Lecture 6:Continuous Time Fourier Transform (CTFT)*

Truc Pham-Dinh
Tam Huynh-Ngoc
Anh Tuan Hoang
Thanh Dinh Vu

Abstract

Extends the notion of the frequency response of a system to the frequency content of a signal. Widely used tool in many areas (communications, control, signal processing, X-ray diffraction, Medical imaging — CAT & PET scan). Continue development of Fourier transform pairs. Illustrate different methods for finding Fourier transforms.

Lecture #6:
CONTINUOUS TIME FOURIER TRANSFORM (CTFT)
Motivation:

- Extends the notion of the frequency response of a system to the frequency content of a signal.
- Widely used tool in many areas (communications, control, signal processing, X-ray diffraction, Medical imaging — CAT & PET scan).

- Continue development of Fourier transform pairs.
- Illustrate different methods for finding Fourier transforms

Outline:

- The continuous time Fourier transform (CTFT)
- Properties of the CTFT
- Simple CTFT pairs

- Fourier transform pairs

– Fourier transform of the unit step function
 – Fourier transform of causal sinusoids
 – Fourier transform of rectangular pulse — the sinc function
 – Fourier transform of triangular pulse

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- Filtering the ECG revisited
- Conclusion

Example — How to filter the ECG?

The recorded activity from the surface of the chest includes the electrical activity of the heart plus extraneous signals or "noise." How can we design a filter that will reduce the noise?

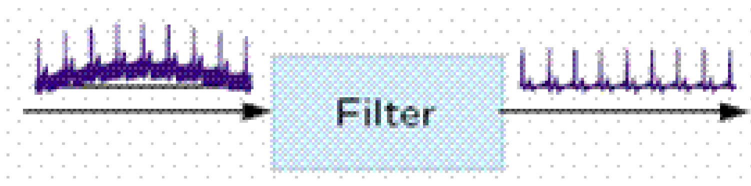


Figure 1

It is most effective to compute the frequency content of the recorded signal and to identify those components that are due to the electrical activity of the heart and those that are noise. Then the filter can be designed rationally. This is one of many motivations for understanding the Fourier transform.

I. THE CONTINUOUS TIME FOURIER TRANSFORM (CTFT)

1/ Definition

The continuous time Fourier transform of x(t) is defined as

$X(f) = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{j}2\pi \mathrm{ft}} dt$

and the inverse transform is defined as

$x(t) = \int_{-\infty}^{\infty} X(f) e^{\mathrm{j}2\pi \mathrm{ft}} df$

2/ Relation of Fourier and Laplace Transforms

The bilateral Laplace transform is defined by the analysis formula

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{st}} dt$

and the inverse transform is defined by the synthesis formula

$x(t) = \frac{1}{\mathrm{j}2\pi} \int_C X(s) e^{\mathrm{st}} ds$

Now if the jω axis is in the region of convergence of X(s), then we can substitute s = jω = j2πf into both relations to obtain

$X(\mathrm{j}2\pi\mathrm{f}) = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{j}2\pi \mathrm{ft}} dt$

$x(t) = \frac{1}{\mathrm{j}2\pi} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} X(\mathrm{j}2\pi\mathrm{f}) e^{\mathrm{j}2\pi \mathrm{ft}} d(\mathrm{j}2\pi\mathrm{f})$

3/ Form of the Fourier transform

Finally, by canceling j2π and changing the variable of integration from j2πf to f we obtain

$X(\mathrm{j}2\pi\mathrm{f}) = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{j}2\pi \mathrm{ft}} dt$

$x(t) = \int_{-\infty}^{\infty} X(\mathrm{j}2\pi\mathrm{f}) e^{\mathrm{j}2\pi \mathrm{ft}} df$

It is clumsy to write X(j2πf). Therefore, from this now on, we rewrite the function X(j2πf) described above as in a simpler form X(f) such that the transform pair would be symmetrical.

X(j2πf) Þ X(f)

The new Fourier transform pair will be in the form

$X(f) = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{j}2\pi \mathrm{ft}} dt$ $x(t) = \int_{-\infty}^{\infty} X(f) e^{\mathrm{j}2\pi \mathrm{ft}} df$

4/ Notation

Notation for the Fourier transform varies appreciably from text to text and in different disciplines. Another common notation is to define the Fourier transform in terms of jω as follows

$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$ $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega$

Differences in notation are largely a matter of taste and different notations result in different locations of factors of 2π. The notation we use minimizes the number of factors of 2π that appear in the expressions we will use in this subject and makes the duality of the Fourier transform with its inverse more transparent.

5/ Why bother with the Fourier transform?

- There are certain simple time functions which are more readily represented by Fourier transforms than by Laplace transforms, e.g., x(t) = 1, x(t) = cos(2πft), periodic time functions, etc.
- Certain important operations on signals are more readily analyzed with Fourier transforms, e.g., sampling, modulation, filtering.
- Examination of both signals and systems in the frequency domain gives insights that complement those obtained in the "time" domain.

6/ Functions that have Laplace transforms but not Fourier transforms

There are some time functions that have a Laplace transform but not a Fourier transform, namely those for which the jω-axis is not inside the region of convergence. For example, x(t) = eαtu(t) for $\alpha > 0$.

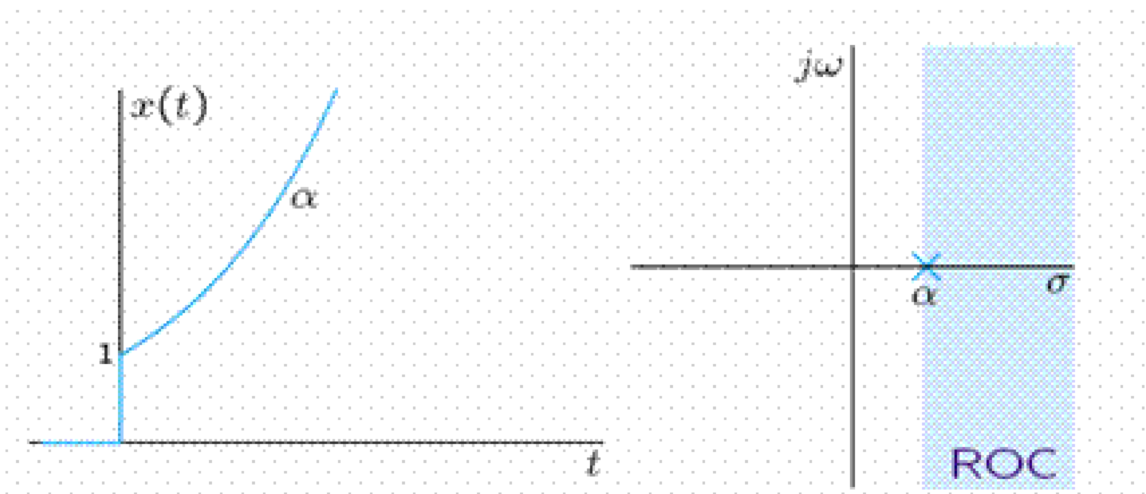


Figure 2

II. PROPERTIES OF THE CTFT

1/ Properties — symmetry

We start with the definition of the Fourier transform of a real time function x(t) and expand both terms in the integrand in terms of odd and even components.

$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt$ $X(f) = \int_{-\infty}^{\infty} (x_e(t) + x_o(t))(\cos(2\pi ft) - j\sin(2\pi ft))\, dt$

The odd components of the integrand contribute zero to the integral. Hence, we obtain

$X(f) = \int_{-\infty}^{\infty} x_e(t)\cos(2\pi ft)\, dt + j\int_{-\infty}^{\infty} - x_o(t)\sin(2\pi ft)\, dt$ $X(f) = X_r(f) + jX_i(f)$

where

$X_r(f) = \int_{-\infty}^{\infty} x_e(t)\cos(2\pi \mathrm{ft})\,dt$ $X_i(f) = -\int_{-\infty}^{\infty} x_o(t)\sin(2\pi \mathrm{ft})\,dt$

We can infer symmetry properties of the Fourier transform of a real time function x(t).

$X_r(f) = \int_{-\infty}^{\infty} x_e(t)\cos(2\pi \mathrm{ft})\,dt$, evenfunctionof f $X_i(f) = -\int_{-\infty}^{\infty} x_o(t)\sin(2\pi \mathrm{ft})\,dt$, oddfunctionof f $|X(f)| = \sqrt{{X_r}^2(f) + {X_i}^2(f)}$, evenfunctionof f. [U+F3A3]

Therefore, if

x(t) X(f)

Real and even function of t Real and even function of f

Real and odd function of t Imaginary and odd function of f

The angle can be computed as follows,

$$\angle \mathrm{X}(f) = \begin{cases} \tan^{-1}\left(\frac{X_i(f)}{X_r(f)}\right) \mathrm{for} X_r(f) > 0 \\ \text{[U+F528]} \\ \pi + \tan^{-1}\left(\frac{X_i(f)}{X_r(f)}\right) \mathrm{for} X_r(f) < 0 \end{cases}$$

But, since $\pm \mathrm{n}2\pi$ can always be added to the angle, and since is an odd function of f,

$$\angle \mathrm{X}(-f) = \begin{cases} -\tan^{-1}\left(\frac{X_i(f)}{X_r(f)}\right) \mathrm{for} X_r(f) > 0 \\ \text{[U+F528]} \\ -\pi - \tan^{-1}\left(\frac{X_i(f)}{X_r(f)}\right) \mathrm{for} X_r(f) < 0 \end{cases}$$

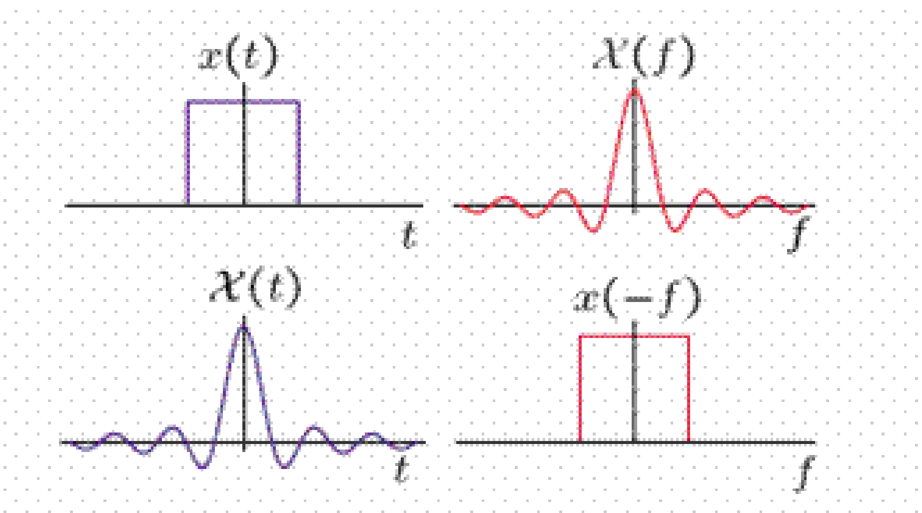
Therefore, ÐX(f) is an odd function of f.

