The Real and Complex Numbers: The Complex Numbers*

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Abstract

Complex numbers are covered, involving i. The fundamental theorem of algebra is referenced. The absolute value of a complex number is defined. The triangle inequality is stated.

It is useful to build from the real numbers another number system called the *complex numbers.* Although the real numbers R have many of the properties we expect, i.e., every positive number has a positive square root, every number has a cube root, and so on, there are somewhat less prominent properties that R fails to possess. For instance, negative numbers do not have square roots. This is actually a property that is missing in any ordered field, since every square is positive in an ordered field. See part (e) of here[1]. One way of describing this shortcoming on the part of the real numbers is to note that the equation $1+x^2=0$ has no solution in the real numbers. Any solution would have to be a number whose square is -1, and no real number has that property. As an initial extension of the set of real numbers, why not build a number system in which this equation has a solution?

We faced a similar kind of problem earlier on. In the set N there is no element j such that $j+n=n$ for all $n\in N$. That is, there was no element like 0 in the natural numbers. The solution to the problem in that case was simply to "create" something called zero, and just adjoin it to our set N. The same kind of solution exists for us now. Let us invent an additional number, this time denoted by i, which has the property that its square i^2 is -1. Because the square of any nonzero real number is positive, this new number i was traditionally referred to as an "imaginary" number. We simply adjoin this number to the set R, and we will then have a number whose square is negative, i.e., -1. Of course, we will require that our new number system should still be a field; we don't want to give up our basic algebraic operations. There are several implications of this requirement: First of all, if y is any real number, then we must also adjoin to R the number $y\times i\equiv yi$, for our new number system should be closed under multiplication. Of course the square of iy will equal $i^2y^2=-y^2$, and therefore this new number iy must also be imaginary, i.e., not a real number. Secondly, if x and y are any two real numbers, we must have in our new system a number called $x+yi$, because our new system should be closed under addition.

Definition 1:
Let i denote an object whose square $i^2=-1$. Let C be the set of all objects that can be represented in the form $z=x+yi$, where both x and y are real numbers.

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[1] "The Real and Complex Numbers: The Real Numbers", Exercise 2 <http://cnx.org/content/m36069/latest/#fs-id1166089624930>

Define two operations $+$ and $\times$ on C as follows:

$$(x+yi)+\left(x^{'}+y^{'}i\right)=x+x^{'}+\left(y+y^{'}\right)i, \tag{1}$$

and

$$(x+iy)\left(x^{'}+iy^{'}\right)=xx^{'}+xiy^{'}+iyx^{'}+iyiy^{'}=xx^{'}-yy^{'}+\left(xy^{'}+yx^{'}\right)i. \tag{2}$$

Theorem 1:

1. The two operations $+$ and $\times$ defined above are commutative and associative, and multiplication is distributive over addition.
2. Each operation has an identity: $(0+0i)$ is the identity for addition, and $(1+0i)$ is the identity for multiplication.
3. The set C with these operations is a field.

Proof:
We leave the proofs of Parts (1) and (2) to the following exercise. To see that C is a field, we need to verify one final condition, and that is to show that if $z=x+yi\neq 0=0+0i$, then there exists a $w=u+vi$ such that $z\times w=1=1+0i$. Thus, suppose $z=x+yi\neq 0$. Then at least one of the two real numbers x and y must be nonzero, so that $x^2+y^2>0$. Define a complex number w by

$$w=\frac{x}{x^2+y^2}+\frac{-y}{x^2+y^2}i. \tag{3}$$

We then have

$$\begin{aligned} z\times w &= (x+yi)\times\left(\frac{x}{x^2+y^2}+\frac{-y}{x^2+y^2}i\right) \\ &= \frac{x^2}{x^2+y^2}-\frac{-y^2}{x^2+y^2}+\left(x\frac{-y}{x^2+y^2}+y\frac{x}{x^2+y^2}\right)i \\ &= \frac{x^2+y^2}{x^2+y^2}+\frac{0}{x^2+y^2}i \\ &= 1+0i \\ &= 1, \end{aligned} \tag{4}$$

as desired.

Exercise 1
Prove parts (1) and (2) of Theorem 1, p. 2.

One might think that these kinds of improvements of the real numbers will go on and on. For instance, we might next have to create and adjoin another object j so that the number i has a square root; i.e., so that the equation $i-z^2=0$ has a solution. Fortunately and surprisingly, this is not necessary, as we will see when we finally come to the Fundamental Theorem of Algebra in here[2].

The subset of C consisting of the pairs $x+0i$ is a perfect (isomorphic) copy of the real number system R. We are justified then in saying that the complex number system extends the real number system, and we will say that a real number x is the same as the complex number $x+0i$. That is, real numbers are special kinds of complex numbers. The complex numbers of the form $0+yi$ are called *purely imaginary numbers*. Obviously, the only complex number that is both real and purely imaginary is the number $0=0+0i$. The set C can also be regarded as a 2-dimensional space, a plane, and it is also helpful to realize that the complex numbers form a 2-dimensional vector space over the field of real numbers.

