THE LIMIT OF A SEQUENCE OF NUMBERS: SUBSEQUENCES AND CLUSTER POINTS*

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Abstract

An introduction of the convergence of subsequences, the bolzano-weierstrass theorem, cluster sets, suprema, infima, and the catchy criterion. Multiple exercises and proofs are included.

Definition 1:

Let $\{a_n\}$ be a sequence of real or complex numbers. A subsequence of $\{a_n\}$ is a sequence $\{b_k\}$ that is determined by the sequence $\{a_n\}$ together with a strictly increasing sequence $\{n_k\}$ of natural numbers. The sequence $\{b_k\}$ is defined by $b_k = a_{n_k}$. That is, the kth term of the sequence $\{b_k\}$ is the n_k th term of the original sequence $\{a_n\}$.

Exercise 1

Prove that a subsequence of a subsequence of $\{a_n\}$ is itself a subsequence of $\{a_n\}$. Thus, let $\{a_n\}$ be a sequence of numbers, and let $\{b_k\} = \{a_{n_k}\}$ be a subsequence of $\{a_n\}$. Suppose $\{c_j\} = \{b_{k_j}\}$ is a subsequence of the sequence $\{b_k\}$. Prove that $\{c_j\}$ is a subsequence of $\{a_n\}$. What is the strictly increasing sequence $\{m_j\}$ of natural numbers for which $c_j = a_{m_j}$?

Here is an interesting generalization of the notion of the limit of a sequence.

Definition 2:

Let $\{a_n\}$ be a sequence of real or complex numbers. A number x is called a cluster point of the sequence $\{a_n\}$ if there exists a subsequence $\{b_k\}$ of $\{a_n\}$ such that $x = limb_k$. The set of all cluster points of a sequence $\{a_n\}$ is called the cluster set of the sequence.

Exercise 2

- a. Give an example of a sequence whose cluster set contains two points. Give an example of a sequence whose cluster set contains exactly n points. Can you think of a sequence whose cluster set is infinite?
- b. Let $\{a_n\}$ be a sequence with cluster set S. What is the cluster set for the sequence $\{-a_n\}$? What is the cluster set for the sequence $\{a_n^2\}$?
- c. If $\{b_n\}$ is a sequence for which $b = limb_n$, and $\{a_n\}$ is another sequence, what is the cluster set of the sequence $\{a_nb_n\}$?
- d. Give an example of a sequence whose cluster set is empty.
- e. Show that if the sequence $\{a_n\}$ is bounded above, then the cluster set S is bounded above. Show also that if $\{a_n\}$ is bounded below, then S is bounded below.

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- f. Give an example of a sequence whose cluster set S is bounded above but not bounded below.
- g. Give an example of a sequence that is not bounded, and which has exactly one cluster point.

Theorem 1:

Suppose $\{a_n\}$ is a sequence of real or complex numbers.

- 1. (Uniqueness of limits) Suppose $\lim a_n = L$, and $\lim a_n = M$. Then L = M. That is, if the limit of a sequence exists, it is unique.
- 2. If $L = lima_n$, and if $\{b_k\}$ is a subsequence of $\{a_n\}$, then the sequence $\{b_k\}$ is convergent, and $limb_k = L$. That is, if a sequence has a limit, then every subsequence is convergent and converges to that same limit.

Proof:

Suppose $\lim a_n = L$ and $\lim a_n = M$. Let ε be a positive number, and choose N_1 so that $|a_n - L| < \varepsilon/2$ if $n \ge N_1$, and choose N_2 so that $|a_n - M| < \varepsilon/2$ if $n \ge N_2$. Choose an n larger than both N_1 and N_2 . Then

$$|L - M| = |L - a_n + a_n - M| \le |L - a_n| + |a_n - M| < \varepsilon. \tag{1}$$

Therefore, since $|L-M| < \varepsilon$ for every positive ε , it follows that L-M=0 or L=M. This proves part (1).