2/ Properties — duality

The Fourier transform and its inverse differ only by a sign in the exponent,

$X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-\mathrm{j}2\pi \mathrm{ft}}dt x(t) = \int_{-\infty}^{\infty} X(f)\,e^{\mathrm{j}2\pi \mathrm{ft}}df$

Therefore, if x(t)X (f) then X(t)x(−f). This means that if we have found one Fourier transform pair, we automatically know another.



3/ List of simple properties

Some of the important properties are summarized here; a more complete list is appended.

Property	$x(t)$	$X(f)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(f) + bX_2(f)$
Shift in t	$x(t - t_o)$	$X(f)\, e^{-j2\pi f t_o}$
Shift in f	$x(t)\, e^{j2\pi f_o t}$	$X(f - f_o)$
Differentiate in t	$\frac{dx(t)}{dt}$	$j2\pi f X(f)$
Differentiate in f	$-j2\pi t x(t)$	$\frac{dX(f)}{df}$
Convolve in t	$x_1(t) * x_2(t)$	$X_1(f) . X_2(f)$
Convolve in f	$x_1(t) . x_2(t)$	$X_1(f) * X_2(f)$

Figure 3

a/ Properties — linearity

Most proofs of Fourier transform properties are simple.

$\mathrm{ax}_1(t) + \mathrm{bx}_2(t)$ $\overset{F}{\text{[U+27FA]}}$ $\mathrm{aX}_1(f) + \mathrm{bX}_2(f)$

The proof follows from the definition of the Fourier transform as a definite integral.

$X(f) = \int_{-\infty}^{\infty} (\mathrm{ax}_1(t) + \mathrm{bx}_2(t))\, e^{-\mathrm{j}2\pi \mathrm{ft}} dt$ $X(f) = a\int_{-\infty}^{\infty} x_1(t)\, e^{-\mathrm{j}2\pi \mathrm{ft}} dt + b\int_{-\infty}^{\infty} x_2(t)\, e^{-\mathrm{j}2\pi \mathrm{ft}} dt$ $X(f) = \mathrm{aX}_1(f) + \mathrm{bX}_2(f)$.

b/ Delay by to

$x(t - t_o)$ $\overset{F}{\text{[U+27FA]}}$ $X(f)\, e^{-\mathrm{j}2\pi \mathrm{ft}_o}$

This result can be seen using the synthesis formula.

$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{\mathrm{j}2\pi \mathrm{ft}} df$ $x(t - t_o) = \int_{-\infty}^{\infty} X(f)\, e^{\mathrm{j}2\pi \mathrm{f}(t - t_o)} df$ $x(t - t_o) = \int_{-\infty}^{\infty} X(f)\, e^{-\mathrm{j}2\pi \mathrm{ft}_o} e^{\mathrm{j}2\pi \mathrm{ft}} df$.

Mnemonic: a delay of the time function multiplies the Fourier transform by a lag factor, i.e., a delay of the time function of to adds -2πfto to the angle of the Fourier transform but does not affect the magnitude.

c/ Differentiate in t

$\frac{\mathrm{dx}(t)}{\mathrm{dt}}$ $\overset{F}{\text{[U+27FA]}}$ $\mathrm{j}2\pi \mathrm{fX}(f)$

This result can be seen using the synthesis formula.

$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{\mathrm{j}2\pi \mathrm{ft}} df$ $\frac{\mathrm{dx}(t)}{\mathrm{dt}} = \int_{-\infty}^{\infty} X(f) \frac{d}{\mathrm{dt}} \left(e^{\mathrm{j}2\pi \mathrm{ft}}\right) df$ $\frac{\mathrm{dx}(t)}{\mathrm{dt}} = \int_{-\infty}^{\infty} \mathrm{j}2\pi \mathrm{f} X(f)\, e^{\mathrm{j}2\pi \mathrm{ft}} df$

Differentiating the time function, adds $\pi/2$ radians to the angle of the Fourier transform and multiplies the magnitude by 2πf.

d/ Multiply by ej2πfot

$x(t)\, e^{\mathrm{j}2\pi \mathrm{f}_o t}$ $\overset{F}{\text{[U+27FA]}}$ $X(f - f_o)$

This result can be seen using the analysis formula.

$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-\mathrm{j}2\pi \mathrm{ft}} dt$ $X(f - f_o) = \int_{-\infty}^{\infty} x(t)\, e^{-\mathrm{j}2\pi(f - f_o)t} dt$ $X(f - f_o) = \int_{-\infty}^{\infty} x(t)\, e^{\mathrm{j}2\pi \mathrm{f}_o t} e^{-\mathrm{j}2\pi \mathrm{ft}} dt$

This result can also be obtained from the delay-in-time property and duality. Mnemonic: multiplying a time function by a complex exponential at frequency fo shifts the Fourier transform to fo.

e/ Convolution in time

If x(t) = x1(t) $*$ x2(t) then X(f) is obtained as follows

$X(f) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t - \tau)\, \mathrm{d}\tau\right) e^{-\mathrm{j}2\pi \mathrm{ft}} \mathrm{dt} = \int_{-\infty}^{\infty} x_1(\tau) \left(\int_{-\infty}^{\infty} x_2(t - \tau)\, e^{-\mathrm{j}2\pi \mathrm{ft}} \mathrm{dt}\right) \mathrm{d}\tau = \int_{-\infty}^{\infty} x_1(\tau)\, X_2(f)\, e^{-\mathrm{j}2\pi \mathrm{f}\tau} \mathrm{d}$

$= X_1(f)\, X_2(f)$

The Fourier transform of the convolution of two time functions is the product of the Fourier transforms.

f/ Properties — zeroth-order moments

From the definition of the Fourier transform and its inverse

$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-\mathrm{j}2\pi \mathrm{ft}} dt \text{and} x(t) = \int_{-\infty}^{\infty} X(f)\, e^{\mathrm{j}2\pi \mathrm{ft}} df$

it follows that

$X(0) = \int_{-\infty}^{\infty} x(t)\, dt \text{and} x(0) = \int_{-\infty}^{\infty} X(f)\, df$

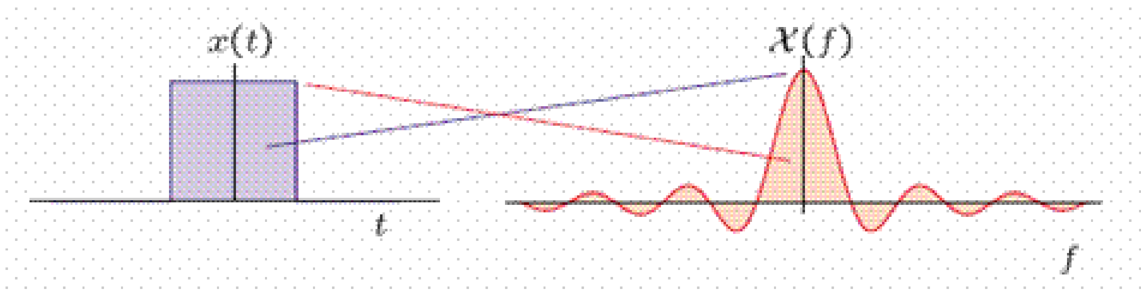


Figure 4

g/ Properties — nth-order moments of time functions

Take the nth derivative of the Fourier transform with respect to f to obtain

$\frac{d^n X(f)}{\mathrm{df}^n} = \int_{-\infty}^{\infty} (-\mathrm{j}2\pi \mathrm{t})^n x(t)\, e^{-\mathrm{j}2\pi \mathrm{ft}} \mathrm{dt}$

If we rearrange terms and evaluate the equation at f = 0, we obtain

$\frac{1}{(-\mathrm{j}2\pi)^n} \frac{d^n X(f)}{\mathrm{df}^n}|_{f=0} = \int_{-\infty}^{\infty} t^n x(t)\, \mathrm{dt}$

Hence, the nth moment of x(t) can be obtained from the nth derivative of X(f) evaluated at f = 0. From duality of the Fourier transform, the nth moment of X(f) can be obtained from derivatives of x(t) evaluated at t = 0.

III. SIMPLE CTFT PAIRS

1/ Unit impulse in time

Suppose x(t) = δ(t). Then the CTFT is

$X(f) = \int_{-\infty}^{\infty} \delta(t)\, e^{-\mathrm{j}2\pi \mathrm{ft}} \mathrm{dt} = 1$

This shows that we can synthesize the impulse as follow

$\delta(t) = \int_{-\infty}^{\infty} 1 \cdot e^{\mathrm{j}2\pi \mathrm{ft}} \mathrm{df}$

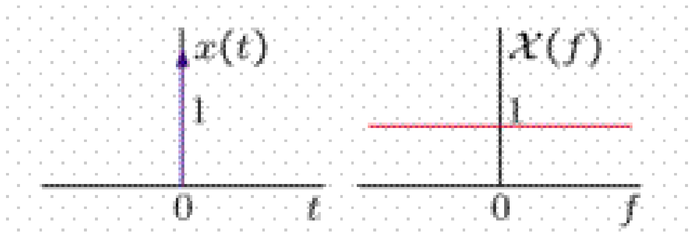


Figure 5

Generalization: a function that is punctuate in time has a Fourier transform that is extensive in frequency.
2/ Unit impulse in frequency
Suppose X(f) = δ(f). Then the inverse CTFT is
$x\left(t\right)=\int_{-\infty}^{\infty}\delta\left(f\right)e^{\mathrm{j}2\pi\mathrm{ft}}\mathrm{df}=1$
This shows that we can synthesize the impulse as follows
$\delta\left(f\right)=\int_{-\infty}^{\infty}1\cdot e^{-\mathrm{j}2\pi\mathrm{ft}}\mathrm{dt}$

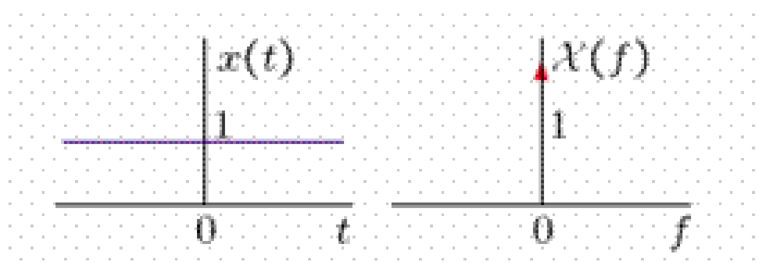


Figure 6

Generalization: a function that is punctuate in frequency has an inverse Fourier transform that is extensive in time.
3/ Unit impulse shifted in time
Suppose x(t) = δ(t−to). Then from the delay property we obtain
$X\left(f\right)=e^{-\mathrm{j}2\pi\mathrm{ft}_o}=\cos\left(2\pi\mathrm{ft}_o\right)-j\sin\left(2\pi\mathrm{ft}_o\right)$

