Orthogonal Bases*

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Definition 1

A collection of vectors B in an inner product space V is called an orthogonal basis if

- 1. span (B) = V
- 2. $v_i \perp v_j$ (i.e., $\langle v_i, v_j \rangle = 0$) $\forall i \neq j$

If, in addition, the vectors are normalized under the induced norm, i.e., $||v_i|| = 1 \forall i$, then we call V an orthonormal basis (or "orthobasis"). If V is infinite dimensional, we need to be a bit more careful with 1. Specifically, we really only need the closure of span (B) to equal V. In this case any $x \in V$ can be written as

$$x = \sum_{i=1}^{\infty} c_i v_i \tag{1}$$

for some sequence of coefficients $\{c_i\}_{i=1}^{\infty}$.

(This last point is a technical one since the span is typically defined as the set of linear combinations of a finite number of vectors. See Young Ch 3 and 4 for the details. This won't affect too much so we will gloss over the details.)

Example 1

• $V = \mathbb{R}^2$, standard basis

$$v_1 = \begin{bmatrix} 1\\0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0\\1 \end{bmatrix} \tag{2}$$

Example 2

• Suppose $V = \{$ piecewise constant functions on $[0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4}), [\frac{3}{4}, 1] \}$. An example of such a function is illustrated below.

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Figure 1

Consider the set



Figure 2

The vectors $\{v_1, v_2, v_3, v_4\}$ form an orthobasis for V. • Suppose $V = L_2 [-\pi, \pi]$. $B = \{\frac{1}{\sqrt{2\pi}} e^{jkt}\}_{k=-\infty}^{\infty}$, i.e, the Fourier series basis vectors, form an

orthobasis for V. To verify the orthogonality of the vectors, note that:

$$<\frac{1}{\sqrt{2\pi}}e^{jkt}, \frac{1}{\sqrt{2\pi}}e^{jkt} > = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j(k_1-k_2)t} = \frac{1}{2\pi}\frac{e^{j(k_1-k_2)t}}{j(k_1-k_2)}\Big|_{-\pi}^{\pi}$$
(3)
$$= \frac{1}{2\pi}\cdot\frac{-1+1}{j(k_1-k_2)} = 0 \quad (k_1 \neq k_2)$$

See Young for proof that the closure of B is $L_2[-\pi,\pi]$, i.e., the fact that $any f \in L_2[-\pi,\pi]$ has a Fourier series representation.