Properties of the Laplace Transform*

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Abstract

Derives basic properties of the Laplace transform.

1 Properties of the Laplace Transform

The properties associated with the Laplace transform are similar to those of the Fourier transform. First, let's set define some notation, we will use the notation $\mathfrak{L}\{\}$ to denote the Laplace transform operation. Therefore we can write $X(s) = \mathfrak{L}\{x(t)\}$ and $x(t) = \mathfrak{L}^{-1}\{X(s)\}$ for the *forward* and *inverse* Laplace transforms, respectively. We can also use the transform pair notation used earlier:

$$x(t) \leftrightarrow X(s) \tag{1}$$

With this notation defined, lets now look at some properties.

1.1 Linearity

Given that $x_1(t) \leftrightarrow X_1(s)$ and $x_2(t) \leftrightarrow X_2(s)$ then for any constants α and β, we have

$$\alpha x_1(t) + \beta x_2(t) \leftrightarrow \alpha X_1(s) + \beta X_2(s) \tag{2}$$

The linearity property follows easily using the definition of the Laplace transform.

1.2 Time Delay

The reason we call this the *time delay* property rather than the time shift property is that the time shift must be positive, i.e. if $\tau > 0$, then $x(t-\tau)$ corresponds to a delay. If $\tau < 0$ then we would not be able to use the single-sided Laplace transform because we would have a lower integration limit of τ, which is less than zero. To derive the property, lets evaluate the Laplace transform of the time-delayed signal

$$\mathfrak{L}\{x(t-\tau)\} = \int_0^\infty x(t-\tau)\, e^{-st} dt \tag{3}$$

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Letting $\gamma = t - \tau$ leads to $t = \gamma + \tau$ and $dt = d\gamma$. Substituting these quantities into (3) gives

$$\begin{aligned} \mathfrak{L}\{x(t-\tau)\} &= \int_{-\tau}^{\infty} x(\gamma) e^{-s(\gamma+\tau)} d\gamma \\ &= e^{-s\tau} \int_{-\tau}^{\infty} x(\gamma) e^{-s\gamma} d\gamma \\ &= e^{-s\tau} \int_{-\tau}^{0} x(\gamma) e^{-s\gamma} d\gamma + e^{-s\tau} \int_{0}^{\infty} x(\gamma) e^{-s\gamma} d\gamma \end{aligned} \tag{4}$$

where we note that the first integral in the last line is zero since $x(t) = 0, t < 0$. Therefore the time delay property is given by

$$\mathfrak{L}\{x(t-\tau)\} = e^{-s\tau} X(s) \tag{5}$$

1.3 s-Shift

This property is the Laplace transform corresponds to the frequency shift property of the Fourier transform. In fact, the derivation of the s-shift property is virtually identical to that of the frequency shift property.

$$\begin{aligned} \mathfrak{L}\{e^{-at}x(t)\} &= \int_{0}^{\infty} e^{-at} x(t) e^{-st} dt \\ &= \int_{0}^{\infty} x(t) e^{-(a+s)t} dt \\ &= \int_{0}^{\infty} x(t) e^{-(a+\sigma+j\Omega)t} dt \\ &= X(s+a) \end{aligned} \tag{6}$$

The s-shift property also alters the region of convergence of the Laplace transform. If the region of convergence for $X(s)$ is $\sigma > \sigma_{min}$, then the region of convergence for $\mathfrak{L}\{e^{-at}x(t)\}$ is $\sigma > \sigma_{min} - \mathrm{Re}(a)$.

1.4 Multiplication by t

Let's begin by taking the derivative of the Laplace transform:

$$\begin{aligned} \frac{dX(s)}{ds} &= \frac{d}{ds} \int_{0}^{\infty} x(t) e^{-st} dt \\ &= \int_{0}^{\infty} x(t) \frac{d}{ds} e^{-st} dt \\ &= -\int_{0}^{\infty} t x(t) e^{-st} dt \end{aligned} \tag{7}$$

So we can write

$$\mathfrak{L}\{tx(t)\} = -\frac{dX(s)}{ds} \tag{8}$$

This idea can be extended to multiplication by t^n. Letting $y(t) = tx(t)$, if follows that

$$\begin{aligned} ty(t) &\leftrightarrow -\frac{dY(s)}{ds} \\ &\leftrightarrow \frac{d^2X(s)}{ds^2} \end{aligned} \tag{9}$$

Proceeding in this manner, we find that

$$t^n x(t) \leftrightarrow (-1)^n \frac{d^n X(s)}{ds^n} \tag{10}$$

1.5 Time Scaling

The time scaling property for the Laplace transform is similar to that of the Fourier transform:

$$\begin{aligned} \mathfrak{L}\{x(\alpha t)\} &= \int_0^\infty x(\alpha t)\, e^{-st} dt \\ &= \frac{1}{\alpha}\int_0^\infty x(\gamma)\, e^{-\frac{s}{\alpha}\gamma} d\gamma \\ &= \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right) \end{aligned} \tag{11}$$

where in the second equality, we made the substitution $t = \frac{\gamma}{\alpha}$ and $dt = \frac{d\gamma}{\alpha}$.

