The Real and Complex Numbers: The Geometric Progression and the Binomial Theorem*

Lawrence Baggett

This work is produced by The Connexions Project and licensed under the
Creative Commons Attribution License †

Abstract

The binomial theorem is introduced, the existences of nth roots of real numbers is explored, the binomial coefficient is defined, and a theorem providing a formula for the sum of a geometric progression is included.

There are two special algebraic identities that hold in R (in fact in any field F whatsoever) that we emphasize. They are both proved by mathematical induction. The first is the formula for the sum of a geometric progression.

Theorem 1: Geometric Progression
Let x be a real number, and let n be a natural number. Then,

1. If $x \neq 1$, then
$$\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x}. \tag{1}$$
2. If $x = 1$, then
$$\sum_{j=0}^{n} x^j = n+1. \tag{2}$$

Proof:
The second claim is clear, since there are $n+1$ summands and each is equal to 1.
We prove the first claim by induction. Thus, if $n = 1$, then the assertion is true, since

$$\sum_{j=0}^{1} x^j = x^0 + x^1 = 1 + x = (1+x)\frac{1-x}{1-x} = \frac{1-x^2}{1-x}. \tag{3}$$

Now, supposing that the assertion is true for the natural number k, i.e., that

$$\sum_{j=0}^{k} x^j = \frac{1-x^{k+1}}{1-x}, \tag{4}$$

*Version 1.2: Dec 9, 2010 5:09 pm -0600
†http://creativecommons.org/licenses/by/3.0/

let us show that the assertion holds for the natural number $k+1$. Thus

$$\begin{array}{rcl} \sum_{j=0}^{k+1} x^j & = & \sum_{j=0}^{k} x^j + x^{k+1} \\ & = & \frac{1-x^{k+1}}{1-x} + x^{k+1} \\ & = & \frac{1-x^{k+1}+x^{k+1}-x^{k+2}}{1-x} \\ & = & \frac{1-x^{k+1+1}}{1-x}, \end{array} \tag{5}$$

which completes the proof.

The second algebraic formula we wish to emphasize is the Binomial Theorem. Before stating it, we must introduce some useful notation.

Definition 1:
Let n be a natural number. As earlier in this chapter, we define $n!$ as follows:

$$n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1. \tag{6}$$

For later notational convenience, we also define $0!$ to be 1.

If k is any integer for which $0 \leq k \leq n$, we define the *binomial coefficient* $\binom{n}{k}$ by

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n \times (n-1) \times (n-2) \times ... \times (n-k+1)}{k!}. \tag{7}$$

Exercise 1

a. Prove that $\binom{n}{0} = 1$, $\binom{n}{1} = n$ and $\binom{n}{n} = 1$.
b. Prove that

$$\binom{n}{k} \leq \frac{2n^k}{2^k} \tag{8}$$

for all natural numbers n and all integers $0 \leq k \leq n$.
c. Prove that

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{9}$$

for all natural numbers n and all integers $1 \leq k \leq n$.

Theorem 2:
If $x, y \in R$ and n is a natural number, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{10}$$

Proof:
We shall prove this theorem by induction. If $n = 1$, then the assertion is true, for $(x+y)^1 = x+y$ and

$$\sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k} = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = x+y. \tag{11}$$

Now, assume that the assertion holds for the natural number j; i.e.,

$$(x+y)^j = \sum_{k=0}^{j} \binom{j}{k} x^k y^{j-k}, \tag{12}$$

and let us prove that the assertion holds for the natural number $j+1$. We will make use of part (c) of Exercise . We have that