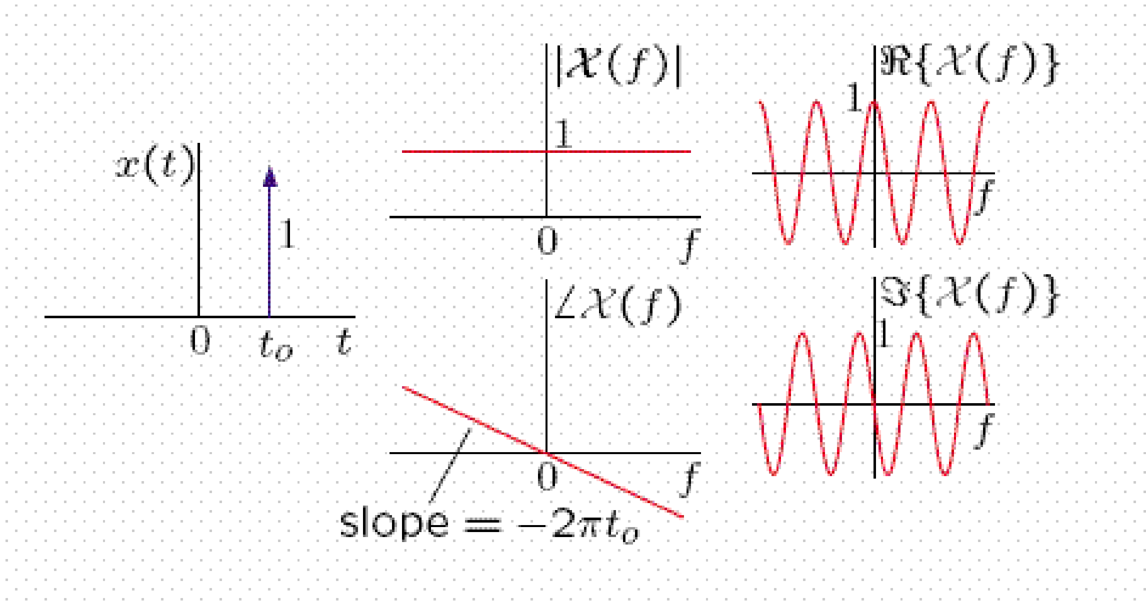


Figure 7

4/ Unit impulse shifted in frequency
Suppose X(f) = δ(f − fo). From the properties, the inverse CTFT is
$x\left(t\right)=e^{\mathrm{j}2\pi \mathrm{f}_{o}t}=\cos\left(2\pi \mathrm{f}_{o}t\right)+j\sin\left(2\pi \mathrm{f}_{o}t\right)$

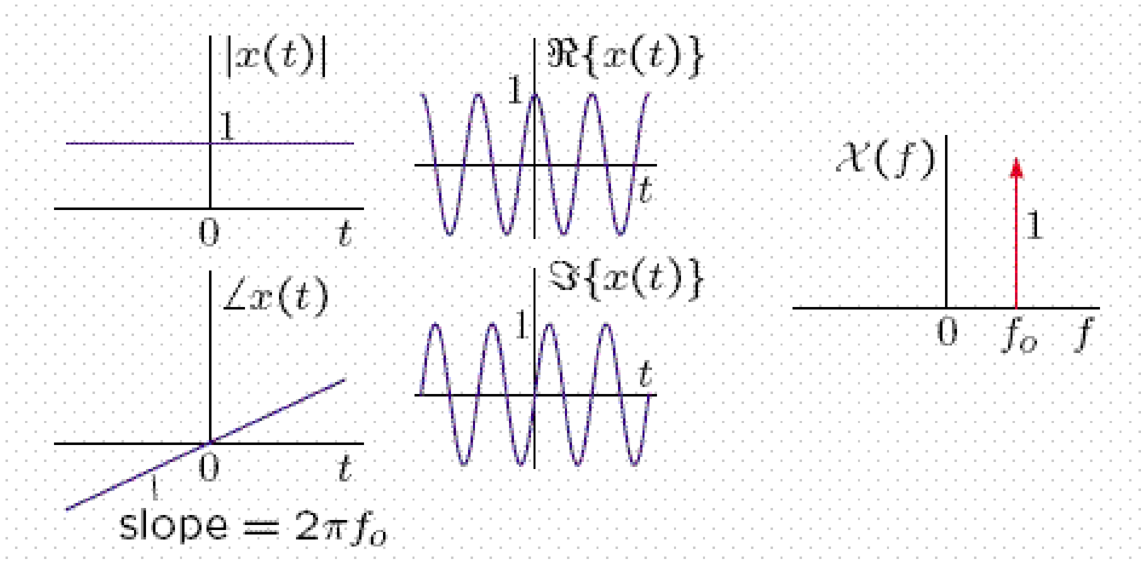


Figure 8

5/ Sinusoidal time functions

$\cos\left(2\pi \mathrm{f}_o t\right) = \left(\frac{e^{\mathrm{j}2\pi \mathrm{f}_o t}+e^{-\mathrm{j}2\pi \mathrm{f}_o t}}{2}\right) \overset{F}{\text{[U+27FA]}} \frac{\delta(f-f_o)+\delta(f+f_o)}{2}$ $\sin\left(2\pi \mathrm{f}_o t\right) = \left(\frac{e^{\mathrm{j}2\pi \mathrm{f}_o t}-e^{-\mathrm{j}2\pi \mathrm{f}_o t}}{2j}\right) \overset{F}{\text{[U+27FA]}} \frac{\delta(f-f_o)-\delta(f+f_o)}{2j}$

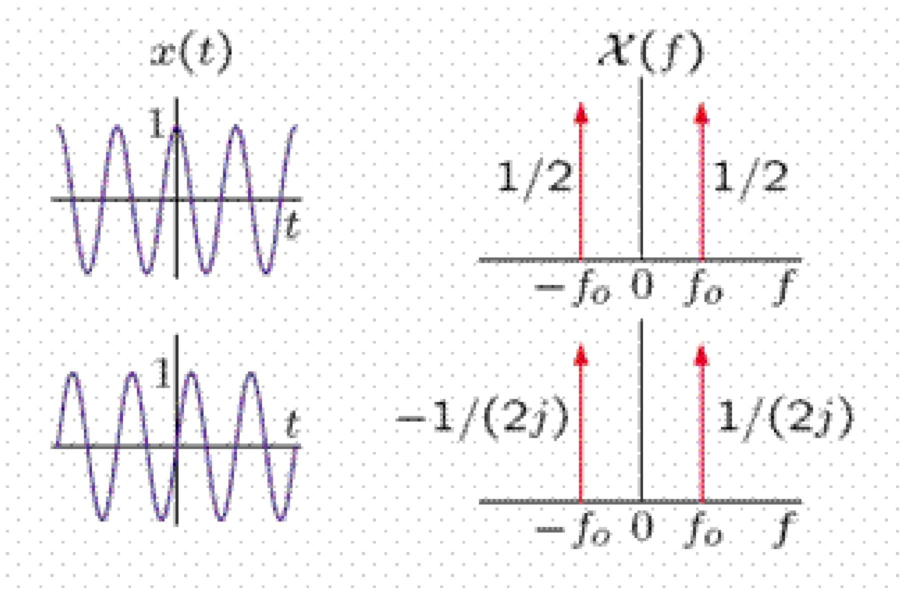


Figure 9

6/ Causal exponential time function

$x(t) = e^{-\alpha t} u(t) \frac{1}{s+\alpha} \text{for} \sigma > -\alpha \, (\alpha > 0)$

Therefore, since the jω axis is in the region of convergence of X(s), we can evaluate X(s) on the jω axis to obtain the Fourier transform.

$X(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega+\alpha}$

which leads to

$X(f) = \frac{1}{\mathrm{j}2\pi \mathrm{f}+\alpha}$

$|X(f)| = \frac{1}{\sqrt{(2\pi \mathrm{f})^2+\alpha^2}} \text{and} \angle \mathrm{X}(f) = -\tan^{-1}\left(\frac{2\pi \mathrm{f}}{\alpha}\right)$

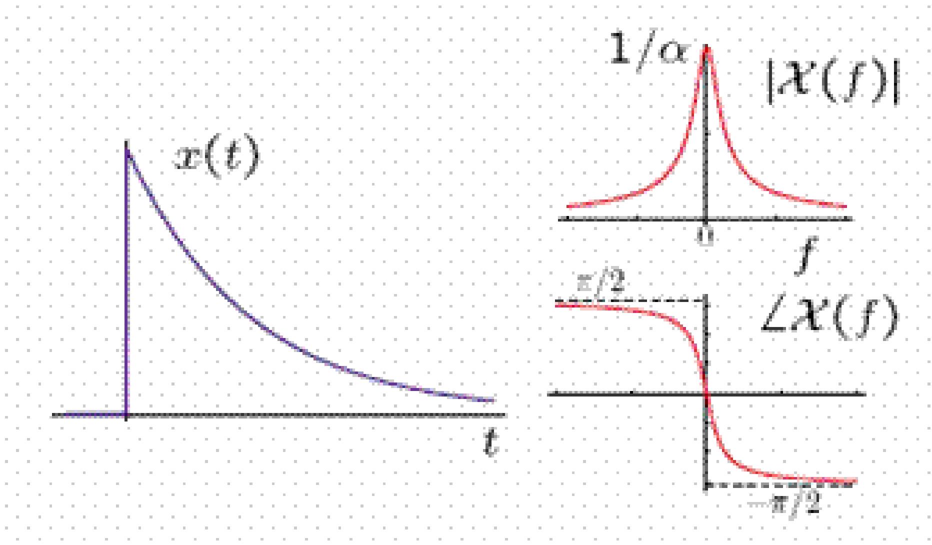


Figure 10

As α increases, the time function decays more rapidly and its duration (as measured by the time constant) decreases. As α increases, the Fourier transform width (as measured by its bandwidth) increases.