Next, suppose $\lim a_n = L$ and let $\{b_k\}$ be a subsequence of $\{a_n\}$. We wish to show that $\lim b_k = L$. Let $\varepsilon > 0$ be given, and choose an N such that $|a_n - L| < \varepsilon$ if $n \ge N$. Choose a K so that $n_K \ge N$. (How?) Then, if $k \ge K$, we have $n_k \ge n_K \ge N$, whence $|b_k - L| = |a_{n_k} - L| < \varepsilon$, which shows that $\lim b_k = L$. This proves part (2).

1:

REMARK The preceding theorem has the following interpretation. It says that if a sequence converges to a number L, then the cluster set of the sequence contains only one number, and that number is L. Indeed, if x is a cluster point of the sequence, then there must be some subsequence that converges to x. But, by part (2), every subsequence converges to L. Then, by part (1), x = L. Part (g) of Exercise shows that the converse of this theorem is not valid. that is, the cluster set may contain only one point, and yet the sequence is not convergent.

We give next what is probably the most useful fundamental result about sequences, the Bolzano-Weierstrass Theorem. It is this theorem that will enable us to derive many of the important properties of continuity, differentiability, and integrability.

Theorem 2: Bolzano-Weierstrass

Every bounded sequence $\{a_n\}$ of real or complex numbers has a cluster point. In other words, every bounded sequence has a convergent subsequence.

The Bolzano-Weierstrass Theorem is, perhaps not surprisingly, a very difficult theorem to prove. We begin with a technical, but very helpful, lemma.

Lemma 1:

Let $\{a_n\}$ be a bounded sequence of real numbers; i.e., assume that there exists an M such that $|a_n| \leq M$ for all n. For each $n \geq 1$, let S_n be the set whose elements are $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$. That is, S_n is just the elements of the tail of the sequence from n on. Define $x_n = \sup_{n \geq n} a_n$. Then

- 1. The sequence $\{x_n\}$ is bounded (above and below).
- 2. The sequence $\{x_n\}$ is non-increasing.
- 3. The sequence $\{x_n\}$ converges to a number x.

4. The limit x of the sequence $\{x_n\}$ is a cluster point of the sequence $\{a_n\}$. That is, there exists a subsequence $\{b_k\}$ of the sequence $\{a_n\}$ that converges to x.

5. If y is any cluster point of the sequence $\{a_n\}$, then $y \leq x$, where x is the cluster point of part (4). That is, x is the maximum of all cluster points of the sequence $\{a_n\}$.

Proof:

Since x_n is the supremum of the set S_n , and since each element of that set is bounded between -M and M, part (1) is immediate.

Since $S_{n+1} \subseteq S_n$, it is clear that

$$x_{n+1} = \sup S_{n+1} \le \sup S_n = x_n,\tag{2}$$

showing part (2).

The fact that the sequence $\{x_n\}$ converges to a number x is then a consequence of here¹.

We have to show that the limit x of the sequence $\{x_n\}$ is a cluster point of $\{a_n\}$. Notice that $\{x_n\}$ may not itself be a subsequence of $\{a_n\}$, each x_n may or may not be one of the numbers a_k , so that there really is something to prove. In fact, this is the hard part of this lemma. To finish the proof of part (4), we must define an increasing sequence $\{n_k\}$ of natural numbers for which the corresponding subsequence $\{b_k\} = \{a_{n_k}\}$ of $\{a_n\}$ converges to x. We will choose these natural numbers $\{n_k\}$ so that $|x-a_{n_k}|<1/k$. Once we have accomplished this, the fact that the corresponding subsequence $\{a_{n_k}\}$ converges to x will be clear. We choose the n_k 's inductively. First, using the fact that $x = \lim x_n$, choose an n so that $|x_n-x| = x_n - x < 1/1$. Then, because $x_n = \sup S_n$, we may choose by here² some $m \ge n$ such that $x_n \ge a_m > x_n - 1/1$. But then $|a_m-x|<1/1$. (Why?) This m we call n_1 . We have that $|a_{n_1}-x|<1/1$.