$$\begin{aligned}
(x+y)^{j+1} &= (x+y)(x+y)^j \\
&= (x+y)\sum_{k=0}^{j}\binom{j}{k}x^k y^{j-k} \\
&= x\sum_{k=0}^{j}\binom{j}{k}x^k y^{j-k} + y\sum_{k=0}^{j}\binom{j}{k}x^k y^{j-k} \\
&= \sum_{k=0}^{j}\binom{j}{k}x^{k+1} y^{j-k} + \sum_{k=0}^{j}\binom{j}{k}x^k y^{j+1-k} \\
&= \sum_{k=0}^{j-1}\binom{j}{k}x^{k+1} y^{j-k} + \binom{j}{j}x^{j+1}y^0 \\
&\quad + \sum_{k=1}^{j}\binom{j}{k}x^k y^{j+1-k} + \binom{j}{0}x^0 y^{j+1} \\
&= x^{j+1} + \sum_{k=1}^{j}\binom{j}{k-1}x^k y^{j+1-k} \\
&\quad + \sum_{k=1}^{j}\binom{j}{k}x^k y^{j+1-k} + y^{j+1} \\
&= x^{j+1} + \sum_{k=1}^{j}\left(\binom{j}{k-1}+\binom{j}{k}\right)x^k y^{j+1-k} + y^{j+1} \\
&= x^{j+1} + \sum_{k=1}^{j}\binom{j+1}{k}x^k y^{j+1-k} + y^{j+1} \\
&= \binom{j+1}{j+1}x^{j+1}y^0 + \sum_{k=1}^{j}\binom{j+1}{k}x^k y^{j+1-k} + \binom{j+1}{0}x^0 y^{j+1} \\
&= \sum_{k=0}^{j+1}\binom{j+1}{k}x^k y^{j+1-k},
\end{aligned} \tag{13}$$

which shows that the assertion of the theorem holds for the natural number $j+1$. This completes the proof.

The next exercise is valid in any ordered field, but, since we are mainly interested in the order field R, we state everything in terms of that field.

Exercise 2

a. If x and y are positive real numbers, and if n and k are natural numbers with $k \leq n$, show that $(x+y)^n \geq \binom{n}{k}x^k y^{n-k}$.
b. For any positive real number x and natural number n, show that $(1+x)^n \geq 1+nx$.
c. For any real number $x > -1$ and natural number n, prove that $(1+x)^n \geq 1+nx$. HINT: Do not try to use the binomial theorem as in part (b); it won't work because the terms are not all positive; prove this directly by induction.

There is one more important algebraic identity, which again can be proved by induction. It is actually just a corollary of the geometric progression formula.

Theorem 3:
If $x, y \in R$ and n is a natural number, then

$$x^n - y^n = (x-y)\left(\sum_{j=0}^{n-1} x^j y^{n-1-j}\right. . \tag{14}$$

Proof:
If $n=1$ the theorem is clear. Suppose it holds for a natural number k, and let us prove the identity

for the natural number $k+1$. We have

$$\begin{aligned} x^{k+1}-y^{k+1} &= x^{k+1}-x^k y+x^k y-y^{k+1} \\ &= (x-y)\,x^k+y\left(x^k-y^k\right) \\ &= (x-y)\,x^k+y\,(x-y)\left(\sum_{j=0}^{k-1} x^j y^{k-1-j}\right) \\ &= (x-y)\,x^k+(x-y)\left(\sum_{j=0}^{k-1} x^j y^{k-j}\right. \\ &= (x-y)\left(x^k y^{k-k}+\sum_{j=0}^{k-1} x^j y^{k-j}\right) \\ &= (x-y)\left(\sum_{j=0}^{k} x^j y^{k-j}\right) \end{aligned} \tag{15}$$

, which shows that the assertion holds for the natural number $k+1$. So, by induction, the theorem is proved.

Exercise 3

Let x and y be real numbers.

a. Let n be an odd natural number; i.e., $n=2k+1$ for some natural number k. Show that

$$x^n+y^n=(x+y)\left(\sum_{j=0}^{n-1}(-1)^j x^j y^{n-1-j}\right. . \tag{16}$$

HINT: Write $x^n+y^n=x^n-(-y)^n$.

b. Show that x^2+y^2 can not be factored into a product of the form $(ax+by)\,(cx+dy)$ for any choices of real numbers a, b, c, and d.

Using the Binomial Theorem together with the preceding theorem, we may now investigate the existence of nth roots of real numbers. This next theorem is definitely not valid in any ordered field, for it again depends on the completeness property.

Theorem 4:

Let n be a natural number and let x be a positive real number. Then there exists a unique positive real number y such that $y^n=x$; i.e., x has a unique positive nth root.