1.6 Convolution

The derivation of the convolution property for the Laplace transform is virtually identical to that of the Fourier transform. We begin with

$$\mathfrak{L}\{\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\} = \int_{-\infty}^{\infty} x(\tau)\, \mathfrak{L}\{h(t-\tau)\} d\tau \tag{12}$$

Applying the time-delay property of the Laplace transform gives

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau)\, \mathfrak{L}\{h(t-\tau)\} d\tau &= H(s) \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau} d\tau \\ &= H(s)\, X(s) \end{aligned} \tag{13}$$

If $h(t)$ is the the impulse response of a linear time-invariant system, then we call $H(s)$ the *system function* of the system. The frequency response results by setting $s = j\Omega$ in $H(s)$. The system function provides us with a very powerful means of determining the output of a linear time-invariant filter given the input signal. It will also enable us to determine a means of establishing the stability[1] of a linear-time invariant filter, something which was not possible with the frequency response.

1.7 Differentiation

The Laplace transform of the derivative of a signal will be used widely. Consider

$$\mathfrak{L}\{\frac{d}{dt} x(t)\} = \int_{0^-}^{\infty} x^{'}(t)\, e^{-st} dt \tag{14}$$

this can be integrated by parts:

$$\begin{aligned} u &= e^{-st} & v^{'} &= x^{'}(t) \\ u^{'} &= -se^{-st} & v &= x(t) \end{aligned} \tag{15}$$

which gives

$$\begin{aligned} \mathfrak{L}\{\tfrac{d}{dt} x(t)\} &= uv|_{0^-}^{\infty} - \int_{0^-}^{\infty} u^{'} v dt \\ &= e^{-st} x(t)|_{0^-}^{\infty} + \int_{0^-}^{\infty} sx(t)\, e^{-st} dt \\ &= -x(0^-) + sX(s) \end{aligned} \tag{16}$$

therefore we have,

$$\frac{d}{dt} x(t) \leftrightarrow sX(s) - x\left(0^-\right) \tag{17}$$

[1] We will discuss stability shortly

1.8 Higher Order Derivatives

The previous derivation can be extended to higher order derivatives. Consider

$$y(t) = \frac{dx(t)}{dt} \leftrightarrow sX(s) - x\left(0^-\right) \tag{18}$$

it follows that

$$\frac{dy(t)}{dt} \leftrightarrow sY(s) - y\left(0^-\right) \tag{19}$$

which leads to

$$\frac{d^2}{dt^2}x(t) \leftrightarrow s^2X(s) - sx\left(0^-\right) - \frac{dx\left(0^-\right)}{dt} \tag{20}$$

This process can be iterated to get the Laplace transform of arbitrary higher order derivatives, giving

$$\frac{d^n x(t)}{dt^n} \quad \leftrightarrow \quad s^n X(s) - s^{n-1}x\left(0^-\right) - \sum_{k=2}^{n} s^{n-k}\frac{d^{k-1}x\left(0^-\right)}{dt^{k-1}} \tag{21}$$

where it should be understood that

$$\frac{d^m x\left(0^-\right)}{dt^m} \equiv \left.\frac{d^m x(t)}{dt^m}\right|_{t=0^-}, m = 1, ..., n-1 \tag{22}$$

1.9 Integration

Let

$$g(t) = \int_{0^-}^{t} x(\tau)\, d\tau \tag{23}$$

it follows that

$$\frac{dg(t)}{dt} = x(t) \tag{24}$$

and $g\left(0^-\right) = 0$. Moreover, we have

$$\begin{aligned} X(s) &= \mathfrak{L}\{\tfrac{dg(t)}{dt}\} \\ &= sG(s) - g\left(0^-\right) \\ &= sG(s) \end{aligned} \tag{25}$$

therefore

$$G(s) = \frac{X(s)}{s} \tag{26}$$

but since

$$G(s) = \mathfrak{L}\{\int_{0^-}^{t} x(\tau)\, d\tau\} \tag{27}$$

we have

$$\int_{0^-}^{t} x(\tau)\, d\tau \leftrightarrow \frac{X(s)}{s} \tag{28}$$

Now suppose $x(t)$ has a non-zero integral over negative values of t. We have

$$\int_{\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{0^-} x(\tau)\, d\tau + \int_{0^-}^{t} x(\tau)\, d\tau \tag{29}$$

The quantity $\int_{-\infty}^{0^-} x(\tau)\, d\tau$ is a constant for positive values of t, and can be expressed as

$$u(t) \int_{-\infty}^{0^-} x(\tau)\, d\tau \tag{30}$$

it follows that

$$\int_{\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{\int_{-\infty}^{0^-} x(\tau)\, d\tau}{s} + \frac{X(s)}{s} \tag{31}$$

where we have used the fact that $u(t) \leftrightarrow \frac{1}{s}$.

