DISCRETE FOURIER TRANSFORM (DFT)*

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In this chapter we have seen that the Continuous-Time Fourier Series (CTFS) relates the continuous periodic time to the discrete frequencies, and the Continuous-Time Fourier transform (CTFT) relates the continuous time to the continuous frequency, then the Discrete Time Fourier Series (DTFS) relates the discrete periodic time to the discrete frequencies, and, finally, the Discrete Time Fourier Transform (DTFT) relates the discrete time to the continuous frequency. The two latter representations differ from the two former in that they are 2π - periodic in the frequency domain due to the effect of the sampling in the time domain.

Although the DTFT is very useful for examing the frequency characteristic of discrete – time signals and systems, but there is a computational problem with it, i.e. its frequency representation is continuous. The Discrete – time Fourier Transform (DFT) fills up the time-frequency picture, it relates the discrete time to the discrete frequency. There is still the Fast Fourier Transform (FFT) which is the algorithm to compute the DFT. Both the DFT and FFT are really important tools to tackle many problems is DSP(DTSP). In this last section we present just a brief introduction to the DFT.

1 From the DTFT to the DFT

While the discrete sequence x(n) is discrete in time , its DTFT $X(\omega)$is continuous and 2π-periodic in frequency ω, which is not convenient for computation on computers. So, the frequency variable ω must be discretized in one period $[0, 2\pi]$ and we then take the transform of these discrete frequencies.

First let's repeat the DTFT pair here for comvenience (Equation (3.39) and (3.40)):

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-\mathrm{j}\omega \mathrm{n}} \tag{1}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{\mathrm{j}\omega \mathrm{n}} \mathrm{d}\omega \tag{2}$$

For the sequence x(n) having N samples (time indices) we also discretize ω into the same N points in the interal $[0, 2\pi]$:

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$$0,\ \ \frac{2\pi}{N},\ \ 2\frac{2\pi}{N},\ \ 3\frac{2\pi}{N},\ \ ...,\ \ (N-1)\frac{2\pi}{N}$$

These are the frequency samples .We take the summation from 0 to N-1 instead of from $-\infty$to ∞ to get the DFT of x(n) :

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad , \quad k = 0,\ 1,\ 2, ...,\ N-1 \quad (DFT) \quad (synthesis\, equation) \tag{3}$$

This is the DFT synthesis equation. X(k) is the spectral coefficients.

Since X(k) is discrete we don't have to take the integral as in Equation (3.77) to recover the time sequence x(n) but rather a summation :

$$X(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \quad , \quad n = 0,\ 1,\ 2, ...,\ N-1 \quad (IDFT) \quad (analysis\, equation) \tag{4}$$

X(k) and x(n) forms the transform pair :

$$x(n) \overset{DFT}{\leftrightarrow} X(k)$$

The transform also applies to a system represented by its impulse response h(n):

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-j\frac{2\pi}{N}kn} \quad , \quad k = 0,\ 1,\ 2, ...,\ N-1$$

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) e^{j\frac{2\pi}{N}kn} \quad , \quad n = 0,\ 1,\ 2, ...,\ N-1$$

Above is N-point DFT . For convenience, many authors write the exponent as 2πkn/N but writing $\frac{2\pi}{N}$kn is more meaningful.

It is interesting to compare the above N-point DFT with the DTFS of N samples (Equation (3.36) and (3.37)). Besides the factor $1/N$ appended to differently , we see that they are just the same with the coefficients a_k in the place of the coefficients $X(k)$. The idea is that for a nonperiodic discrete sequence x(n) of N time samples , for this we must use the DTFT, we consider a long periodic sequence having the original x(n) as its period and then use the DTFS but with the new name DFT. If we compute the DFT equations outside the interval $0 \leq k \leq N-1$and $0 \leq n \leq N-1$we will see repeated values.

By reason of computational convenience the number N is usually taken as the integer power of 2 (i.e. $N = 2^n$ with n integer), e.g. $N = 256$. But then one question might arise, when the number of samples (data samples) is below 256 . In that case the technique called zero filling or zero padding is used, whereby we will fill up the vacant samples with zero.