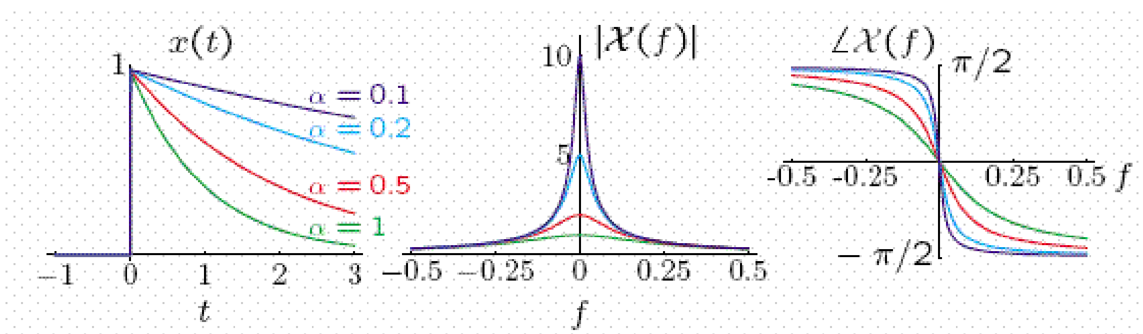


Figure 11

Thus, the width of the time function is inversely proportional to the width of the Fourier transform. $X(f) = \frac{1}{j2\pi f + \alpha} = \frac{\alpha}{(2\pi f)^2 + \alpha^2} + j\frac{-2\pi f}{(2\pi f)^2 + \alpha^2}$

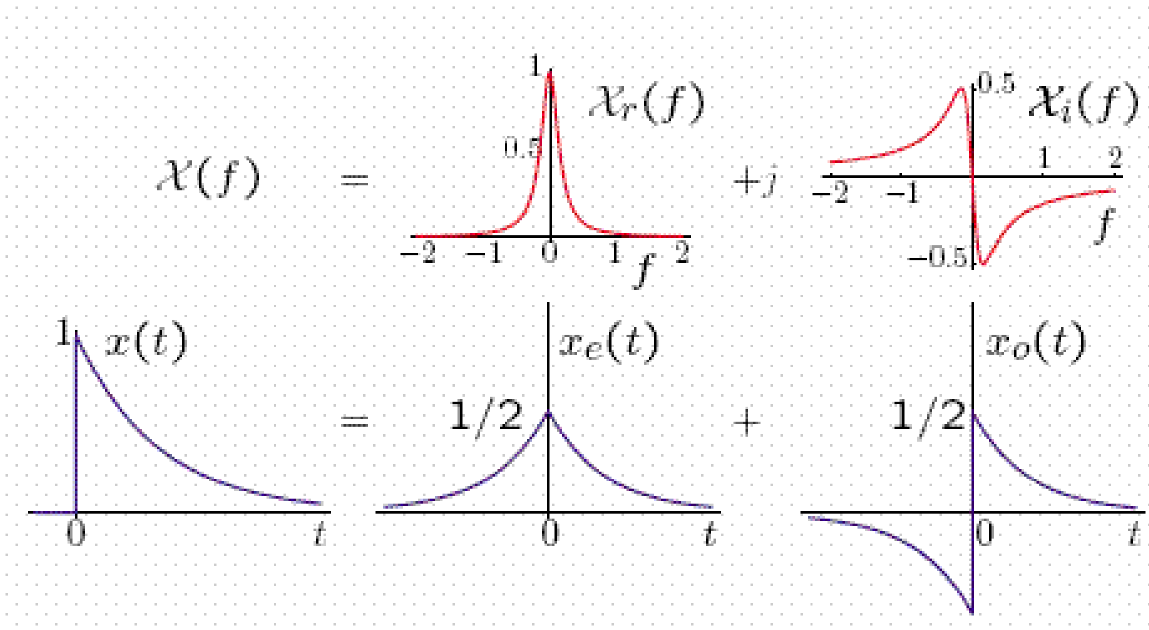


Figure 12

Therefore, we also have that

$x_e(t) = (1/2)\,e^{-\alpha|t|} \overset{F}{\text{[U+27FA]}} X_r(f) = \frac{\alpha}{(2\pi \mathrm{f})^2+\alpha^2}$

$x_o(t) = (1/2)\,e^{-\alpha|t|}\mathrm{sgn}(t) \overset{F}{\text{[U+27FA]}} \mathrm{jX}_i(f) = j\frac{-2\pi \mathrm{f}}{(2\pi \mathrm{f})^2+\alpha^2}$

Where

$$\mathrm{sgn}(t) = \{ \begin{array}{c} 1\mathrm{for}t>0 \\ -1\mathrm{for}t<0 \end{array}$$

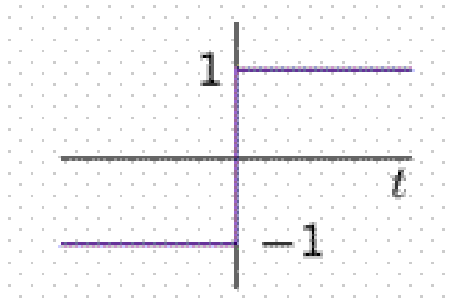


Figure 13

Two-minute miniquiz problem

Problem 16-1 — Fourier transform of a unit step

The Laplace transform of a unit step is

$x(t) = u(t) \overset{F}{\text{[U+27FA]}} X(s) = \frac{1}{s}\text{for}\sigma > 0$

This suggests that the Fourier transform of the unit step is

$x(t) = u(t) \overset{F}{\text{[U+27FA]}} X(f) = \frac{1}{\mathrm{j}2\pi\mathrm{f}}$

The Fourier transform of a causal exponential is

$x(t) = e^{-\alpha t}u(t) \overset{F}{\text{[U+27FA]}} X(f) = \frac{1}{\mathrm{j}2\pi\mathrm{f}+\alpha}$

This also suggests that the Fourier transform of the unit step is

$x(t) = u(t) \overset{F}{\text{[U+27FA]}} X(f) = \frac{1}{\mathrm{j}2\pi\mathrm{f}}$

Explain why this cannot be the Fourier transform of a unit step.

Solution

$X(f) = \frac{1}{\mathrm{j}2\pi\mathrm{f}}$

is an imaginary odd function of f. Hence, it must be the Fourier transform of an odd function of t. The unit step is neither an odd nor an even function of t.

The argument based on the Laplace transform of a step is fallacious because the Laplace transform of the step has a region of convergence that does not include the jω axis. Hence, we cannot simply substitute s = j2πf into the Laplace transform to obtain the Fourier transform. The second argument is fallacious because care has to be taken in evaluating.

$\frac{1}{\mathrm{j}2\pi\mathrm{f}+\alpha}\text{as}\alpha \rightarrow 0\text{at}f = 0$

7/ Unit step

To obtain the Fourier transform of the unit step we start with the Fourier transform of a causal exponential

$x(t) = e^{-\alpha t}u(t) \overset{F}{\text{[U+27FA]}} X(f) = \frac{1}{\mathrm{j}2\pi\mathrm{f}+\alpha}$

and examine the solution as $\alpha \rightarrow 0$.

Note that

$x(t) = e^{-\alpha t}u(t) = x_e(t) + x_o(t) = (1/2)\, e^{-\alpha|t|} + (1/2)\, e^{-\alpha|t|}\text{sgn}(t)$

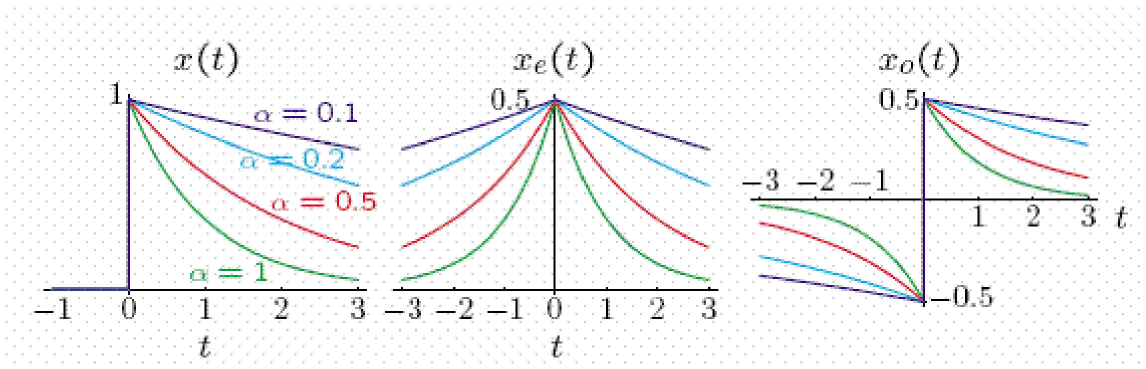


Figure 14

Note that
$X(f) = \frac{1}{\mathrm{j}2\pi\mathrm{f}+\alpha} = X_r(f) + \mathrm{j}\mathrm{X}_i(f) = \frac{\alpha}{(2\pi\mathrm{f})^2+\alpha^2} + j\frac{-2\pi\mathrm{f}}{(2\pi\mathrm{f})^2+\alpha^2}$

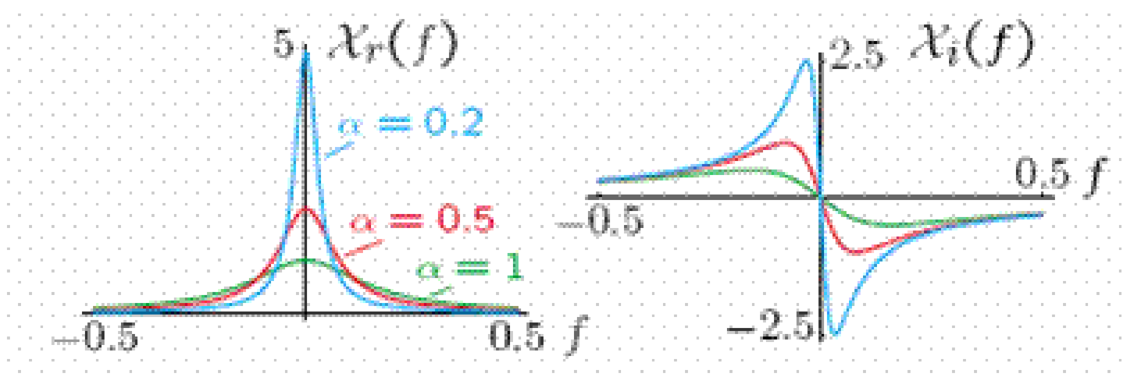


Figure 15

As $\alpha \to 0$, Xr(f) becomes tall and narrow and, as we shall see, its area is 1/2. Hence, as $\alpha \to 0$, $X_r(f) \to \frac{1}{2}\delta(f)\,\mathrm{and}\mathrm{j}\mathrm{X}_i(f) \to \frac{1}{\mathrm{j}2\pi\mathrm{f}}$
We need to determine the area of Xr(f). Because
$x_e(t)$ $\overset{F}{\text{[U+27FA]}}$ $X_r(f)$ $x_e(0) = \int_{-\infty}^{\infty} X_r(f)\,\mathrm{df}$.

