Z-TRANSFORM EXAMPLES*

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1 *z*-transform examples

Example 1

Consider the z-transform given by H(z) = z, as illustrated below.





The corresponding DTFT has magnitude and phase given below.

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Figure 2



Figure 3

What could the system H be doing? It is a perfect all-pass, linear-phase system. But what does this mean?

Suppose $h[n] = \delta[n - n_0]$. Then

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

=
$$\sum_{n=-\infty}^{\infty} \delta[n-n_0] z^{-n}$$

=
$$z^{-n_0}.$$
 (1)

Thus, $H(z) = z^{-n_0}$ is the z-transform of a system that simply delays the input by n_0 . $H(z) = z^{-1}$ is the z-transform of a unit-delay.

Example 2

Now consider $x[n] = \alpha^n u[n]$



Figure 4

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$
$$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$
$$= \frac{1}{1 - \frac{\alpha}{z}} \quad \text{(if } |\alpha/z| < 1) \text{ (Geometric Series)}$$
$$= \frac{z}{z - \alpha} \tag{2}$$

What if $\left|\frac{a}{z}\right| \geq 1$? Then $\sum_{n=0}^{\infty} \left(\frac{\alpha}{n}\right)^n$ does not converge! Therefore, whenever we compute a *z*-transform, we must also specify the set of *z*'s for which the *z*-transform exists. This is called the region of convergence (ROC). In the above example, the ROC= $\{z : |z| > |\alpha|\}$.



Figure 5

Example 3 What about the "evil twin" $x[n] = -\alpha^n u [-1 - n]$?

$$X(z) = \sum_{n=-\infty}^{\infty} -\alpha^n u \left[-1-n\right] z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n}$$
$$= -\sum_{n=-\infty}^{-1} \left(\frac{z}{\alpha}\right)^{-n}$$
$$= -\sum_{n=1}^{\infty} \left(\frac{z}{\alpha}\right)^n$$
(3)
$$= 1 - \sum_{n=0}^{\infty} \left(\frac{z}{\alpha}\right)^n$$
 (converges if $|z/\alpha| < 1$)
$$= 1 - \frac{1}{1-\frac{z}{\alpha}} = \frac{\alpha-z-\alpha}{\alpha-z} = \frac{z}{z-\alpha}$$

We get the exact same result but with ROC={ $z: |z| < |\alpha|$ }.