[2]"Fundamental Theorem of Algebra, Analysis: The Fundamental Theorem of Algebra", Theorem 1: Fundamental Theorem of Algebra <http://cnx.org/content/m36238/latest/#fs-id1172549276378>

Definition 2:
If $z = x + yi$, we say that the real number x is the *real part* of z and write $x = \Re(z)$. We say that the real number y is the *imaginary part* of z and write $y = \Im(z)$.

If $z = x + yi$ is a complex number, define the *complex conjugate*$\overline{z}$ of z by $\overline{z} = x - yi$.

The complex number i satisfies $i^2 = -1$, showing that the negative number -1 has a square root in C, or equivalently that the equation $1 + z^2 = 0$ has a solution in C. We have thus satisfied our initial goal of extending the real numbers. But what about other complex numbers? Do they have square roots, cube roots, nth roots? What about solutions to other kinds of equations than $1 + z^2$?

Exercise 2

a. Prove that every complex number has a square root. HINT: Let $z = a+bi$. Assume $w = x+yi$ satisfies $w^2 = z$, and just solve the two equations in two unknowns that arise.
b. Prove that every quadratic equation $az^2 + bz + c = 0$, for a, b, and c complex numbers, has a solution in C. HINT: If $a = 0$, it is easy to find a solution. If $a \neq 0$, we need only find a solution to the equivalent equation

$$z^2 + \frac{b}{a}z + \frac{c}{a} = 0. \tag{5}$$

Justify the following algebraic manipulations, and then solve the equation.

$$\begin{array}{rcl} z^2 + \frac{b}{a}z + \frac{c}{a} & = & z^2 + \frac{b}{a}z + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} \\ & = & \left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}. \end{array} \tag{6}$$

What about this new field C? Does every complex number have a cube root, a fourth root, does every equation have a solution in C? A natural instinct would be to suspect that C takes care of square roots, but that it probably does not necessarily have higher order roots. However, the content of the Fundamental Theorem of Algebra, to be proved in here[3], is that every equation of the form $P(z) = 0$, where P is a nonconstant polynomial, has a solution in C. This immediately implies that every complex number c has an nth root, for any solution of the equation $z^n - c = 0$ would be an nth root of c.

The fact that the Fundamental Theorem of Algebra is true is a good indication that the field C is a "good" field. But it's not perfect.

Theorem 2:
In no way can the field C be made into an ordered field. That is, there exists no subset P of C that satisfies the two positivity axioms.

Proof:
Suppose C were an ordered field, and write P for its set of positive elements. Then, since every square in an ordered field must be in P (part (e) of here[4]), we must have that $-1 = i^2$ must be in P. But, by part (a) of here[5], we also must have that 1 is in P, and this leads to a contradiction of the law of tricotomy. We can't have both 1 and -1 in P. Therefore, C is not an ordered field.

Although we may not define when one complex number is smaller than another, we can define the absolute value of a complex number and the distance between two of them.

[3] "Fundamental Theorem of Algebra, Analysis: The Fundamental Theorem of Algebra" <http://cnx.org/content/m36238/latest/>
[4] "The Real and Complex Numbers: The Real Numbers", Exercise 2 <http://cnx.org/content/m36069/latest/#fs-id1166089624930>
[5] "The Real and Complex Numbers: The Real Numbers", Exercise 2 <http://cnx.org/content/m36069/latest/#fs-id1166089624930>

Definition 3:
If $z = x + yi$ is in C, we define the *absolute value* of z by

$$|z| = \sqrt{x^2 + y^2}. \tag{7}$$

We define the *distanced* (z, w) between two complex numbers z and w by
$d(z, w) = |z - w|$.
If $c \in C$ and $r > 0$, we define the *open disk of radius r around c*, and denote it by $B_r(c)$, by

$$B_r(c) = \{z \in C : |z - c| < r\}. \tag{8}$$

The *closed disk* of radius r around c is denoted by $\overline{B}_r(c)$ and is defined by

$$\overline{B}_r(c) = \{z \in C : |z - c| \le r\}. \tag{9}$$

We also define open and closed *punctured* disks $B_r^{'}(c)$ and $\overline{B}_r^{'}(c)$ around c by

$$B_r^{'}(c) = \{z : 0 < |z - c| < r\} \tag{10}$$

and

$$\overline{B}_r^{'}(c) = \{z : 0 < |z - c \le r\}. \tag{11}$$

These punctured disks are just like the regular disks, except that they do not contain the central point c.

More generally, if S is any subset of C, we define the *open neighborhood of radius r* around S, denoted by $N_r(S)$, to be the set of all z such that there exists a $w \in S$ for which $|z - w| < r$. That is, $N_r(S)$ is the set of all complex numbers that are within a distance of r of the set S. We define the *closed neighborhood* of radius r around S, and denote it by $\overline{N}_r(S)$, to be the set of all $z \in C$ for which there exists a $w \in S$ such that $|z - w| \le r$.