Next, again using the fact that $x = \lim x_n$, choose another n so that $n > n_1$ and so that $|x_n - x| = x_n - x < 1/2$. Then, since this $x_n = \sup S_n$, we may choose another $m \ge n$ such that $x_n \ge a_m > x_n - 1/2$. This m we call n_2 . Note that we have $|a_{n_2} - x| < 1/2$.

Arguing by induction, if we have found an increasing set $n_1 < n_2 < ... < n_j$, for which $|a_{n_i} - x| < 1/i$ for $1 \le i \le j$, choose an n larger than n_j such that $|x_n - x| < 1/(j+1)$. Then, since $x_n = \sup S_n$, choose an $m \ge n$ so that $x_n \ge a_m > x_n - 1/(j+1)$. Then $|a_m - x| < 1/(j+1)$, and we let n_{j+1} be this m. It follows that $|a_{n_{j+1}} - x| < 1/(j+1)$.

So, by recursive definition, we have constructed a subsequence of $\{a_n\}$ that converges to x, and this completes the proof of part (4) of the lemma.

Finally, if y is any cluster point of $\{a_n\}$, and if $y = \lim a_{n_k}$, then $n_k \geq k$, and so $a_{n_k} \leq x_k$, implying that $x_k - a_{n_k} \geq 0$. Hence, taking limits on k, we see that $x - y \geq 0$, and this proves part (5).

Now, using the lemma, we can give the proof of the Bolzano-Weierstrass Theorem.

Proof:

If $\{a_n\}$ is a sequence of real numbers, this theorem is an immediate consequence of part (4) of the preceding lemma.

If $a_n = b_n + c_n i$ is a sequence of complex numbers, and if $\{a_n\}$ is bounded, then $\{b_n\}$ and $\{c_n\}$ are both bounded sequences of real numbers. See here³. So, by the preceding paragraph, there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ that converges to a real number b. Now, the subsequence $\{c_{n_k}\}$ is itself a bounded sequence of real numbers, so there is a subsequence $\{c_{n_{k_j}}\}$ that converges to a real number c. By part (2) of Theorem 1, p. 2, we also have that the subsequence $\{b_{n_{k_j}}\}$ converges

¹"The Limit of a Sequence of Numbers: Existence of Certain Fundamental Limits", Theorem 1 http://cnx.org/content/m36120/latest/#fs-id1170684226813

²"The Real and Complex Numbers: Properties of the Real Numbers", Theorem 3

< http://cnx.org/content/m36085/latest/#fs-id1170177642504>

^{3&}quot;The Real and Complex Numbers: The Complex Numbers", Exercise 5 http://cnx.org/content/m36113/latest/#fs-id1165800080539

to b. So the subsequence $\{a_{n_{k_j}}\}=\{b_{n_{k_j}}+c_{n_{k_j}}i\}$ of $\{a_n\}$ converges to the complex number b+ci; i.e., $\{a_n\}$ has a cluster point. This completes the proof.

There is an important result that is analogous to the Lemma above, and its proof is easily adapted from the proof of that lemma.

Exercise 3

Let $\{a_n\}$ be a bounded sequence of real numbers. Define a sequence $\{y_n\}$ by $y_n = inf_{k \ge n} a_k$. Prove that:

- a. $\{y_n\}$ is nondecreasing and bounded above.
- b. $y = \lim y_n$ is a cluster point of $\{a_n\}$.
- c. If z is any cluster point of $\{a_n\}$, then $y \leq z$. That is, y is the minimum of all the cluster points of the sequence $\{a_n\}$. HINT: Let $\{\alpha_n\} = \{-a_n\}$, and apply the preceding lemma to $\{\alpha_n\}$. This exercise will then follow from that.

The Bolzano-Wierstrass Theorem shows that the cluster set of a bounded sequence $\{a_n\}$ is nonempty. It is also a bounded set itself.

The following definition is only for sequences of real numbers. However, like the Bolzano-Weierstrass Theorem, it is of very basic importance and will be used several times in the sequel.

