The Limit of a Sequence of Numbers: Properties of Convergent Sequences*

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Abstract

A discussion of the properties of convergent sequences, such as when they form a bounded set, the squeeze theorem, and the properties of convergence when sequences are combined algebraically.

Often, our goal is to show that a given sequence is convergent. However, as we study convergent sequences, we would like to establish various properties that they have in common. The first theorem of this section is just such a result.

Theorem 1:
Suppose $\{a_n\}$ is a convergent sequence of real or complex numbers. Then the sequence $\{a_n\}$ forms a bounded set.

Proof:
Write $L = lim a_n$. Let ε be the positive number 1. Then, there exists a natural number N such that $|a_n - L| < 1$ for all $n \geq N$. By the backward triangle inequality, this implies that $||a_n| - |L|| < 1$ for all $n \geq N$, which implies that $|a_n| \leq |L| + 1$ for all $n \geq N$. This shows that at least the tail of the sequence is bounded by the constant $|L| + 1$.

Next, let K be a number larger than the finitely many numbers $|a_1|, ..., |a_{N-1}|$. Then, for any n,$|a_n|$ is either less than K or $|L| + 1$. Let M be the larger of the two numbers K and $|L| + 1$. Then $|a_n| < M$ for all n. Hence, the sequence $\{a_n\}$ is bounded.

Note that the preceding theorem is a partial converse to ; i.e., a convergent sequence is necessarily bounded. Of course, not every convergent sequence must be either nondecreasing or nonincreasing, so that a full converse to is not true. For instance, take $z = -1/2$ in part (1) of . It converges to 0 all right, but it is neither nondecreasing nor nonincreasing.

Exercise 1

a. Suppose $\{a_n\}$ is a sequence of real numbers that converges to a number a, and assume that $a_n \geq c$ for all n. Prove that $a \geq c$. HINT: Suppose not, and let ε be the positive number $c - a$. Let N be a natural number corresponding to this choice of ε, and derive a contradiction.

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b. If $\{a_n\}$ is a sequence of real numbers for which $lim a_n = a$, and if $a \neq 0$, then prove that $a_n \neq 0$ for all large enough n. Show in fact that there exists an N such that $|a_n| > |a|/2$ for all $n \geq N$. HINT: Make use of the positive number $\varepsilon = |a|/2$.

Exercise 2

a. If $\{a_n\}$ is a sequence of positive real numbers for which $lim a_n = a > 0$, prove that $lim \sqrt{a_n} = \sqrt{a}$. HINT: Multiply the expression $\sqrt{a_n} - \sqrt{a}$ above and below by $\sqrt{a_n} + \sqrt{a}$.
b. If $\{a_n\}$ is a sequence of complex numbers, and $lim a_n = a$, prove that $lim|a_n| = |a|$. HINT: Use the backward triangle inequality.

Exercise 3

Suppose $\{a_n\}$ is a sequence of real numbers and that $L = lim a_n$. Let M_1 and M_2 be real numbers such that $M_1 \leq a_n \leq M_2$ for all n. Prove that $M_1 \leq L \leq M_2$.

HINT: Suppose, for instance, that $L > M_2$. Make use of the positive number $L - M_2$ to derive a contradiction.

We are often able to show that a sequence converges by comparing it to another sequence that we already know converges. The following exercise demonstrates some of these techniques.

Exercise 4

Let $\{a_n\}$ be a sequence of complex numbers.

a. Suppose that, for each n,$|a_n| < 1/n$. Prove that $0 = lim a_n$.
b. Suppose $\{b_n\}$ is a sequence that converges to 0, and suppose that, for each n,$|a_n| < |b_n|$. Prove that $0 = lim a_n$.

The next result is perhaps the most powerful technique we have for showing that a given sequence converges to a given number.

Theorem 2: Squeeze Theorem

Suppose that $\{a_n\}$ is a sequence of real numbers and that $\{b_n\}$ and $\{c_n\}$ are two sequences of real numbers for which $b_n \leq a_n \leq c_n$ for all n. Suppose further that $lim b_n = lim c_n = L$. Then the sequence $\{a_n\}$ also converges to L.

Proof:

We examine the quantity $|a_n - L, |$ employ some add and subtract tricks, and make the following computations:

$$\begin{aligned} rcl|a_n - L| &\leq & |a_n - b_n + b_n - L| \\ &\leq & |a_n - b_n| + |b_n - L| \\ &= & a_n - b_n + |b_n - L| \\ &\leq & c_n - b_n + |b_n - L| \\ &= & |c_n - b_n| + |b_n - L| \\ &\leq & |c_n - L| + |L - b_n| + |b_n - L|. \end{aligned} \tag{1}$$

So, we can make $|a_n - L| < \varepsilon$ by making $|c_n - L| < \varepsilon/3$ and $|b_n - L| < \varepsilon/3$. So, let N_1 be a positive integer such that $|c_n - L| < \varepsilon/3$ if $n \geq N_1$, and let N_2 be a positive integer so that $|b_n - L| < \varepsilon/3$ if $n \geq N_2$. Then set $N = max\,(N_1, N_2)$. Clearly, if $n \geq N$, then both inequalities $|c_n - L| < \varepsilon/3$ and $|b_n - L| < \varepsilon/3$, and hence $|a_n - L| < \varepsilon$. This finishes the proof.