Exercise 3

a. Prove that the absolute value of a complex number z is a nonnegative real number. Show in addition that $|z|^2 = z\overline{z}$.
b. Let x be a real number. Show that the absolute value of x is the same whether we think of x as a real number or as a complex number.
c. Prove that $max(|\Re(z)|, |\Im(z)|) \le |z| \le |\Re(z)| + |\Im(z)|$. Note that this just amounts to verifying that

$$max(|x|, |y|) \le \sqrt{x^2 + y^2} \le |x| + |y| \tag{12}$$

for any two real numbers x and y.
d. For any complex numbers z and w, show that $\overline{z + w} = \overline{z} + \overline{w}$, and that $\overline{\overline{z}} = z$.
e. Show that $z + \overline{z} = 2\Re(z)$ and $z - \overline{z} = 2i\Im(z)$.
f. If $z = a + bi$ and $w = a^{'} + b^{'}i$, prove that $|zw| = |z||w|$. HINT: Just compute $|(a + bi)(a^{'} + b^{'}i)|^2$.

The next theorem is in a true sense the most often used inequality of mathematical analysis. We have already proved the triangle inequality for the absolute value of real numbers, and the proof was not very difficult in that case. For complex numbers, it is not at all simple, and this should be taken as a good indication that it is a deep result.

Theorem 3: Triangle Inequality
If z and $z^{'}$ are two complex numbers, then

$$|z + z^{'}| \le |z| + |z^{'}| \tag{13}$$

and

$$|z - z^{'}| \geq ||z| - |z^{'}||. \tag{14}$$

Proof:
We use the results contained in Exercise .

$$\begin{array}{ccc} |z + z^{'}|^2 & = & \left(z + z^{'}\right)\overline{\left(z + z^{'}\right)} \\ & = & \left(z + z^{'}\right)\left(\overline{z} + \overline{z^{'}}\right) \\ & = & z\overline{z} + z^{'}\overline{z} + z\overline{z^{'}} + z^{'}\overline{z^{'}} \\ & = & |z|^2 + z^{'}\overline{z} + \overline{z^{'}\overline{z}} + |z^{'}|^2 \\ & = & |z|^2 + 2\Re\left(z^{'}\overline{z}\right) + |z^{'}|^2 \\ & \leq & |z|^2 + 2|\Re\left(z^{'}\overline{z}\right)| + |z^{'}|^2 \\ & \leq & |z|^2 + 2|z^{'}\overline{z}| + |z^{'}|^2 \\ & = & |z|^2 + 2|z^{'}||z| + |z^{'}|^2 \\ & = & \left(|z| + |z^{'}|\right)^2. \end{array} \tag{15}$$

The Triangle Inequality follows now by taking square roots.

1:
REMARK The Triangle Inequality is often used in conjunction with what's called the "add and subtract trick." Frequently we want to estimate the size of a quantity like $|z - w|$, and we can often accomplish this estimation by adding and subtracting the same thing within the absolute value bars:

$$|z - w| = |z - v + v - w| \leq |z - v| + |v - w|. \tag{16}$$

The point is that we have replaced the estimation problem of the possibly unknown quantity $|z - w|$ by the estimation problems of two other quantities $|z - v|$ and $|v - w|$. It is often easier to estimate these latter two quantities, usually by an ingenious choice of v of course.

Exercise 4

a. Prove the second assertion of the preceding theorem.
b. Prove the Triangle Inequality for the distance function. That is, prove that

$$d(z, w) \leq d(z, v) + d(v, w) \tag{17}$$

for all $z, w, v \in C$.
c. Use mathematical induction to prove that

$$|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|. \tag{18}$$

It may not be necessary to point out that part (b) of the preceding exercise provides a justification for the name "triangle inequality." Indeed, part (b) of that exercise is just the assertion that the length of one side of a triangle in the plane is less than or equal to the sum of the lengths of the other two sides. Plot the three points $z, w,$ and v, and see that this interpretation is correct.

Definition 4:
A subset S of C is called *Bounded* if there exists a real number M such that $|z| \leq M$ for every z in S.

Exercise 5

a. Let S be a subset of C. Let S_1 be the subset of R consisting of the real parts of the complex numbers in S, and let S_2 be the subset of R consisting of the imaginary parts of the elements of S. Prove that S is bounded if and only if S_1 and S_2 are both bounded.
HINT: Use Part (c) of Exercise ..
b. Let S be the unit circle in the plane, i.e., the set of all complex numbers $z = x + iy$ for which $|z| = 1$. Compute the sets S_1 and S_2 of part (a).

Exercise 6

a. Verify that the formulas for the sum of a geometric progression and the binomial theorem (here[6] and here[7]) are valid for complex numbers z and z'. HINT: Check that, as claimed, the proofs of those theorems work in any field.
b. Prove here[8] for complex numbers z and z'.

[6]"The Real and Complex Numbers: The Geometric Progression and the Binomial Theorem", Theorem 1: Geometric Progression <http://cnx.org/content/m36104/latest/#fs-id1168615573590>
[7]"The Real and Complex Numbers: The Geometric Progression and the Binomial Theorem", Theorem 2 <http://cnx.org/content/m36104/latest/#fs-id1170975717247>
[8]"The Real and Complex Numbers: The Geometric Progression and the Binomial Theorem", Theorem 3 <http://cnx.org/content/m36104/latest/#fs-id1170978519676>