Definition 3:

Let $\{a_n\}$ be a sequence of real numbers and let S denote its cluster set.

If S is nonempty and bounded above, we define $\lim \sup a_n$ to be the supremum $\sup S$ of S.

If S is nonempty and bounded below, we define $\lim \inf a_n$ to be the infimum $\inf S$ of S.

If the sequence $\{a_n\}$ of real numbers is not bounded above, we define $\lim \sup a_n$ to be ∞ , and if $\{a_n\}$ is not bounded below, we define $\lim \inf a_n$ to be $-\infty$.

If $\{a_n\}$ diverges to ∞ , then we define $\lim \sup a_n$ and $\lim \inf a_n$ both to be ∞ . And, if $\{a_n\}$ diverges to $-\infty$, we define $\lim \sup a_n$ and $\lim \inf a_n$ both to be $-\infty$.

Exercise 4

- a. Suppose $\{a_n\}$ is a bounded sequence of real numbers. Prove that the sequence $\{x_n\}$ of the lemma following Theorem 2, Bolzano-Weierstrass, p. 2 converges to $\lim \sup a_n$. Show also that the sequence $\{y_n\}$ of Exercise converges to $\lim \inf a_n$.
- b. Let $\{a_n\}$ be a not necessarily bounded sequence of real numbers. Prove that

$$\lim \sup_{n} \sup_{k \ge n} \sup_{n} \sup_{k \ge n} a_k. \tag{3}$$

and

$$\lim \inf_{n} \inf_{k \ge n} fa_k = \lim_{n} \inf_{k \ge n} fk \ge na_k. \tag{4}$$

HINT: Check all cases, and use Lemma 1, p. 2 and Exercise.

c. Let $\{a_n\}$ be a sequence of real numbers. Prove that

$$\lim \sup a_n = -\lim \inf \left(-a_n \right). \tag{5}$$

- d. Give examples to show that all four of the following possibilities can happen.
 - a. $\lim \sup a_n$ is finite, and $\lim \inf a_n = -\infty$.
 - b. $\lim \sup a_n = \infty$ and $\lim \inf a_n$ is finite.

- c. $\lim \sup a_n = \infty$ and $\lim \inf a_n = -\infty$.
- d. both $\lim \sup a_n$ and $\lim \inf a_n$ are finite.

The notions of limsup and liminf are perhaps mysterious, and they are in fact difficult to grasp. The previous exercise describes them as the result of a kind of two-level process, and there are occasions when this description is a great help. However, the limsup and liminf can also be characterized in other ways that are more reminiscent of the definition of a limit. These other ways are indicated in the next exercise.

Exercise 5

Let $\{a_n\}$ be a bounded sequence of real numbers with $\lim \sup a_n = L$ and $\lim \inf a_n = l$. Prove that L and l satisfy the following properties.

- a. For each $\varepsilon > 0$, there exists an N such that $a_n < L + \varepsilon$ for all $n \ge N$. HINT: Use the fact that $\lim \sup a_n = L$ is the number x of the lemma following Theorem 2.8, and that x is the limit of a specific sequence $\{x_n\}$.
- b. For each $\varepsilon > 0$, and any natural number k, there exists a natural number $j \ge k$ such that $a_j > L \varepsilon$. Same hint as for part (a).
- c. For each $\varepsilon > 0$, there exists an N such that $a_n > l \varepsilon$ for all $n \ge N$.
- d. For each $\varepsilon > 0$, and any natural number k, there exists a natural number j > k such that $a_j < l + \varepsilon$.
- e. Suppose L' is a number that satisfies parts (a) and (b). Prove that L' is the limsup of $\{a_n\}$. HINT: Use part (a) to show that L' is greater than or equal to every cluster point of $\{a_n\}$. Then use part (b) to show that L' is less than or equal to some cluster point.
- f. If l' is any number that satisfies parts (c) and (d), show that l' is the liminf of the sequence $\{a_n\}$.