Relating to the DFT we might ask if the sampling of the frequency ω at N equal-distance points represents corectly the frequency variation . In parallel t_0 the sampling theorem time domain (section ...) , we have the sampling theorem in frequency domain which is stated as follows.

The continuous frequency spectrum of a signal existing in a finite time interval T_0 seconds can be represented completely by the frequency samples separated by a distance no more than $1/T_0$ Hertzs. The frequency sepectrum can then be recovered completely from the samples.

It can be checked that the previons sampling of N points does satisfy this theorem. Another way to check the legitimacy of the DFT pair is from the given transform $X(k)$we can recover $x(n)$. The problem is very similar to the case of CTFT (section 3.4), the difference is that in this case we use the orthogonality of discrete-time exponentials.

Example 1

Find the N-point DFT of the following signal

(a) $x(n) = \delta(n)$
(b) $x(n) = 1$
(c) $x(n) = a^n$
(d) $x(n) = 2\cos\omega_0 n$

Solution

(a)

$$X(k) = \sum_{n=0}^{N-1} \delta(n) e^{-j\frac{2\pi}{N}\mathrm{kn}} = 1e^{-j\frac{2\pi}{N}k_0} = 1, \qquad k = 0, 1, 2, ..., N-1$$

(b)

$$X(k) = \sum_{n=0}^{N-1} 1e^{-j\frac{2\pi}{N}\mathrm{kn}}$$

The summation has a value of N for k=0 and of zero for $k \neq 0$, thus

$$X(k) = \mathrm{N}\delta(k)$$

(c)

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi}{N}\mathrm{kn}} = \sum_{n=0}^{N-1} \left(\mathrm{a}e^{-j\frac{2\pi}{N}k}\right)^n$$

Using the formula of finite geometric series we get

$$X(k) = \frac{1-b^n}{1-b}, \qquad k = 0, 1, 2, ..., N-1$$

Where b stands for $e^{-j\frac{2\pi}{N}k}$.

(d) As usual, we express the cosinusoid in terms of exponentials:

$$x(n) = 2\cos\omega_0 n = e^{\mathrm{j}\omega_0 n} + e^{-\omega_0 n}$$

The DFT is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}\mathrm{kn}} = \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi}{N}k-\omega_0\right)n} + \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi}{k}+\omega_0\right)n}$$

For convenience, let's call a spectral coefficient $k_0$corresponding to the given frequency ω_0, satisfying :

$$\frac{2\pi}{N}k_0 = \omega_0 \qquad or \qquad k_0 = \frac{N}{2\pi}N_0$$

Rewrite the transform as

$$X(k) = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} + \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+k_0)n}$$

The first summation equals to N when $k = k_0$, and to zero when $k \neq k_0$. The second summation equals to N when $k = N - k_0$, and to zero when $k \neq N - k_0$. Thus

$$X(k) = N \quad , \quad k = k_0 \quad and \qquad k = N - k_0$$

$$= 0 \quad , \quad otherwise$$

For example with $\omega_0 = \pi/4$ radians/sample then

$$k_0 = \frac{N}{2\pi}\frac{\pi}{4} = \frac{N}{8} \quad , \qquad N - k_0 = \frac{7N}{8}$$

Example 2
[1] A filter has impulse response as

$$h(0) = 0, \qquad h(1) = 1, \qquad h(2) = 2, \qquad h(3) = 3, \qquad h(n) = 0$$

otherwise
(a) Find the 4-point DFT of the impulse response.
(b) Taking the inverse DFT to see if we can recover the impulse response.

Solution
(a) The impulse response is sampled at four points. Its DFT is

$$H(k) = \sum h(n) e^{-j\frac{2\pi}{4}\mathrm{kn}} = \sum \mathrm{ne}^{-j\frac{2\pi}{4}\mathrm{kn}}, \qquad k = 0, 1, 2, 3$$

The computation is proceeded as follows.

$$H(0) = \sum_{n=0}^{3} ne^{-j\frac{\pi}{2}n_0} = 1 + 2 + 3 = 6$$

$$H(1) = \sum_{n=0}^{3} ne^{-j\frac{\pi}{2}n_1} = e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + 3e^{-j\frac{3\pi}{2}} = -2 + j2$$

$$H(2) = \sum_{n=0}^{3} ne^{-j\frac{\pi}{2}n_2} = e^{-j\pi} + 2e^{-j2\pi} + 3e^{-j3\pi} = -2$$

$$H(3) = \sum_{n=0}^{3} ne^{-j\frac{\pi}{2}n_3} = e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + 3e^{-j\frac{3\pi}{2}} = -2 - j2$$

The result is depicted in Fig.3.35 The four values of DFT are the samples of the corresponding continuous-frequency frequency response $H(\omega)$given by the DTFT of the impulse response.

