The z-Transform*

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1 The z-transform

We introduced the z-transform before as

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} \tag{1}$$

where z is a complex number. When $H(z)$ exists (the sum converges), it can be interpreted as the "response" of an LSI system with impulse response $h[n]$ to the input of z^n. The z-transform is useful mostly due to its ability to simplify system analysis via the following result.

Theorem
If $y = h * x$, then $Y(z) = H(z)\, X(z)$.
Proof
First observe that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} y[n]\, z^{-n} &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \right) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x[k] \left(\sum_{n=-\infty}^{\infty} h[n-k]\, z^{-n} \right) \end{aligned} \tag{2}$$

Let $m = n - k$, and note that $z^{-n} = z^{-m} \cdot z^{-k}$. Thus we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} y[n]\, z^{-n} &= \sum_{k=-\infty}^{\infty} x[k] \left(\sum_{n=-\infty}^{\infty} h[m]\, z^{-m} \right) z^{-k} \\ &= \sum_{k=-\infty}^{\infty} x[k]\, H(z)\, z^{-k} \\ &= H(z) \left(\sum_{k=-\infty}^{\infty} x[k]\, z^{-k} \right) \\ &= H(z)\, X(z) \end{aligned} \tag{3}$$

This yields the "transfer function"

$$H(z) = \frac{Y(z)}{X(z)}. \tag{4}$$

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