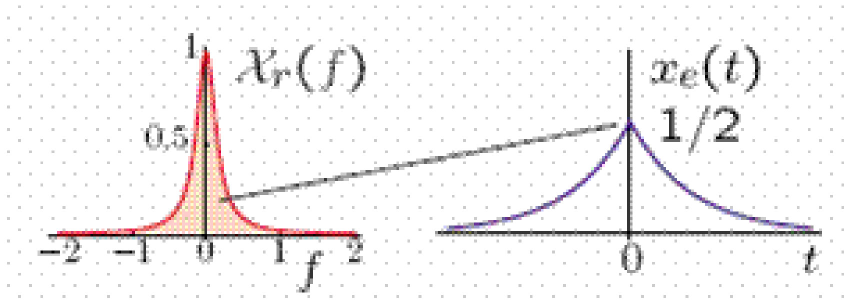


Figure 16

Hence,

$x_e(0) = \int_{-\infty}^{\infty} X_r(f)\,\mathrm{df} = \frac{1}{2}$

Thus, we have

$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t) \overset{F}{\text{[U+27FA]}} F\{u(t)\} = \frac{1}{2}\delta(f) + \frac{1}{\mathrm{j}2\pi\mathrm{f}}$

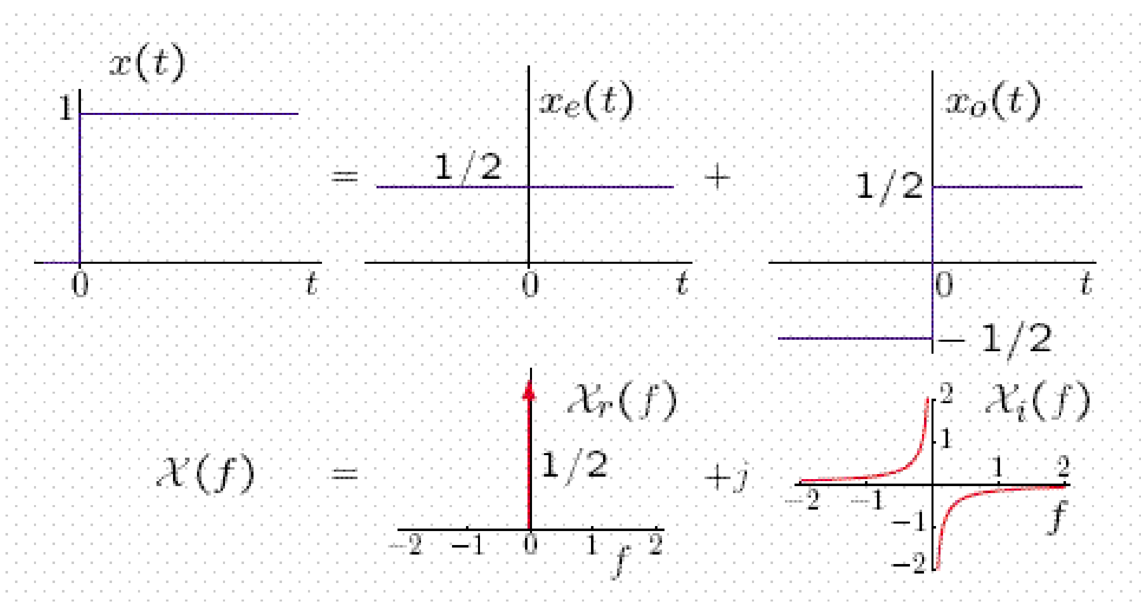


Figure 17

Unit step, bottom line

To summarize,

$u(t)$ $\overset{F}{\text{[U+27FA]}}$ $\frac{1}{s}\text{for}R\{s\} > 0$ $u(t)$ $\overset{F}{\text{[U+27FA]}}$ $\frac{1}{2}\delta(f) + \frac{1}{\text{j}2\pi\text{f}}$.

Thus, if the Laplace transform is evaluated on the edge of the region of convergence, on the jω axis, then there is an impulse in the real part at the location of the pole.

8/ Signum and unit step function — another approach

We illustrate a method for finding Fourier transforms using the Fourier transform properties and the Fourier transforms of simple time functions. Consider the time function x(t) = (1/2)sgn(t).

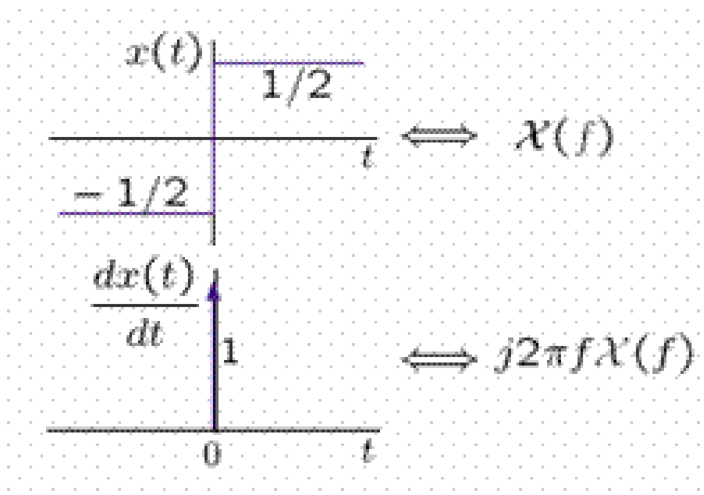


Figure 18

The derivative of x(t) is a unit impulse in time. Differentiation in time is equivalent to multiplying the transform by j2πf.

Hence,

$x(t) = \text{sgn}(t)$ $\overset{F}{\text{[U+27FA]}}$ $X(f) = \frac{1}{\text{j}2\pi\text{f}}$

The same approach can be used to find the Fourier transform of the step, x(t) = u(t), if some care is exercised. Note that u(t) = 1/2 + (1/2)sgn(t).

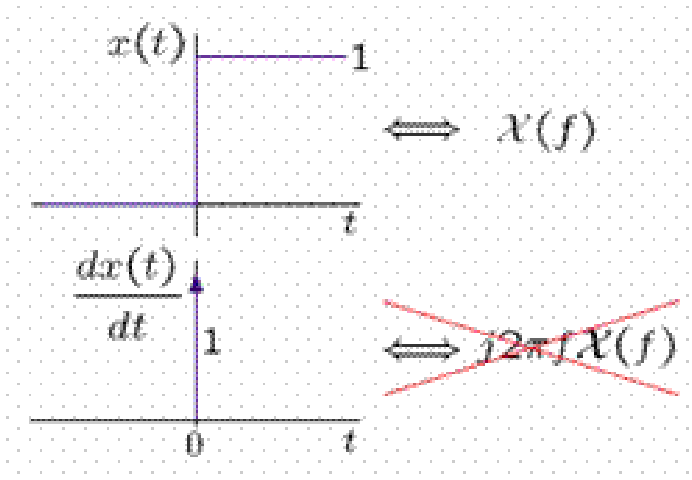


Figure 19

The derivative of x(t) is a unit impulse in time but the constant is lost. Thus, it must be included. Hence,

$x(t) = u(t) = 1/2 + (1/2)\,\mathrm{sgn}(t)\, X(f) = (1/2)\,\delta(f) + \frac{1}{\mathrm{j}2\pi\mathrm{f}}$

Two-minute miniquiz problem

Problem 17-1 — Step with a dc offset

Find the Fourier transform of x(t).

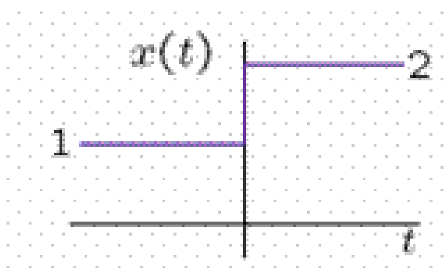


Figure 20

Solution

We can represent x(t) as

x(t) = 3/2 + (1/2)sgn(t).

Hence,

$X(f) = (3/2)\,\delta(f) + \frac{1}{\mathrm{j}2\pi\mathrm{f}}$

9/ Causal cosinusoidal time function

A causal cosinusoid is a cosinusoid that starts at t = 0,

$x(t) = \cos(2\pi \mathrm{f}_o t)\, u(t) = \cos(2\pi \mathrm{f}_o t)\,(1/2 + (1/2)\,\mathrm{sgn}(t))$

Therefore,

$X(f) = F\{\cos(2\pi \mathrm{f}_o t)\} * F\{u(t)\} = (1/2)\,(\delta(f - f_o) + \delta(f + f_o)) * \left((1/2)\,\delta(f) + \frac{1}{\mathrm{j}2\pi \mathrm{f}}\right) = (1/4)\,(\delta(f - f_o) + \delta(f + f_o)) + \frac{1}{\mathrm{j}4\pi(f - f_o)} + \frac{1}{\mathrm{j}4\pi(f + f_o)} = (1/4)\,(\delta(f - f_o) + \delta(f + f_o)) + \frac{f}{\mathrm{j}2\pi(f^2 - {f_o}^2)}$

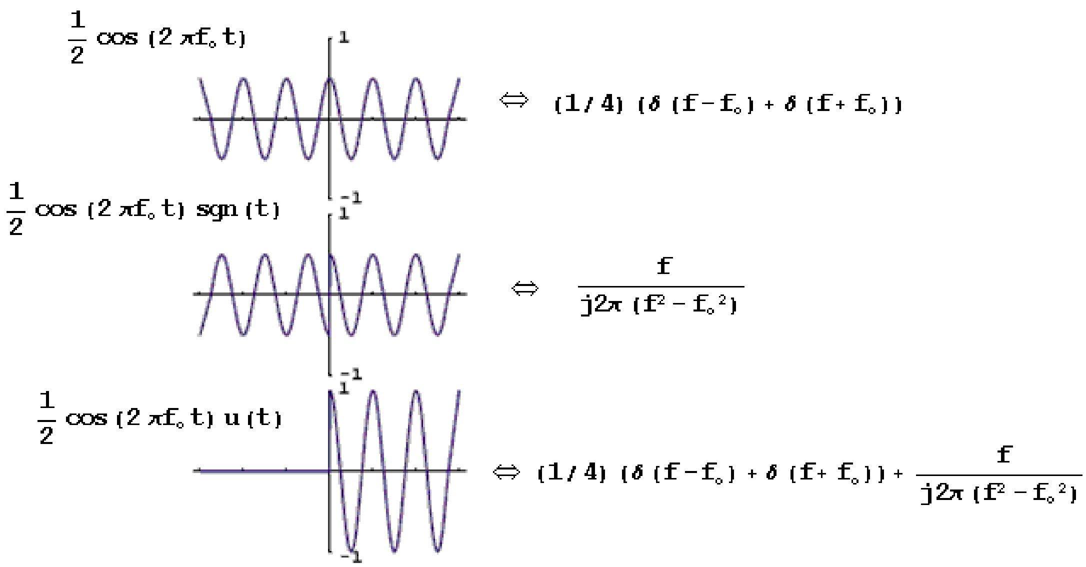


Figure 21

The Fourier and Laplace transforms of the causal cosinusoid are:

$\cos(2\pi \mathrm{f}_o t)\, u(t)$ $\overset{F}{\text{[U+27FA]}}$ $(1/4)\,(\delta(f - f_o) + \delta(f + f_o)) + \frac{f}{\mathrm{j}2\pi(f^2 - {f_o}^2)}$ $\cos(2\pi \mathrm{f}_o t)\, u(t)$ $\overset{F}{\text{[U+27FA]}}$ $\frac{s}{s^2 + (2\pi \mathrm{f}_o)^2} \mathrm{for} R\{s\} > 0$

Therefore, just as with the unit step, the causal cosinusoid has poles on the jω axis, and the Fourier transform contains impulses at the frequencies of the poles.