[1] This example is adapter from

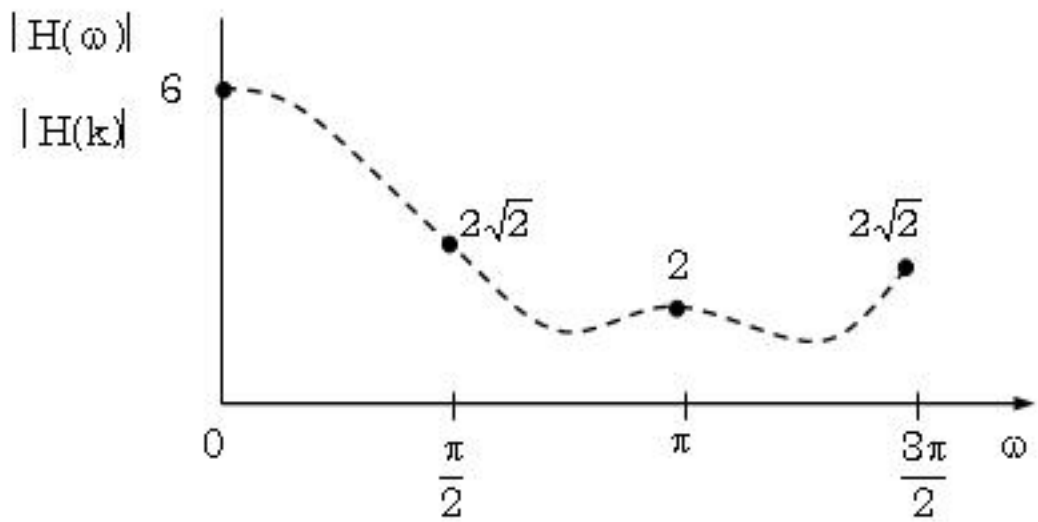

Figure 1: 4-point DFT of the impulse response

(b) The IDFT of the above DFT is given by

$$h(n) = \frac{1}{4}\sum_{k=0}^{3} H(k)\, e^{j\frac{2\pi}{4}kn} \qquad n = 0,\ 1,\ 2,\ 3$$

Thus

$$h(0) = \frac{1}{4}\sum_{k=0}^{3} H(k)\, e^{j\frac{2\pi}{4}k0} = \frac{1}{4}\left[6 + (-2+j2) - 2 + (-2-j2)\right] = 0$$

$$h(1) = \tfrac{1}{4}\textstyle\sum_{k=0}^{3} H(k)\, e^{j\frac{2\pi}{4}k_1}$$

$$= \tfrac{1}{4}\left[6e^{j\frac{\pi}{2}0} + (-2+j2)\, e^{j\frac{\pi}{2}} - 2e^{j\pi} + (-2-j2)\, e^{j\frac{3\pi}{2}}\right] = 1$$

Continuing we will get back h(2) and h(3) as expected.

In part (a) if we evalue H(4) we will see it is equal to H(1), and in part (b) if we evaluate h(4) we will see it is equal to h(1). The reason is that H(k) is periodic and h(n) is assumed periodic

2 Properties of the DFT

The discrete Fourier transform DFT has many properties similar to those of the discrete-time Fourier transform DTFT but with one basic difference as follows. Since the sequence x(n) is assumed to repeat itself indefinitely (this long sequence is periodic with x(n) its period) and the frequency sampling points in a frequency interval $[0, 2\pi]$ are repeated in other 2π-intervals, we will be concerned with circular shift instead of linear shift , and circular convolution instead of linear convolution. The properties and other aspects of the DFT , and the related FFT algorithm will be elaborated in a separate chapter.