Inner Product Spaces*

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Where normed vector spaces incorporate the concept of length into a vector space, inner product spaces incorporate the concept of angle.

Definition 1
Let V be a vector space over K. An *inner product* is a function $< \cdot,\cdot >: V \times V \rightarrow K$ such that for all $x, y, z \in V, \alpha \in K$

IP1. $< x, y >= \overline{< y, x >}$
IP2. $< \alpha x, y >= \alpha < x, y >$
IP3. $< x + y, z >=< x, z > + < y, z >$
IP4. $< x, x >\geq 0$ with equality iff $x = 0$.

A vector space together with an inner product is called an *inner product space*.

Example 1

- $V = \mathbb{C}^N$, $< x, y >:= \sum_{i=1}^{N} x_i \overline{y_i} = y^* x$
- $V = C\left[a, b\right]$, $< x, y >:= \int_a^b x\left(t\right) \overline{y\left(t\right)} dt$

Note that a valid inner product space induces a normed vector space with norm $\|x\| = \sqrt{< x, x >}$. (Proof relies on Cauchy-Schwartz inequality.) In $\mathbb{R}^N$ or $\mathbb{C}^N$, the standard inner product induces the ℓ_2-norm. We summarize the relationships between the various spaces introduced over the last few lectures in Figure 1.

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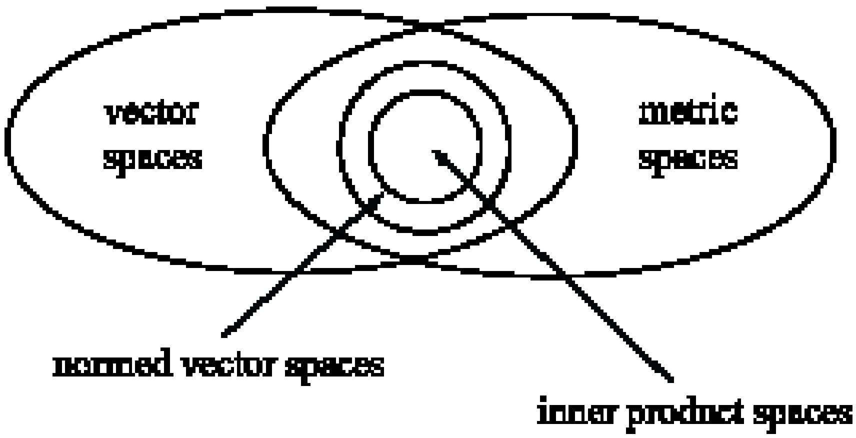

Figure 1: Venn diagram illustrating the relationship between vector and metric spaces.