Exercise 6

- a. Let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of real numbers, and write $L = \lim \sup a_n$ and $M = \lim \sup b_n$. Prove that $\lim \sup (a_n + b_n) \le \lim \sup a_n + \lim \sup b_n$. HINT: Using part (a) of the preceding exercise, show that for every $\varepsilon > 0$ there exists a N such that $a_n + b_n < L + M + \varepsilon$ for all $n \ge N$, and conclude from this that every cluster point y of the sequence $\{a_n + b_n\}$ is less than or equal to L + M. This will finish the proof, since $\lim \sup (a_n + b_n)$ is a cluster point of that sequence.
- b. Again, let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of real numbers, and write $l = \lim \inf a_n$ and $m = \lim \inf b_n$. Prove that $\lim \inf (a_n + b_n) \ge \lim \inf a_n + \lim \inf b_n$. HINT: Use part (c) of the previous exercise.
- c. Find examples of sequences $\{a_n\}$ and $\{b_n\}$ for which $\lim \sup a_n = \lim \sup b_n = 1$, but $\lim \sup (a_n + b_n) = 0$.

We introduce next another property that a sequence can possess. It looks very like the definition of a convergent sequence, but it differs in a crucial way, and that is that this definition only concerns the elements of the sequence $\{a_n\}$ and not the limit L.

Definition 4:

A sequence $\{a_n\}$ of real or complex numbers is a Cauchy sequence if for every $\varepsilon > 0$, there exists a natural number N such that if $n \geq N$ and $m \geq N$ then $|a_n - a_m| < \varepsilon$.

2:

REMARK No doubt, this definition has something to do with limits. Any time there is a positive ε and an N, we must be near some kind of limit notion. The point of the definition of a Cauchy sequence is that there is no explicit mention of what the limit is. It isn't that the terms of the sequence are getting closer and closer to some number L, it's that the terms of the sequence are getting closer and closer to each other. This subtle difference is worth some thought.

Exercise 7

Prove that a Cauchy sequence is bounded. (Try to adjust the proof of here⁴ to work for this situation.)

The next theorem, like the Bolzano-Weierstrass Theorem, seems to be quite abstract, but it also turns out to be a very useful tool for proving theorems about continity, differentiability, etc. In the proof, the completeness of the set of real numbers will be crucial. This theorem is not true in ordered fields that are not complete.

Theorem 3: Cauchy Criterion

A sequence $\{a_n\}$ of real or complex numbers is convergent if and only if it is a Cauchy sequence. **Proof:**

If $\lim a_n = a$ then given $\varepsilon > 0$, choose N so that $|a_k - a| < \varepsilon/2$ if $k \ge N$. From the triangle inequality, and by adding and subtracting a, we obtain that $|a_n - a_m| < \varepsilon$ if $n \ge N$ and $m \ge N$. Hence, if $\{a_n\}$ is convergent, then $\{a_n\}$ is a Cauchy sequence.

Conversely, if $\{a_n\}$ is a cauchy sequence, then $\{a_n\}$ is bounded by the previous exercise. Now we use the fact that $\{a_n\}$ is a sequence of real or complex numbers. Let x be a cluster point of $\{a_n\}$. We know that one exists by the Bolzano-Weierstrass Theorem. Let us show that in fact this number x not only is a cluster point but that it is in fact the limit of the sequence $\{a_n\}$. Given $\varepsilon > 0$, choose Nso that $|a_n - a_m| < \varepsilon/2$ whenever both n and $m \ge N$. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$ that converges to x. Because $\{n_k\}$ is strictly increasing, we may choose a k so that $n_k > N$ and also so that $|a_{n_k} - x| < \varepsilon/2$. Then, if $n \ge N$, then both n and this particular n_k are larger than or equal to N. Therefore, $|a_n - x| \le |a_n - a_{n_k}| + |a_{n_k} - x| < \varepsilon$, this completes the proof that $x = \lim a_n$.

⁴"The Limit of a Sequence of Numbers: Properties of Convergent Sequences", Theorem 1 http://cnx.org/content/m36126/latest/#fs-id1164146425836>