Quadratic Functions and Graphs*

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1 Introduction

In Grade 10, you studied graphs of many different forms. In this chapter, you will learn a little more about the graphs of quadratic functions.

2 Functions of the Form $y = a(x+p)^2 + q$

This form of the quadratic function is slightly more complex than the form studied in Grade 10, $y = ax^2 + q$. The general shape and position of the graph of the function of the form $f(x) = a(x+p)^2 + q$ is shown in Figure 1.

Figure 1: Graph of $f(x) = \frac{1}{2}(x+2)^2 - 1$

2.1 Investigation : Functions of the Form $y = a(x+p)^2 + q$

1. On the same set of axes, plot the following graphs:
   a. $a(x) = (x-2)^2$
   b. $b(x) = (x-1)^2$
   c. $c(x) = x^2$
   d. $d(x) = (x+1)^2$
   e. $e(x) = (x+2)^2$

   Use your results to deduce the effect of p.

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2. On the same set of axes, plot the following graphs:
   a. $f(x) = (x-2)^2 + 1$
   b. $g(x) = (x-1)^2 + 1$
   c. $h(x) = x^2 + 1$
   d. $j(x) = (x+1)^2 + 1$
   e. $k(x) = (x+2)^2 + 1$

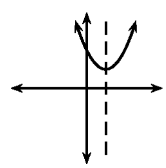
   Use your results to deduce the effect of q.
3. Following the general method of the above activities, choose your own values of p and q to plot 5 different graphs (on the same set of axes) of $y = a(x+p)^2 + q$ to deduce the effect of a.

From your graphs, you should have found that a affects whether the graph makes a smile or a frown. If $a < 0$, the graph makes a frown and if $a > 0$ then the graph makes a smile. This was shown in Grade 10.

You should have also found that the value of q affects whether the turning point of the graph is above the x-axis ($q < 0$) or below the x-axis ($q > 0$).

You should have also found that the value of p affects whether the turning point is to the left of the y-axis ($p > 0$) or to the right of the y-axis ($p < 0$).

These different properties are summarised in Table 1. The axes of symmetry for each graph is shown as a dashed line.

	$p < 0$
	$a > 0$
$q \geq 0$	 **Figure 2**

continued on next page

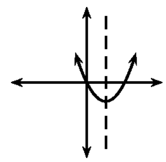
$q \leq 0$	 **Figure 6**

Table 1: Table summarising general shapes and positions of functions of the form $y = a(x+p)^2 + q$. The axes of symmetry are shown as dashed lines.

Phet simulation for graphing

This media object is a Flash object. Please view or download it at
<http://cnx.org/content/m30843/1.4/equation-grapher.swf>

Figure 10

2.2 Domain and Range

For $f(x) = a(x+p)^2 + q$, the domain is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $f(x)$ is undefined.

The range of $f(x) = a(x+p)^2 + q$ depends on whether the value for a is positive or negative. We will consider these two cases separately.

If $a > 0$ then we have:

$$\begin{array}{rcl} (x+p)^2 & \geq & 0 \quad \text{(The square of an expression is always positive)} \\ a(x+p)^2 & \geq & 0 \quad \text{(Multiplication by a positive number maintains the nature of the inequality)} \\ a(x+p)^2 + q & \geq & q \\ f(x) & \geq & q \end{array} \tag{1}$$

This tells us that for all values of x, $f(x)$ is always greater than or equal to q. Therefore if $a > 0$, the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in [q, \infty)\}$.

Similarly, it can be shown that if $a < 0$ that the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in (-\infty, q]\}$. This is left as an exercise.

For example, the domain of $g(x) = (x-1)^2 + 2$ is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $g(x)$ is undefined. The range of $g(x)$ can be calculated as follows:

$$\begin{array}{rcl} (x-p)^2 & \geq & 0 \\ (x+p)^2 + 2 & \geq & 2 \\ g(x) & \geq & 2 \end{array} \tag{2}$$

Therefore the range is $\{g(x) : g(x) \in [2, \infty)\}$.

2.2.1 Domain and Range

1. Given the function $f(x) = (x-4)^2 - 1$. Give the range of $f(x)$.
2. What is the domain of the equation $y = 2x^2 - 5x - 18$?

2.3 Intercepts

For functions of the form, $y = a(x+p)^2 + q$, the details of calculating the intercepts with the x and y axes is given.

The y-intercept is calculated as follows:

$$\begin{array}{rcl} y & = & a(x+p)^2 + q \\ y_{int} & = & a(0+p)^2 + q \\ & = & ap^2 + q \end{array} \quad (3)$$

If $p = 0$, then $y_{int} = q$.