A causal sinusoid is a sinusoid that starts at t = 0,

$x(t) = \sin(2\pi \mathrm{f}_o t)\, u(t) = \sin(2\pi \mathrm{f}_o t)\,(1/2 + (1/2)\,\mathrm{sgn}(t))$

Therefore,

$X(f) = F\{\sin(2\pi \mathrm{f}_o t)\} * F\{u(t)\} = (1/2j)\,(\delta(f - f_o) - \delta(f + f_o)) * \left((1/2)\,\delta(f) + \frac{1}{\mathrm{j}2\pi \mathrm{f}}\right) = (1/4)\,(\delta(f - f_o) - \delta(f + f_o)) - \frac{-1}{4\pi(f - f_o)} - \frac{-1}{4\pi(f + f_o)} = (1/4j)\,(\delta(f - f_o) - \delta(f + f_o)) + \frac{-f_o}{2\pi(f^2 - {f_o}^2)}$

10/ Rectangular pulse

a/ Rectangular pulse — derivation

The transform of the rectangular pulse x(t) is:

$X(f) = \int_{-T/2}^{T/2} \mathrm{A}e^{-\mathrm{j}2\pi \mathrm{ft}} dt = \frac{\mathrm{A}e^{-\mathrm{j}2\pi \mathrm{ft}}}{-\mathrm{j}2\pi \mathrm{f}}\Big|_{-T/2}^{T/2} = \mathrm{AT}\frac{e^{\mathrm{j}\pi \mathrm{fT}} - e^{-\mathrm{j}\pi \mathrm{fT}}}{\mathrm{j}2\pi \mathrm{fT}} = \mathrm{AT}\left(\frac{\sin \pi \mathrm{fT}}{\pi \mathrm{fT}}\right)$

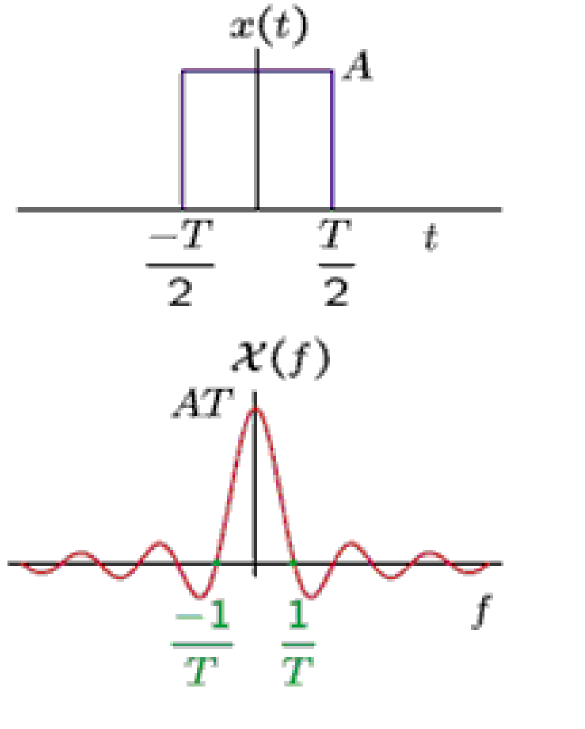


Figure 22

$\frac{\sin \pi \mathrm{fT}}{\pi \mathrm{fT}} = \begin{cases} 1 \mathrm{for} f=0 \\ 0 \mathrm{for} f=n/T, n\neq 0 \end{cases}$
where n is an integer.
b/ Use of moment properties
From the moment properties we know that
$X(0) = \int_{-\infty}^{\infty} x(t)\,\mathrm{dt} = \mathrm{AT} \mathrm{and} x(0) = A = \int_{-\infty}^{\infty} X(f)\,\mathrm{df}$

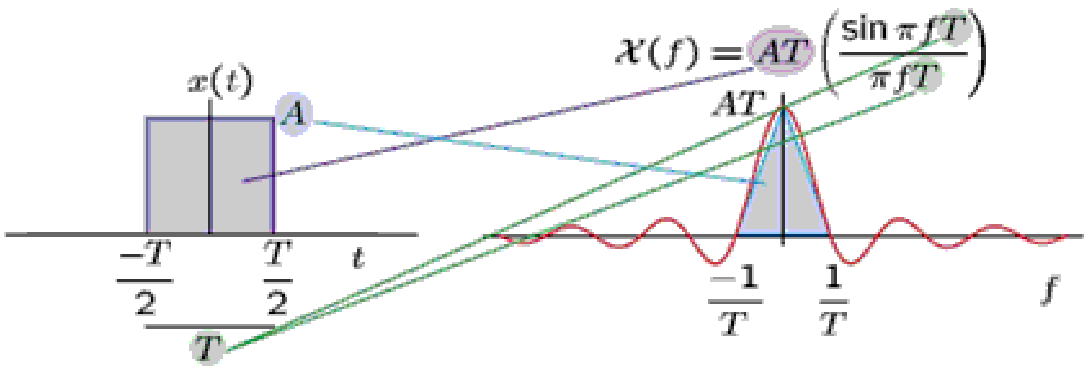


Figure 23

Note thatwhich is just the area of the inscribed triangle.
c/ Sinc function
The type of function arises so frequently that it is useful to define the sinc function,
$\text{sinc}\left(x\right) = \frac{\text{sin}\pi \text{x}}{\pi \text{x}}$
Therefore,
$X\left(f\right) = \text{AT}\left(\frac{\text{sin}\pi \text{fT}}{\pi \text{fT}}\right) = \text{ATsinc}\left(\text{fT}\right)$
d/ Source of zeros in FT

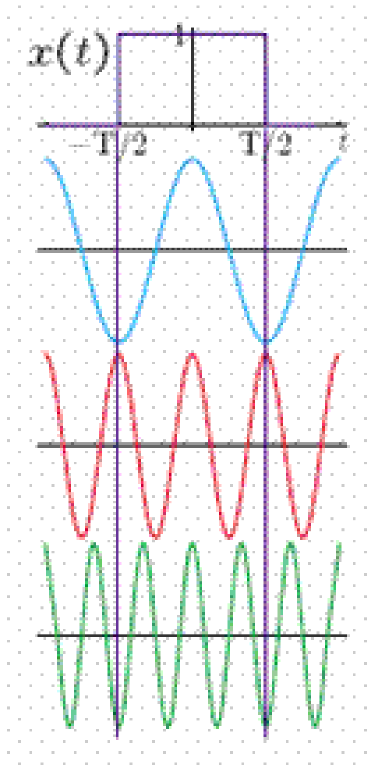


Figure 24

has zeros at f = n/T for n ≠ 0. What causes these zeros?

Note from the definition of the Fourier transform

$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-\mathrm{j}2\pi \mathrm{ft}} dt \ = \int_{-\infty}^{\infty} x(t)\left(\cos(2\pi\mathrm{ft}) - j\sin(2\pi\mathrm{ft})\right) dt$

The duration of x(t) is T. Hence, the integral is zero for those frequencies whose periods are submultiples of T. These are the frequencies f = n/T where n ≠ 0. The examples show the frequencies f = 1/T , f = 2/T , and f = 3/T .

e/ Alternate derivation of FT

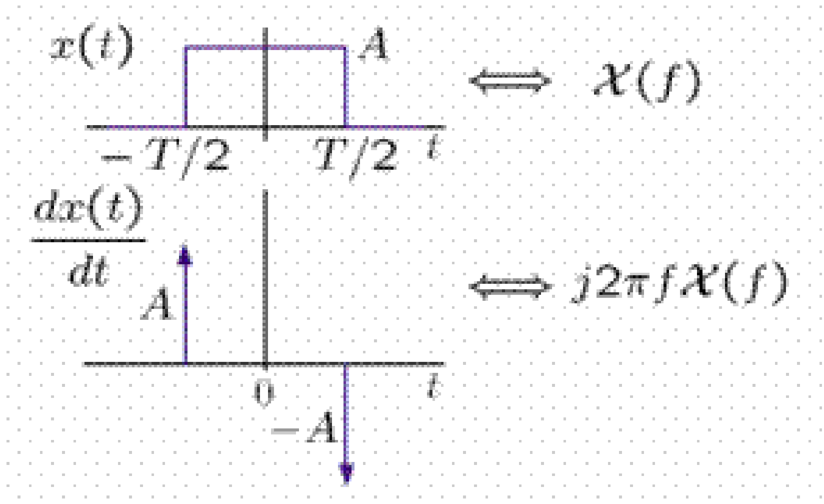


Figure 25

The Fourier transform can be found from
$j2\pi f X(f) = Ae^{j2\pi f(T/2)} - Ae^{-j2\pi f(T/2)} = 2jA\sin(\pi fT) \; X(f) = AT\left(\frac{\sin(\pi fT)}{\pi fT}\right)$
f/ Effect of duration
Consider the sequence of rectangular pulses of the form
$x(t) = \{^{1/T \text{for} t < |T/2|}_{0 \text{otherwise}} \; \overset{F}{\text{[U+27FA]}} \; X(f) = \frac{\sin(\pi fT)}{\pi fT}$

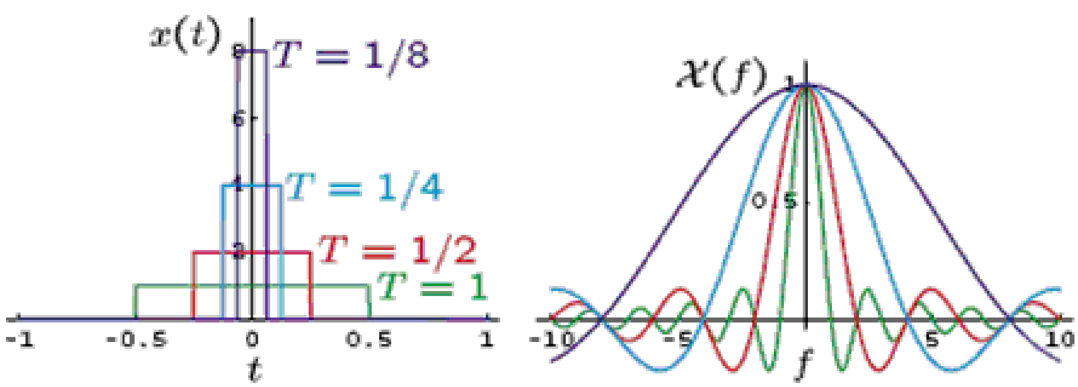


Figure 26

Note that as T decreases x(t) becomes tall and narrow and X(f) gets broader in frequency. Interpreted as generalized functions, x(t) $\to \delta$(t) and X(f) $\to$ 1.

11/ Triangular pulse

A triangular pulse is obtained by convolving a rectangular pulse with itself.