Proof:

Note first that if $0 \le t < s$, then $t^n < s^n$. (To see this, argue by induction, and use part (e) of here[1].) Using this, we mimic the proof of here[2]. Thus, let S be the set of all positive real numbers t for which $t^n \le x$. Then S is nonempty and bounded above. Indeed, if $x \ge 1$, then $1 \in S$, while if $x<1$, then x itself is in S. Therefore, S is nonempty. Also, using part (b) of Exercise , we see that $1+(x/n)$ is an upper bound for S. For, if $t>1+x/n$, then

$$t^n > (1+(x/n))^n \ge 1+n\,(x/n) > x. \tag{17}$$

Now let $y=supS$, and let us show that $y^n=x$. We rule out the other two possibilities. First, if $y^n>x$, let ε be the positive number y^n-x, and define $\varepsilon^{'}$ to be the positive number $\varepsilon/\left(ny^{n-1}\right)$.

[1] "The Real and Complex Numbers: The Real Numbers", Exercise 1 <http://cnx.org/content/m36069/latest/#fs-id1166093017647>

[2] "The Real and Complex Numbers: Properties of the Real Numbers", Theorem 4 <http://cnx.org/content/m36085/latest/#fs-id1170178787854>

Then, using here[3], choose $t \in S$ so that $y - \varepsilon^{'} < t \leq y$. (here[4] is where the completeness of the ordered field R is crucial.) We have

$$\begin{aligned} \varepsilon &= y^n - x \\ &= y^n - t^n + t^n - x \\ &\leq y^n - t^n \\ &= (y-t)\left(\sum_{j=0}^{n-1} y^j t^{n-1-j}\right) \\ &\leq (y-t)\left(\sum_{j=0}^{n-1} y^j y^{n-1-j}\right) \\ &= (y-t)\left(\sum_{j=0}^{n-1} y^{n-1}\right. \\ &< \varepsilon^{'} n y^{n-1} \\ &= \varepsilon, \end{aligned} \tag{18}$$

and this is a contradiction. Therefore, y^n is not greater than x.

Now, if $y^n < x$, let ε be the positive number $x - y^n$, and choose a $\delta > 0$ such that $\delta < 1$ and $\delta < \varepsilon/(y+1)^n$. Then, using the Binomial Theorem, we have that

$$\begin{aligned} (y+\delta)^n &= \sum_{k=0}^{n} \binom{n}{k} y^k \delta^{n-k} \\ &= y^n + \sum_{k=0}^{n-1} \binom{n}{k} y^k \delta^{n-k} \\ &= y^n + \delta \sum_{k=0}^{n-1} \binom{n}{k} y^k \delta^{n-1-k} \\ &< y^n + \delta \sum_{k=0}^{n} \binom{n}{k} y^k 1^{n-k} \\ &= y^n + \delta (y+1)^n \\ &= x - \varepsilon + \delta (y+1)^n \\ &< x - \varepsilon + \varepsilon \\ &= x, \end{aligned} \tag{19}$$

implying that $y + \delta \in S$. But this is a contradiction, since $y = supS$. Therefore, y^n is not less than x, and so $y^n = x$.

We have shown the existence of a positive nth root of x. To see the uniqueness, suppose y and $y^{'}$ are two positive nth roots of x. Then

$$\begin{aligned} 0 &= y^n - y^{'n} \\ &= \left(y - y^{'}\right)\left(\sum_{j=0}^{n-1} y^j y^{'n-j-1}\right., \end{aligned} \tag{20}$$

which implies that either $y - y^{'} = 0$ or $\sum_{j=0}^{n-1} y^j y^{'n-j-1} = 0$. Since this latter sum consists of positive terms, it cannot be 0, whence $y = y^{'}$. This shows that there is but one positive nth root of x, and the theorem is proved.

Exercise 4

a. Show that if $n = 2k$ is an even natural number, then every positive real number has exactly two distinct nth roots.

[3] "The Real and Complex Numbers: Properties of the Real Numbers", Theorem 3 <http://cnx.org/content/m36085/latest/#fs-id1170177642504>

[4] "The Real and Complex Numbers: Properties of the Real Numbers", Theorem 3 <http://cnx.org/content/m36085/latest/#fs-id1170177642504>

b. If $n = 2k + 1$ is an odd natural number, show that every real number has exactly one nth root.
c. If n is a natural number greater than 1, prove that there is no rational number whose nth power equals 2, i.e., the nth root of 2 is not a rational number.