1.10 The Initial Value Theorem

The initial value theorem makes it possible to determine $x(t)$ at $t = 0^+$ from $X(s)$. From the derivative property of the Laplace transform, we can write

$$\mathfrak{L}\{\frac{dx(t)}{dt}\} = sX(s) - x\left(0^-\right) \tag{32}$$

Taking the limit $s \to \infty$

$$\begin{aligned} \lim_{s\to\infty} \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt &= \lim_{s\to\infty} \left[sX(s) - x(0^-)\right] \\ \int_{0^-}^{\infty} \lim_{s\to\infty} \frac{dx(t)}{dt} e^{-st} dt &= \lim_{s\to\infty} \left[sX(s) - x(0^-)\right] \end{aligned} \tag{33}$$

There are two cases, the first is when $x(t)$ is continuous at $t = 0$. In this case it is clear that $\frac{dx(t)}{dt} e^{-st} \to 0$ as $s \to \infty$, so (33) can be written as

$$0 = \lim_{s\to\infty} \left[sX(s) - x\left(0^-\right)\right] \tag{34}$$

Since $x(t)$ is continuous at $t = 0$, $x(0^-) = x(0^+)$, the Initial Value Theorem follows,

$$x\left(0^+\right) = \lim_{s\to\infty} sX(s) \tag{35}$$

The second case is when $x(t)$ is discontinuous at $t = 0$. In this case, we use the fact that

$$\left.\frac{dx(t)}{dt}\right|_{t=0} = \left[x\left(0^+\right) - x\left(0^-\right)\right] \delta(t) \tag{36}$$

For example, if we integrate the right-hand side of (36) with $x(0^-) = 0$ and $x(0^+) = 1$, we get the unit step function, $u(t)$. Proceeding as before, we have

$$\lim_{s\to\infty} \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \lim_{s\to\infty} \left[sX(s) - x\left(0^-\right)\right] \tag{37}$$

The left-hand side of (37) can be written as

$$\lim_{s\to\infty} \int_{0^+}^{0^-} \left[x\left(0^+\right) - x\left(0^-\right)\right] \delta(t)\, e^{-st} dt + \lim_{s\to\infty} \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt \tag{38}$$

From the sifting property of the unit impulse, the first term in (38) is

$$\left[x\left(0^{+}\right)-x\left(0^{-}\right)\right] \tag{39}$$

while the second term is zero since in the limit, the real part of s goes to infinity. Substituting these results into the left-hand side of (37) again leads to the initial value theorem, in (35).

1.11 The Final Value Theorem

The Final Value Theorem allows us to determine

$$\lim_{t\to\infty} x\left(t\right) \tag{40}$$

from $X\left(s\right)$. Taking the limit as s approaches zero in the derivative property gives

$$\lim_{s\to 0}\int_{0^{-}}^{\infty}\frac{dx\left(t\right)}{dt}e^{-st}dt=\lim_{s\to 0}\left[sX\left(s\right)-x\left(0^{-}\right)\right] \tag{41}$$

The left-hand-side of (41) can be written as

$$\int_{0^{-}}^{\infty}\lim_{s\to 0}\frac{dx\left(t\right)}{dt}e^{-st}dt=\int_{0^{-}}^{\infty}\frac{dx\left(t\right)}{dt}dt=x\left(\infty\right)-x\left(0^{-}\right) \tag{42}$$

Substituting this result back into (41) leads to the Final Value Theorem

$$x\left(\infty\right)=\lim_{s\to 0}sX\left(s\right) \tag{43}$$

which is only valid as long as the limit $x\left(\infty\right)$ exists.

Property	$y\left(t\right)$	$Y\left(s\right)$
Linearity	$\alpha x_1\left(t\right)+\beta x_2\left(t\right)$	$\alpha X_1\left(s\right)+\beta X_2\left(s\right)$
Time Delay	$x\left(t-\tau\right)$	$X\left(s\right)e^{-s\tau}$
s-Shift	$x\left(t\right)e^{-at}$	$X\left(s+a\right)$
Multiplication by t	$tx\left(t\right)$	$-\frac{dX(s)}{ds}$
Multiplication by t^n	$t^n x\left(t\right)$	$\left(-1\right)^n\frac{d^n X(s)}{ds^n}$
Convolution	$x\left(t\right)*h\left(t\right)$	$X\left(s\right)H\left(s\right)$
Differentiation	$\frac{dx(t)}{dt}$	$sX\left(s\right)-x\left(0^{-}\right)$
	$\frac{d^2x(t)}{dt^2}$	$s^2X\left(s\right)-sx\left(0^{-}\right)-\frac{dx\left(0^{-}\right)}{dt}$
	$\frac{d^nx(t)}{dt^n}$	$s^nX\left(s\right)-s^{n-1}x\left(0^{-}\right)-\sum_{k=2}^{n}s^{n-k}\frac{d^{k-1}x\left(0^{-}\right)}{dt^{k-1}}$
Integration	$\int_{\infty}^{t}x\left(\tau\right)d\tau$	$\frac{\int_{-\infty}^{0^{-}}x(\tau)d\tau}{s}+\frac{X(s)}{s}$
Initial Value Theorem	$x\left(0^{+}\right)=\lim_{s\to\infty}sX\left(s\right)$	
Final Value Theorem	$x\left(\infty\right)=\lim_{s\to 0}sX\left(s\right)$	

Table 1: Laplace Transform properties.