Orthogonal Bases*

Mark A. Davenport

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Definition 1
A collection of vectors B in an inner product space V is called an *orthogonal basis* if

1. $\text{span}(B) = V$
2. $v_i \perp v_j$ (i.e., $< v_i, v_j >= 0$) $\forall\ i \neq j$

If, in addition, the vectors are normalized under the induced norm, i.e., $\| v_i \| = 1\ \forall\ i$, then we call V an *orthonormal basis* (or "*orthobasis*"). If V is infinite dimensional, we need to be a bit more careful with 1. Specifically, we really only need the closure of $\text{span}(B)$ to equal V. In this case any $x \in V$ can be written as

$$x = \sum_{i=1}^{\infty} c_i v_i \tag{1}$$

for some sequence of coefficients $\{c_i\}_{i=1}^{\infty}$.

(This last point is a technical one since the span is typically defined as the set of linear combinations of a finite number of vectors. See Young Ch 3 and 4 for the details. This won't affect too much so we will gloss over the details.)

Example 1

- $V = \mathbb{R}^2$, standard basis

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2}$$

Example 2

- Suppose $V = \{$ piecewise constant functions on $\left[0, \frac{1}{4}\right), \left[\frac{1}{4}, \frac{1}{2}\right), \left[\frac{1}{2}, \frac{3}{4}\right), \left[\frac{3}{4}, 1\right]\}$. An example of such a function is illustrated below.

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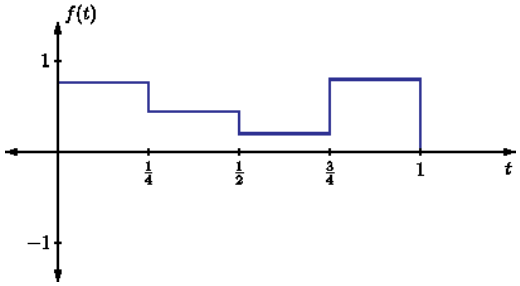

Figure 1

Consider the set

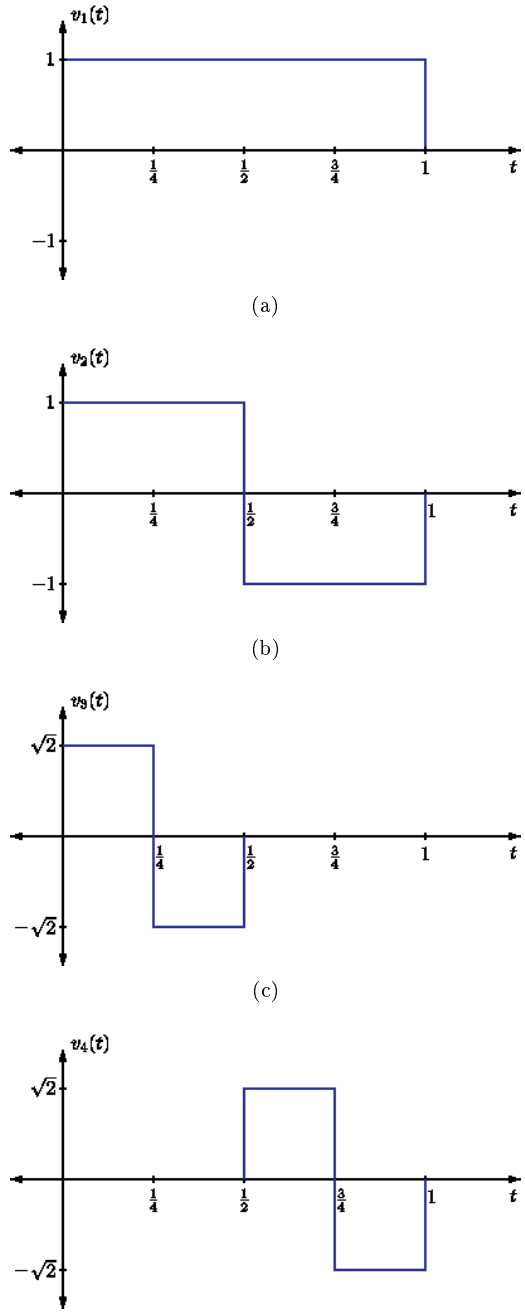

Figure 2

The vectors $\{v_1, v_2, v_3, v_4\}$ form an orthobasis for V.

- Suppose $V = L_2[-\pi, \pi]$. $B = \{\frac{1}{\sqrt{2\pi}} e^{jkt}\}_{k=-\infty}^{\infty}$, i.e, the Fourier series basis vectors, form an

orthobasis for V. To verify the orthogonality of the vectors, note that:

$$\begin{aligned} < \frac{1}{\sqrt{2\pi}} e^{jkt}, \frac{1}{\sqrt{2\pi}} e^{jkt} > \quad &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(k_1-k_2)t} \\ &= \frac{1}{2\pi} \frac{e^{j(k_1-k_2)t}}{j(k_1-k_2)} \Big|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \cdot \frac{-1+1}{j(k_1-k_2)} = 0 \ \ (k_1 \neq k_2) \end{aligned} \tag{3}$$

See Young for proof that the closure of B is $L_2[-\pi, \pi]$, i.e., the fact that $any\, f \in L_2[-\pi, \pi]$ has a Fourier series representation.