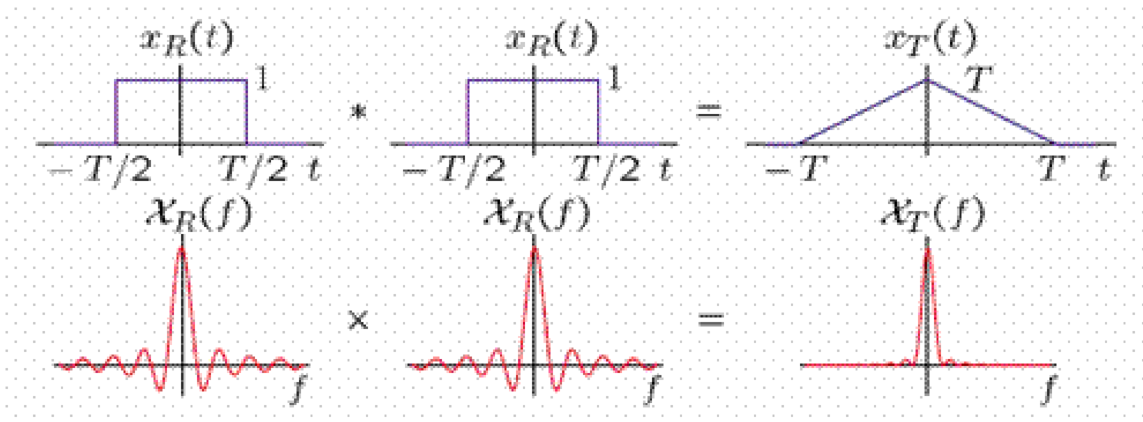


Figure 27

Since xT (t)=xR(t)*xR(t), the triangular pulse has the transform

$X_T(f) = X_R(f) \times X_R(f) = \left(T\frac{\sin(\pi \mathrm{fT})}{\pi \mathrm{fT}}\right)^2$

IV. FILTERING REVISITED

For an LTI system with impulse response h(t) with arbitrary input x(t), the output is given by the convolution of the input with the impulse response.

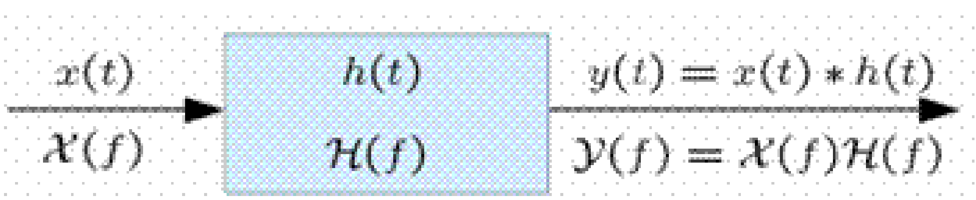


Figure 28

The Fourier transform of the output equals the product of the Fourier transform of the input and the Fourier transform of the impulse response which is just the frequency response of the LTI system. Filtering is an example of a signal processing operation that is effectively viewed in the frequency domain in terms of Fourier transforms of the input, impulse response, and output.

1/ Use of LPF and HPF to separate sinusoids

The signal consists of a sum of sine waves that we wish to separate.

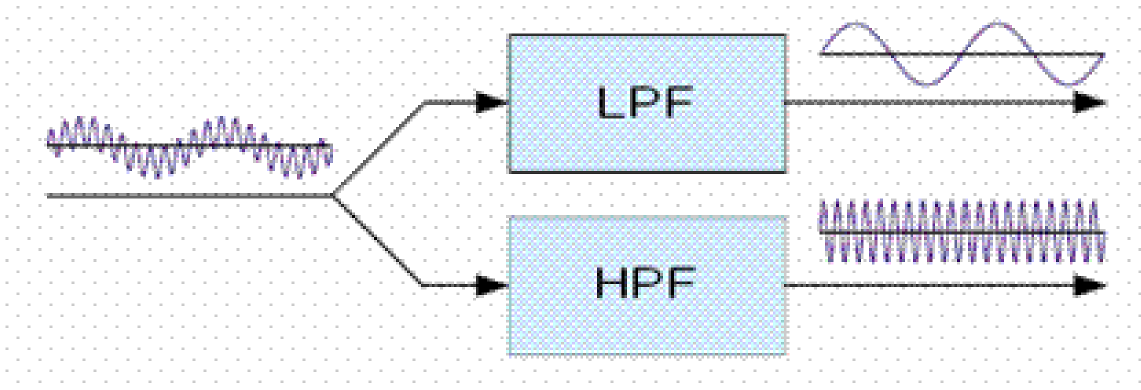


Figure 29

To design the filters we need to specify precisely what we mean by separation. These specifications are easily evaluated in the frequency domain.

The Fourier transform (spectrum) of the signal consists of impulses at the frequencies of the sinusoids. These are superimposed on the frequency responses of the LPF and HPF.

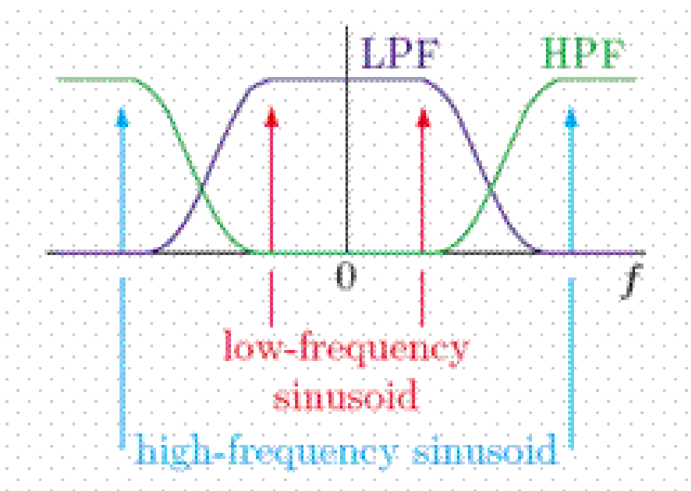


Figure 30

As the frequency separation between the two sinusoids is decreased, the order of the filter required to meet a given specification must be increased.

2/ Use of BPF to extract narrow-band signal from wide-band noise

The Fourier transform (spectrum) of the signal consists of a narrow-band signal, i.e., such as a sinusoid, and wide-band noise, i.e., an undesirable signal. The problem is to extract the signal from the noise using a BPF.

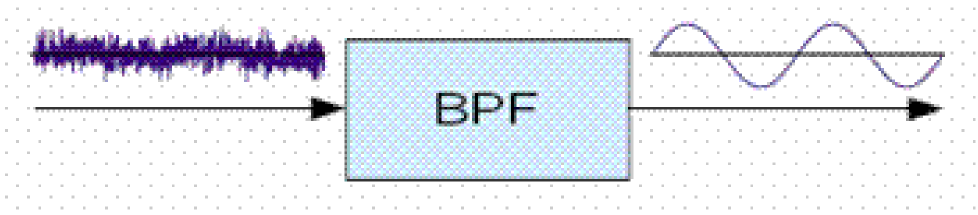


Figure 31

Once again we superimpose the spectrum of the signal and the frequency response of the system.

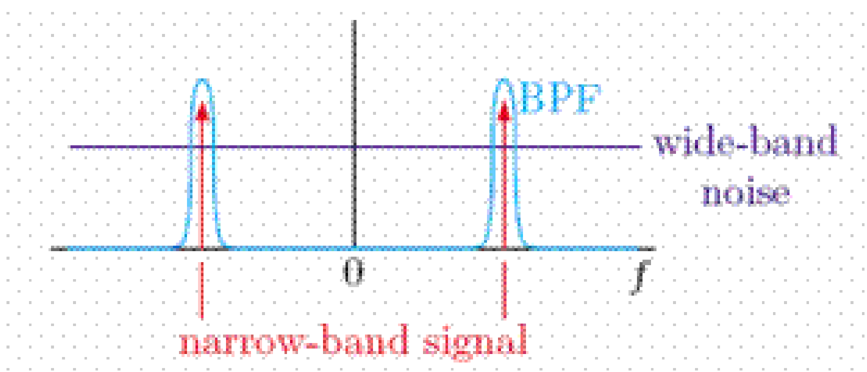


Figure 32

As the bandwidth of the BPF decreases, the amount of noise in the output decreases.

3/ Extraction of signal from noise

The input consists of an ECG signal recorded from the surface of the chest. The recording consists of the signal (the electrical activity of the heart) plus noise from various sources (e.g., the power lines, electrical activity of other muscles, noise from the electrodes). The objective is to extract the signal from the noise.

Figure 33

To design an appropriate filter, we examine the spectrum of the recorded signal.

The spectrum of the recorded signal is shown below.

The spectrum of the electrical activity of the heart is predominantly in the range 0.5 to 35 Hz. The rest is "noise." Note the large peak at 60 Hz.

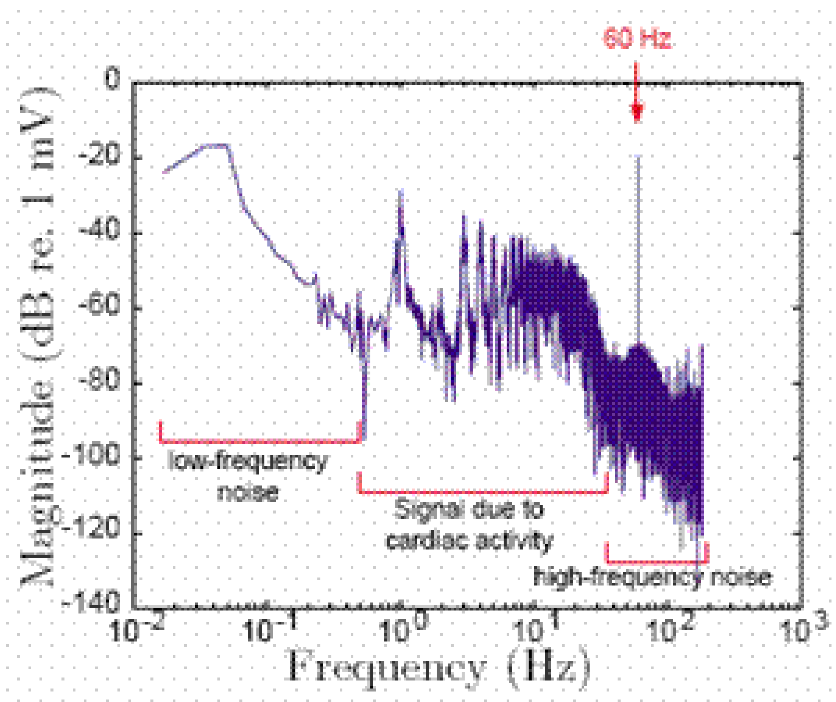


Figure 34

The recorded ECG signal is passed through a filter.

