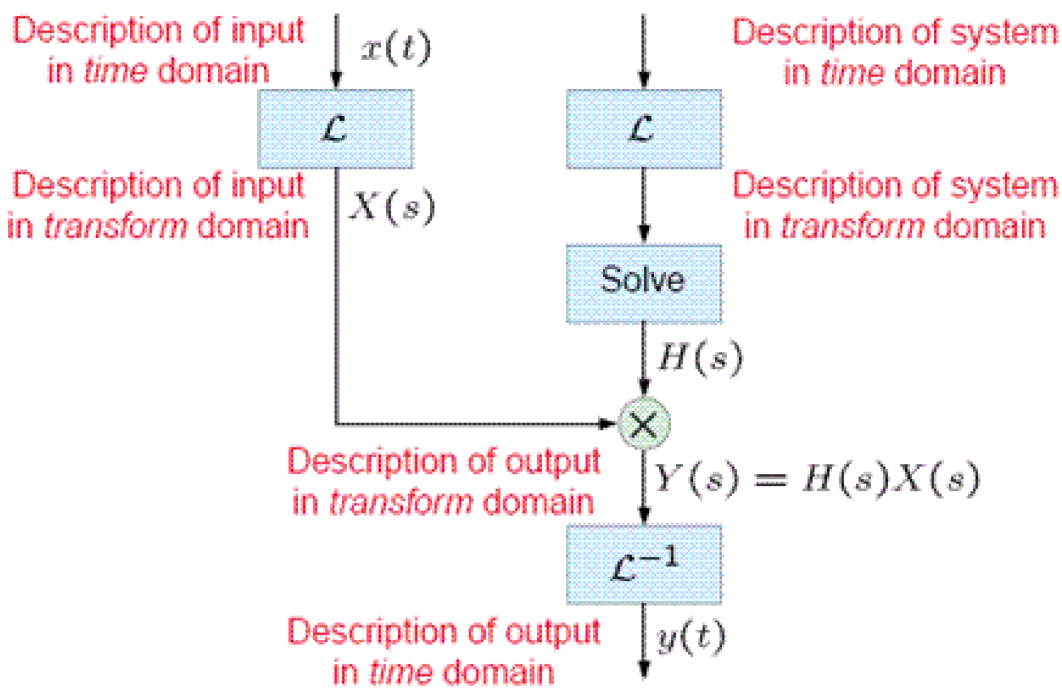

Figure 11

Historical perspective

Oliver Heaviside (1850-1925)

Figure 12

James Clerk Maxwell (1831-1879) died of cancer at age 48 before his ideas on electromagnetic theory could be completely worked out and disseminated. That job was left to three younger men known as the Maxwellians — Oliver Lodge, George Francis FitzGerald, and Oliver Heaviside (shown on the left).

Oliver Heaviside, cont'd

- Born in London on May 18, 1850.
- Nephew of Charles Wheatstone a pioneer in telegraphy who sparked Oliver's interest in electrical science.
- He had a serious hearing defect and difficulties in school which he quit at age 16. He was largely self-taught.
- Worked as a telegrapher from age 18 to 24 at which time he retired.

Oliver Heaviside, cont'd

- He was supported by his parents first and then his brother. His needs were modest and his family regarded him as a genius.
- He had no academic appointment, attended scientific meetings very rarely, and published largely in an electrical trade journal The Electrician.
- He was a recluse, worked in a small room that he kept extremely hot and filled with pipe smoke. He was combative with a caustic wit — "a first-rate oddity". He was devoid of social skills and avoided social contacts.

Oliver Heaviside, cont'd

- He made many important contributions to science, mathematics, and especially to electrical engineering, including:

- He introduced the concepts of inductance, capacitance, and impedance (labelled it Z).
- He was first to write Maxwell's equations in the modern (vector) form.
- He solved problems of signal propagation in the atmosphere and in cables.
- He used operational calculus to solve differential equations and electric networks. He defined his resistance operator p = d/dt to calculate impedances directly from circuits.

Oliver Heaviside, cont'd

- He was a contemporary of James Clerk Maxwell, Charles Darwin, Michael Faraday, George Stokes, William Thomson (Lord Kelvin). He corresponded with many of these and other scientists and was highly respected by the leading scientists of his day.

- He died February 3, 1925.

Exercises.[1]

Solution of exercises.[2]

[1] See the file at <http://cnx.org/content/m27519/latest/ps4sol.pdf>
[2] See the file at <http://cnx.org/content/m27519/latest/ps4.pdf>