The next result establishes what are frequently called the "limit theorems." Basically, these results show how convergence interacts with algebraic operations.

Theorem 3:
Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers with $a = lim a_n$ and $b = lim b_n$. Then

1. The sequence $\{a_n + b_n\}$ converges, and

$$lim\,(a_n + b_n) = lim a_n + lim b_n = a + b. \tag{2}$$

2. The sequence $\{a_n b_n\}$ is convergent, and

$$lim\,(a_n b_n) = lim a_n lim b_n = ab. \tag{3}$$

3. If all the b_n's as well as b are nonzero, then the sequence $\{a_n/b_n\}$ is convergent, and

$$lim\left(\frac{a_n}{b_n} = \frac{lim a_n}{lim b_n} = \frac{a}{b}.\right. \tag{4}$$

Proof:
Part (1) is exactly the same as . Let us prove part (2).

By Theorem 1, p. 1, both sequences $\{a_n\}$ and $\{b_n\}$ are bounded. Therefore, let M be a number such that $|a_n| \leq M$ and $|b_n| \leq M$ for all n. Now, let $\varepsilon > 0$ be given. There exists an N_1 such that $|a_n - a| < \varepsilon/\,(2M)$ whenever $n \geq N_1$, and there exists an N_2 such that $|b_n - b| < \varepsilon/\,(2M)$ whenever $n \geq N_2$. Let N be the maximum of N_1 and N_2. Here comes the add and subtract trick again.

$$\begin{array}{rcl} rcl|a_n b_n - ab| & = & |a_n b_n - ab_n + ab_n - ab| \\ & \leq & |a_n b_n - ab_n| + |ab_n - ab| \\ & = & |a_n - a||b_n| + |a||b - b_n| \\ & \leq & |a_n - a|M + M|b_n - b| \\ & < & \varepsilon \end{array} \tag{5}$$

if $n \geq N$, which shows that $lim\,(a_n b_n) = ab$.

To prove part (3), let M be as in the previous paragraph, and let $\varepsilon > 0$ be given. There exists an N_1 such that $|a_n - a| < \left(\varepsilon|b|^2\right)/\,(4M)$ whenever $n \geq N_1$; there also exists an N_2 such that $|b_n - b| < \left(\varepsilon|b|^2\right)/\,(4M)$ whenever $n \geq N_2$; and there exists an N_3 such that $|b_n| > |b|/2$ whenever $n \geq N_3$. (See Exercise .) Let N be the maximum of the three numbers N_1, N_2 and N_3. Then:

$$\begin{array}{rcl} rcl|\frac{a_n}{b_n} - \frac{a}{b}| & = & |\frac{a_n b - b_n a}{b_n b}| \\ & = & |a_n b - b_n a|\frac{1}{|b_n b|} \\ & < & |a_n b - b_n a|\frac{1}{|b|^2/2} \\ & \leq & \left(|a_n - a||b| + |a||b_n - b|\right)\frac{2}{|b|^2} \\ & < & \left(M|a_n - a| + M|b_n - b|\right)\frac{2}{|b|^2} \\ & < & \varepsilon \end{array} \tag{6}$$

if $n \geq N$. This completes the proof.

1:
REMARK The proof of part (3) of the preceding theorem may look mysterious. Where, for instance, does this number $\varepsilon|b|^2/4M$ come from? The answer is that one begins such a proof by

examining the quantity $|a_n/b_n - a/b|$ to see if by some algebraic manipulation one can discover how to control its size by using the quantities $|a_n - a|$ and $|b_n - b|$. The assumption that $a = lim a_n$ and $b = lim b_n$ mean exactly that the quantities $|a_n - a|$ and $|b_n - b|$ can be controlled by requiring n to be large enough. The algebraic computation in the proof above shows that

$$|\frac{a_n}{b_n} - \frac{a}{b}| \leq (M|a_n - a| + M|b_n - b|)\,\frac{2}{|b|^2}, \tag{7}$$

and one can then see exactly how small to make $|a_n - a|$ and $|b_n - b|$ so that $|a_n/b_n - a/b| < \varepsilon$. Indeed, this is the way most limit proofs work.

Exercise 5

If possible, determine the limits of the following sequences by using , , Theorem 3, p. 2, and the squeeze theorem Theorem 2, Squeeze Theorem, p. 2.

a. $\{n^{1/n^2}\}$.
b. $\{\left(n^2\right)^{1/n}\}$.
c. $\{(1+n)^{1/n}\}$.
d. $\{\left(1+n^2\right)^{1/n^3}\}$.
e. $\{(1+1/n)^{2/n}\}$.
f. $\{(1+1/n)^{2n}\}$.
g. $\{(1+1/n)^{n^2}\}$.
h. $\{(1-1/n)^n\}$. HINT: Note that

$$1 - 1/n = \frac{n-1}{n} = \frac{1}{\frac{n}{n-1}} = \frac{1}{\frac{n-1+1}{n-1}} = \frac{1}{1+\frac{1}{n-1}}. \tag{8}$$

i. $\{(1-1/(2n))^{3n}\}$.
j. $\{(n!)^{1/n}\}$.