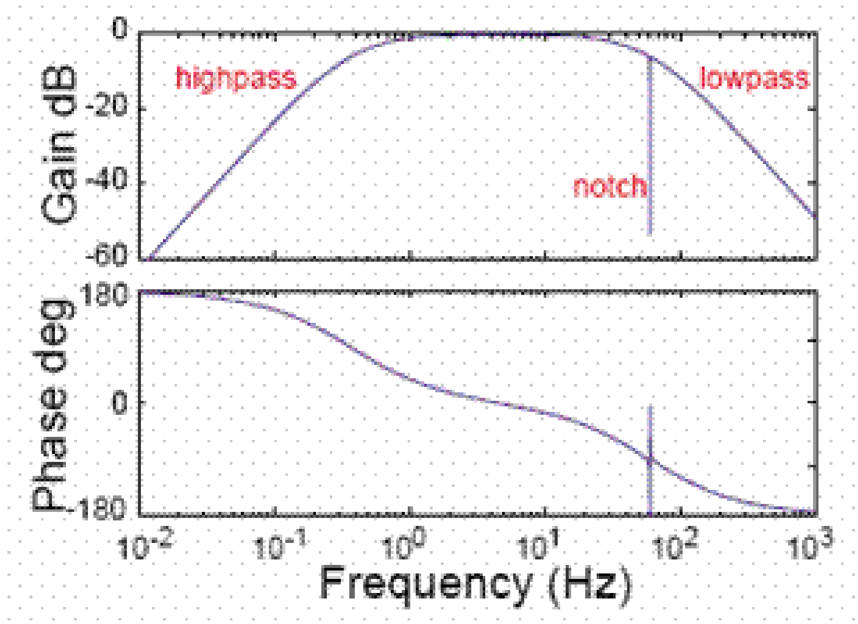


Figure 35

Frequency response of a second-order bandpass filter in cascade with a notch filter (at 60 Hz). The passband of the bandpass filter is between 0.38 and 60 Hz (as suggested in Problem 6 of Problem Set 4).

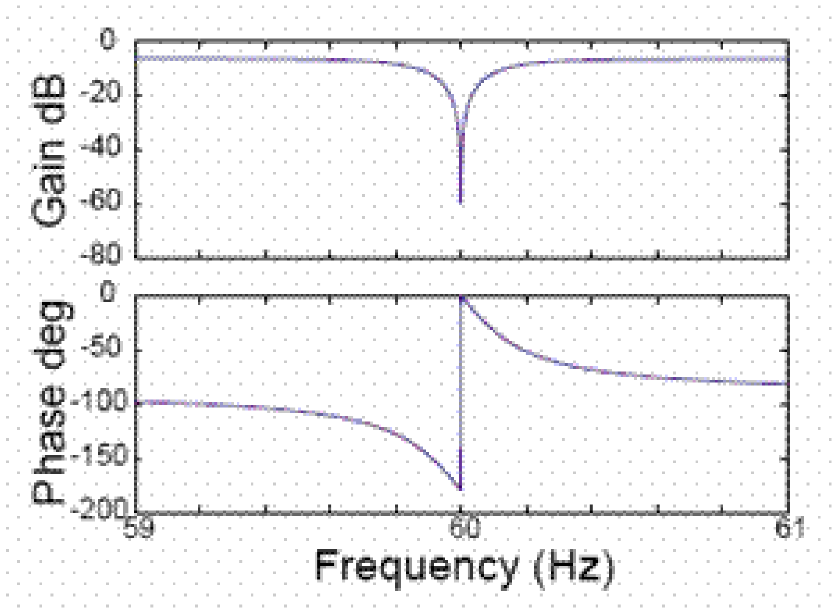


Figure 36

The filter characteristics are shown on an expanded and linear frequency scale centered on 60 Hz to show the effect of the notch filter.

The filtered and unfiltered ECG waveforms and spectra are shown below.

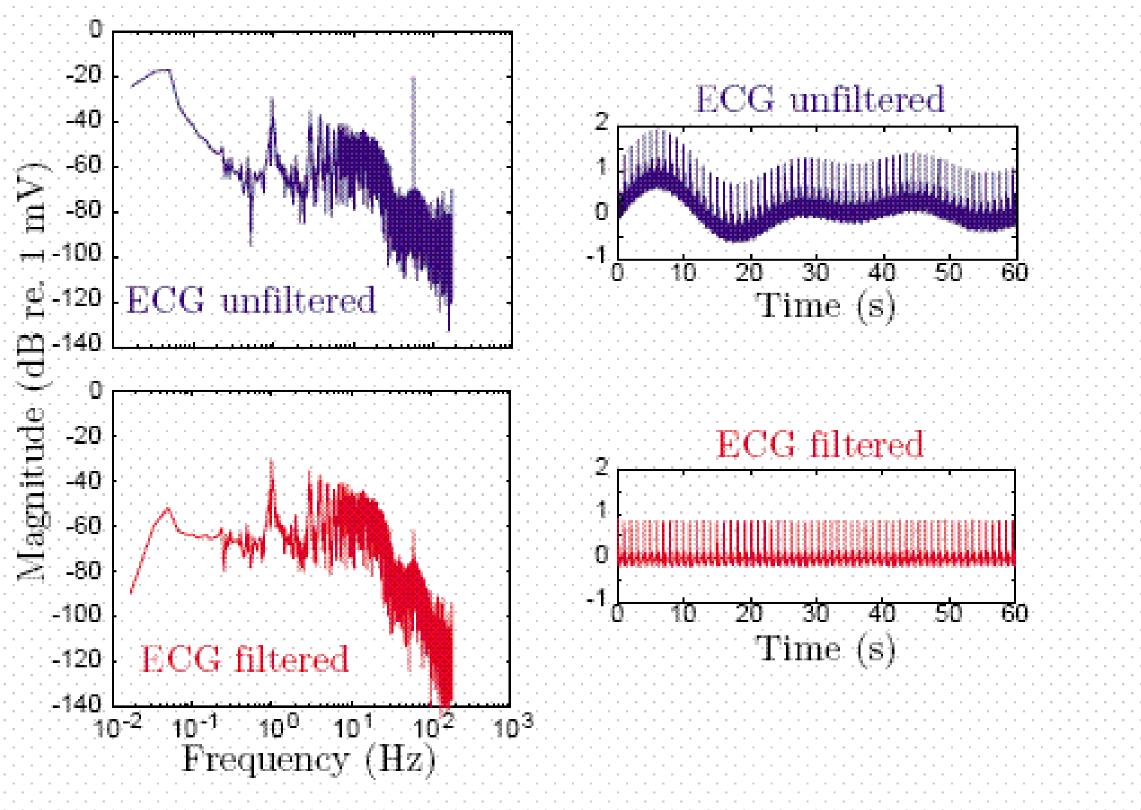


Figure 37

The first 10s of the filtered and unfiltered ECG waveforms are shown below.

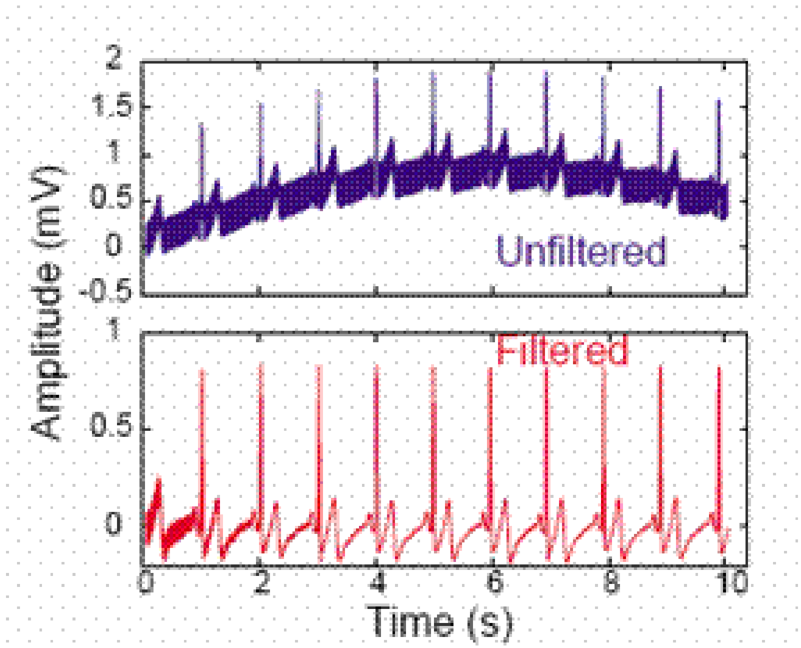


Figure 38

Why does the filtered waveform appear thick in the first two seconds?
The impulse response of the filter is shown below.

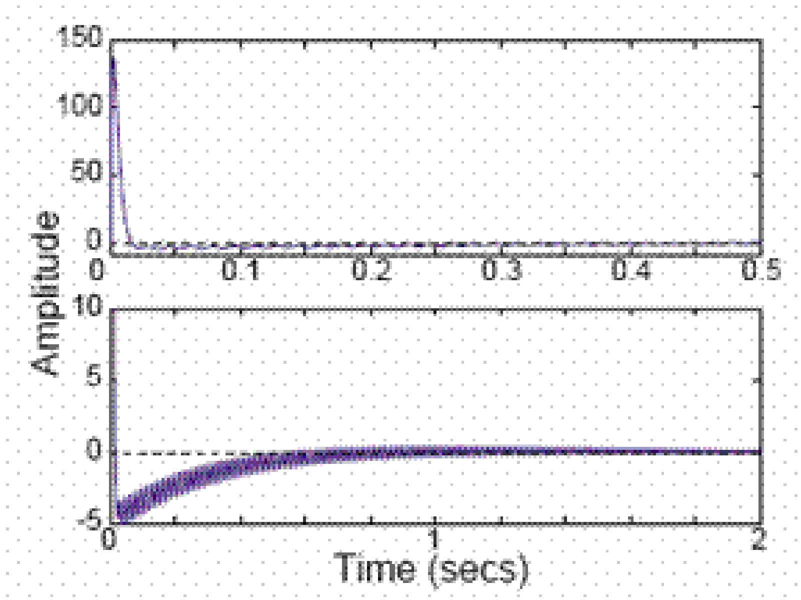


Figure 39

The impulse response is shown on two amplitude and time scales.
It takes about 1-2 seconds for the impulse response to attenuate appreciably?
V. CONCLUSIONS:

- The combination of Fourier transform properties and the Fourier transforms of simple time functions yields a rich collection of Fourier transform pairs.
- If the jω axis is in the region of convergence of the Laplace transform, then the Fourier transform equals the Laplace transform evaluated along the jω axis.
- If there are poles on the jω axis, so that the Laplace transform does not include the jω axis, the Fourier transform can still be defined with the use of singularity functions. This situation will be explored further in the next lecture.
- The filter has attenuated the low-frequency noise as can be seen from the spectrum. Note also that the baseline of the waveform of the filtered ECG signal has been flattened.