The Sinusoidal Steady State Response*

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It is useful to see what the effect of the filter is on a sinusoidal signal, say $x(t) = cos(\Omega_0 t)$. If $y(t)$ is the output of the filter, then we can write

$$y(t) = \int_{-\infty}^{\infty} cos(\Omega_0 (t-\tau)) h(\tau) d\tau \qquad (1)$$

Using the Euler formula for $cos(\Omega_0 t)$, right hand side of (1) can be written as:

$$\frac{1}{2}\int_{-\infty}^{\infty} \left(e^{j(\Omega_0 (t-\tau))} + e^{-j(\Omega_0 (t-\tau))}\right) h(\tau) d\tau \qquad (2)$$

This integral can be split into two separate integrals, and written as:

$$\frac{e^{j\Omega_0 t}}{2}\int_{-\infty}^{\infty} e^{-j\Omega_0 \tau} h(\tau) d\tau + \frac{e^{-j\Omega_0 t}}{2}\int_{-\infty}^{\infty} e^{j\Omega_0 \tau} h(\tau) d\tau \qquad (3)$$

The first of the two integrals can be recognizes as the Fourier Transform of the impulse response evaluated at $\Omega = \Omega_0$. The second integral is just the complex conjugate of the first integral. Therefore (3) can be written as:

$$\frac{e^{j\Omega_0 t}}{2} H(j\Omega_0) + \frac{e^{-j\Omega_0 t}}{2} H^*(j\Omega_0) \qquad (4)$$

Since the second term in (4) is the complex conjugate of the first term, we can express (4) as:

$$Re\{e^{j\Omega_0 t} H(j\Omega_0)\} \qquad (5)$$

or expressing $H(j\Omega_0)$ in terms of polar coordinates:

$$Re\{e^{j\Omega_0 t}|H(j\Omega_0)|e^{j\angle H(j\Omega_0)}\} = Re\{|H(j\Omega_0)|e^{j(\Omega_0 t + \angle H(j\Omega_0))}\} \qquad (6)$$

Therefore, we find that the filter output is given by

$$y(t) = |H(j\Omega_0)| cos(\Omega_0 t + \angle H(j\Omega_0)) \qquad (7)$$

This is called the *sinusoidal steady state response*. It tells us that when the input to a linear, time-invariant filter is a cosine, the filter output is a cosine whose amplitude has been scaled by $|H(j\Omega_0)|$ and that has

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been phase shifted by $\angle H(j\Omega_0)$. The same result applies to an input that is an arbitrarily phase shifted cosine (e.g. a sine wave).

Example 3.1 Find the output of a filter whose impulse response is $h(t) = e^{-5t}u(t)$ and whose input is given by $x(t) = cos(2t)$. It can be readily seen that the frequency response of the filter is

$$H(j\Omega) = \frac{1}{5 + j\Omega} \tag{8}$$

and therefore $|H(j2)| = 0.1857$ and $\angle H(j2) = -0.3805$. Therefore, using (7):

$$y(t) = 0.1857cos(2t - 0.3805) \tag{9}$$