For example, the y-intercept of $g(x) = (x-1)^2 + 2$ is given by setting $x = 0$ to get:

$$\begin{array}{rcl} g(x) & = & (x-1)^2 + 2 \\ y_{int} & = & (0-1)^2 + 2 \\ & = & (-1)^2 + 2 \\ & = & 1 + 2 \\ & = & 3 \end{array} \quad (4)$$

The x-intercepts are calculated as follows:

$$\begin{array}{rcl} y & = & a(x+p)^2 + q \\ 0 & = & a(x_{int}+p)^2 + q \\ a(x_{int}+p)^2 & = & -q \\ x_{int} + p & = & \pm\sqrt{-\frac{q}{a}} \\ x_{int} & = & \pm\sqrt{-\frac{q}{a}} - p \end{array} \quad (5)$$

However, (5) is only valid if $-\frac{q}{a} > 0$ which means that either $q < 0$ or $a < 0$ but not both. This is consistent with what we expect, since if $q > 0$ and $a > 0$ then $-\frac{q}{a}$ is negative and in this case the graph lies above the x-axis and therefore does not intersect the x-axis. If however, $q > 0$ and $a < 0$, then $-\frac{q}{a}$ is positive and the graph is hat shaped with turning point above the x-axis and should have two x-intercepts. Similarly, if $q < 0$ and $a > 0$ then $-\frac{q}{a}$ is also positive, and the graph should intersect with the x-axis twice.

For example, the x-intercepts of $g(x) = (x-1)^2 + 2$ are given by setting $y = 0$ to get:

$$\begin{array}{rcl} g(x) & = & (x-1)^2 + 2 \\ 0 & = & (x_{int}-1)^2 + 2 \\ -2 & = & (x_{int}-1)^2 \end{array} \quad (6)$$

which has no real solutions. Therefore, the graph of $g(x) = (x-1)^2 + 2$ does not have any x-intercepts.

2.3.1 Intercepts

1. Find the x- and y-intercepts of the function $f(x) = (x-4)^2 - 1$.
2. Find the intercepts with both axes of the graph of $f(x) = x^2 - 6x + 8$.
3. Given: $f(x) = -x^2 + 4x - 3$. Calculate the x- and y-intercepts of the graph of f.

2.4 Turning Points

The turning point of the function of the form $f(x) = a(x+p)^2 + q$ is given by examining the range of the function. We know that if $a > 0$ then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in [q, \infty)\}$ and if $a < 0$ then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in (-\infty, q]\}$.

So, if $a > 0$, then the lowest value that $f(x)$ can take on is q. Solving for the value of x at which $f(x) = q$ gives:

$$\begin{array}{rcl} q & = & a(x+p)^2 + q \\ 0 & = & a(x+p)^2 \\ 0 & = & (x+p)^2 \\ 0 & = & x+p \\ x & = & -p \end{array} \tag{7}$$

$\therefore x = -p$ at $f(x) = q$. The co-ordinates of the (minimal) turning point is therefore $(-p, q)$.

Similarly, if $a < 0$, then the highest value that $f(x)$ can take on is q and the co-ordinates of the (maximal) turning point is $(-p, q)$.

2.4.1 Turning Points

1. Determine the turning point of the graph of $f(x) = x^2 - 6x + 8$.
2. Given: $f(x) = -x^2 + 4x - 3$. Calculate the co-ordinates of the turning point of f.
3. Find the turning point of the following function by completing the square: $y = \frac{1}{2}(x+2)^2 - 1$.

2.5 Axes of Symmetry

There is only one axis of symmetry for the function of the form $f(x) = a(x+p)^2 + q$. This axis passes through the turning point and is parallel to the y-axis. Since the x-coordinate of the turning point is $x = -p$, this is the axis of symmetry.

2.5.1 Axes of Symmetry

1. Find the equation of the axis of symmetry of the graph $y = 2x^2 - 5x - 18$.
2. Write down the equation of the axis of symmetry of the graph of $y = 3(x-2)^2 + 1$.
3. Write down an example of an equation of a parabola where the y-axis is the axis of symmetry.

2.6 Sketching Graphs of the Form $f(x) = a(x+p)^2 + q$

In order to sketch graphs of the form $f(x) = a(x+p)^2 + q$, we need to determine five characteristics:

1. sign of a
2. domain and range
3. turning point

4. y-intercept
5. x-intercept

For example, sketch the graph of $g(x) = -\frac{1}{2}(x+1)^2 - 3$. Mark the intercepts, turning point and axis of symmetry.

