The DTFT as an "Eigenbasis"*

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1 The DTFT as an "Eigenbasis"

We saw Parseval/Plancherel in the context of orthonormal basis expansions. This begs the question, do $\mathcal{F}$ and $\mathcal{F}^{-1}$ just take signals and compute their representation in another basis?

Let's look at $\mathcal{F}^{-1} : L_2[-\pi,\pi] \to \ell_2(\mathbb{Z})$ first:

$$\mathcal{F}^{-1}\left(X\left(e^{jw}\right)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega. \tag{1}$$

Recall that $X\left(e^{j\omega}\right)$ is really just a function of ω, so if we replace ω with t, we get

$$\mathcal{F}^{-1}\left(X(t)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} X(t)\,e^{jtn}dt. \tag{2}$$

Does this seem familiar? If $X(t)$ is a periodic function defined on $[-\pi,\pi]$, then $\mathcal{F}^{-1}(X(t))$ is just computing (up to a reversal of the indicies) the continuous-time Fourier series of $X(t)$!

We said before that the Fourier series is a representation in an orthobasis, the sequence of coefficients that we get are just the weights of the different basis elements. Thus we have $\to x[n] = \mathcal{F}\mathcal{F}^{-1}(X(t))$ and

$$X(t) = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{e^{-jtn}}{\sqrt{2\pi}}\right). \tag{3}$$

What about $\mathcal{F}$? In this case we are taking an $x \in \ell_2(\mathbb{Z})$ and mapping it to an $X \in L_2[-\pi,\pi]$. X represents an infinite set of numbers, and when we weight the functions $e^{j\omega n}$ by $X(\omega)$ and sum them all up, we get back the original signal

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\left(\frac{e^{j\omega n}}{\sqrt{2\pi}}\right)d\omega. \tag{4}$$

Unfortunately, $\left|\left|\frac{e^{j\omega n}}{\sqrt{2\pi}}\right|\right| = \infty\ (\neq 1)$ so technically, we can't really think of this as a change of basis.

However, as a unitary transformation, $\mathcal{F}$ has everything we would ever want in a basis and more: We can represent any $x \in \ell_2(\mathbb{Z})$ using $\{e^{j\omega n}\}_{\omega\in[-\pi,\pi]}$, and since it is unitary, we have Parseval and Plancherel Theorems as well. On top of that, we already showed that the set of vectors $\{e^{j\omega n}\}_{\omega\in[-\pi,\pi]}$ are eigenvectors of LSI systems – if this really were a basis, it would be called an *eigenbasis*.

Eigenbases are useful because once we represent a signal using an eigenbasis, to compute the output of a system we just need to know what it does to its eigenvectors (i.e., its eigenvalues). For an LSI system, $H\left(e^{j\omega}\right)$ represents a set of eigenvalues that provide a complete characterization of the system.

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