Firstly, we determine that $a < 0$. This means that the graph will have a maximal turning point.

The domain of the graph is $\{x : x \in \mathbb{R}\}$ because $f(x)$ is defined for all $x \in \mathbb{R}$. The range of the graph is determined as follows:

$$\begin{array}{rcl} (x+1)^2 & \geq & 0 \\ -\frac{1}{2}(x+1)^2 & \leq & 0 \\ -\frac{1}{2}(x+1)^2 - 3 & \leq & -3 \\ \therefore f(x) & \leq & -3 \end{array} \tag{8}$$

Therefore the range of the graph is $\{f(x) : f(x) \in (-\infty, -3]\}$.

Using the fact that the maximum value that $f(x)$ achieves is -3, then the y-coordinate of the turning point is -3. The x-coordinate is determined as follows:

$$\begin{array}{rcl} -\frac{1}{2}(x+1)^2 - 3 & = & -3 \\ -\frac{1}{2}(x+1)^2 - 3 + 3 & = & 0 \\ -\frac{1}{2}(x+1)^2 & = & 0 \\ \text{Divide both sides by } -\frac{1}{2}: \quad (x+1)^2 & = & 0 \\ \text{Take square root of both sides}: \quad x+1 & = & 0 \\ \therefore \quad x & = & -1 \end{array} \tag{9}$$

The coordinates of the turning point are: $(-1, -3)$.

The y-intercept is obtained by setting $x = 0$. This gives:

$$\begin{array}{rcl} y_{int} & = & -\frac{1}{2}(0+1)^2 - 3 \\ & = & -\frac{1}{2}(1) - 3 \\ & = & -\frac{1}{2} - 3 \\ & = & -\frac{1}{2} - 3 \\ & = & -\frac{7}{2} \end{array} \tag{10}$$

The x-intercept is obtained by setting $y = 0$. This gives:

$$\begin{array}{rcl} 0 & = & -\frac{1}{2}(x_{int}+1)^2 - 3 \\ 3 & = & -\frac{1}{2}(x_{int}+1)^2 \\ -3 \cdot 2 & = & (x_{int}+1)^2 \\ -6 & = & (x_{int}+1)^2 \end{array} \tag{11}$$

which has no real solutions. Therefore, there are no x-intercepts.

We also know that the axis of symmetry is parallel to the y-axis and passes through the turning point.

Figure 11: Graph of the function $f\left(x\right) = -\frac{1}{2}(x+1)^2 - 3$

Khan academy video on graphing quadratics

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/RjkTEyO5Zso&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 12

2.6.1 Sketching the Parabola

1. Draw the graph of $y = 3(x-2)^2 + 1$ showing all the intercepts with the axes as well as the coordinates of the turning point.
2. Draw a neat sketch graph of the function defined by $y = ax^2 + bx + c$ if $a > 0$; $b < 0$; $b^2 = 4ac$.

2.7 Writing an equation of a shifted parabola

Given a parabola with equation $y = x^2 - 2x - 3$. The graph of the parabola is shifted one unit to the right. Or else the y-axis shifts one unit to the left i.e. x becomes $x - 1$. Therefore the new equation will become:

$$\begin{aligned} y &= (x-1)^2 - 2\left(x-1\right) - 3 \\ &= x^2 - 2x + 1 - 2x + 2 - 3 \\ &= x^2 - 4x \end{aligned} \quad (12)$$

If the given parabola is shifted 3 units down i.e. y becomes $y + 3$. The new equation will be:
(Notice the x-axis then moves 3 units upwards)

$$\begin{aligned} y + 3 &= x^2 - 2x - 3 \\ y &= x^2 - 2x - 6 \end{aligned} \quad (13)$$

3 End of Chapter Exercises

1. Show that if $a < 0$, then the range of $f\left(x\right) = a(x+p)^2 + q$ is $\{f\left(x\right) : f\left(x\right) \in (-\infty, q]\}$.
2. If (2,7) is the turning point of $f\left(x\right) = -2x^2 - 4ax + k$, find the values of the constants a and k.
3. The graph in the figure is represented by the equation $f\left(x\right) = ax^2 + bx$. The coordinates of the turning point are (3,9). Show that $a = -1$ and $b = 6$.

Figure 13

4. Given: $y = x^2 - 2x + 3$. Give the equation of the new graph originating if:
    a. The graph of f is moved three units to the left.
    b. The x-axis is moved down three units.
5. A parabola with turning point (-1,-4) is shifted vertically by 4 units upwards. What are the coordinates of the turning point of the shifted parabola?