INTEGRATION OVER SMOOTH CURVES IN THE PLANE: SMOOTH CURVES IN THE PLANE^{*}

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Abstract

Our first project is to make a satisfactory definition of a smooth curve in the plane, for there is a good bit of subtlety to such a definition. In fact, the material in this chapter is all surprisingly tricky, and the proofs are good solid analytical arguments, with lots of ε 's and references to earlier theorems.

Our first project is to make a satisfactory definition of a smooth curve in the plane, for there is a good bit of subtlety to such a definition. In fact, the material in this chapter is all surprisingly tricky, and the proofs are good solid analytical arguments, with lots of ε 's and references to earlier theorems.

Whatever definition we adopt for a curve, we certainly want straight lines, circles, and other natural geometric objects to be covered by our definition. Our intuition is that a curve in the plane should be a "1-dimensional" subset, whatever that may mean. At this point, we have no definition of the dimension of a general set, so this is probably not the way to think about curves. On the other hand, from the point of view of a physicist, we might well define a curve as the trajectory followed by a particle moving in the plane, whatever that may be. As it happens, we do have some notion of how to describe mathematically the trajectory of a moving particle. We suppose that a particle moving in the plane proceeds in a continuous manner relative to time. That is, the position of the particle at time t is given by a continuous function $f(t) = x(t) + iy(t) \equiv (x(t), y(t))$, as t ranges from time a to time b. A good first guess at a definition of a curve joining two points z_1 and z_2 might well be that it is the range C of a continuous function f that is defined on some closed bounded interval [a, b]. This would be a curve that joins the two points $z_1 = f(a)$ and $z_2 = f(b)$ in the plane. Unfortunately, this is also not a satisfactory definition of a curve, because of the following surprising and bizarre mathematical example, first discovered by Guiseppe Peano in 1890.

1:

THE PEANO CURVE The so-called "Peano curve" is a continuous function f defined on the interval [0, 1], whose range is the entire unit square $[0, 1] \times [0, 1]$ in \mathbb{R}^2 .

Be careful to realize that we're talking about the "range" of f and not its graph. The graph of a realvalued function could never be the entire square. This Peano function is a complex-valued function of a real variable. Anyway, whatever definition we settle on for a curve, we do not want the entire unit square to be a curve, so this first attempt at a definition is obviously not going to work.

Let's go back to the particle tracing out a trajectory. The physicist would probably agree that the particle should have a continuously varying velocity at all times, or at nearly all times, i.e., the function f should

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be continuously differentiable. Recall that the velocity of the particle is defined to be the rate of change of the position of the particle, and that's just the derivative f' of f. We might also assume that the particle is never at rest as it traces out the curve, i.e., the derivative f'(t) is never 0. As a final simplification, we could suppose that the curve never crosses itself, i.e., the particle is never at the same position more than once during the time interval from t = a to t = b. In fact, these considerations inspire the formal definition of a curve that we will adopt below.

Recall that a function f that is continuous on a closed interval [a, b] and continuously differentiable on the open interval (a, b) is called a smooth function on [a, b]. And, if there exists a partition $\{t_0 < t_1 < ... < t_n\}$ of [a, b] such that f is smooth on each subinterval $[t_{i-1}, t_i]$, then f is called piecewise smooth on [a, b]. Although the derivative of a smooth function is only defined and continuous on the open interval (a, b), and hence possibly is unbounded, it follows from part (d) of here¹ that this derivative is improperly-integrable on that open interval. We recall also that just because a function is improperly-integrable on an open interval, its absolute value may not be improperly-integrable. Before giving the formal definition of a smooth curve, which apparently will be related to smooth or piecewise smooth functions, it is prudent to present an approximation theorem about smooth functions. here² asserts that every continuous function on a closed bounded interval is the uniform limit of a sequence of step functions. We give next a similar, but stronger, result about smooth functions. It asserts that a smooth function can be approximated "almost uniformly" by piecewise linear functions.

Theorem 1:

Let f be a smooth function on a closed and bounded interval [a, b], and assume that |f'| is improperly-integrable on the open interval (a, b). Given an $\varepsilon > 0$, there exists a piecewise linear function p for which

1. $|f(x) - p(x)| < \varepsilon$ for all $x \in [a, b]$.

2.
$$\int_a^b |f'(x) - p'(x)| \, dx < \varepsilon.$$

That is, the functions f and p are close everywhere, and their derivatives are close on average in the sense that the integral of the absolute value of the difference of the derivatives is small. **Proof:**

Because f is continuous on the compact set [a, b], it is uniformly continuous. Hence, let $\delta > 0$ be such that if $x, y \in [a, b]$, and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon/2$.

Because |f'| is improperly-integrable on the open interval (a, b), we may use part (b) of here³ to find a $\delta' > 0$, which may also be chosen to be $< \delta$, such that $\int_{a}^{a+\delta'} |f'| + \int_{b-\delta'}^{b} |f'| < \varepsilon/2$, and we fix such a δ' .

Now, because f' is uniformly continuous on the compact set $[a + \delta', b - \delta']$, there exists an $\alpha > 0$ such that $|f'(x) - f'(y)| < \varepsilon/4 (b - a)$ if x and y belong to $[a + \delta', b - \delta']$ and $|x - y| < \alpha$. Choose a partition $\{x_0 < x_1 < \ldots < x_n\}$ of [a, b] such that $x_0 = a, x_1 = a + \delta', x_{n-1} = b - \delta', x_n = b$, and $x_i - x_{i-1} < \min(\delta, \alpha)$ for $2 \le i \le n - 1$. Define p to be the piecewise linear function on [a, b] whose graph is the polygonal line joining the n + 1 points $(a, f(x_1)), \{(x_i, f(x_i))\}$ for $1 \le i \le n - 1$, and $(b, f(x_{n-1}))$. That is, p is constant on the outer subintervals $[a, x_1]$ and $[x_{n-1}, b]$ determined by the partition, and its graph between x_1 and x_{n-1} is the polygonal line joining the points $\{(x_1, f(x_1)), ..., (x_{n-1}, f(x_{n-1}))\}$. For example, for $2 \le i \le n - 1$, the function p has the form

$$p(x) = f(x_{i-1}) + \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} (x - x_{i-1})$$
(1)

 $^{^1}$ "Integration, Average Behavior: Extending the Definition of Integrability", Exercise 2 <http://cnx.org/content/m36222/latest/#fs-id1164267204474>

²"Functions and Continuity: Power Series Functions", Exercise 5

 $<\!\! http://cnx.org/content/m36165/latest/\#fs-id1171756794069\!\!>$

 $^{^3&}quot;Integration, Average Behavior: Extending the Definition of Integrability", Exercise 2 <math display="inline"><\!http://cnx.org/content/m36222/latest/\#fs-id1164267204474>$

on the interval $[x_{i-1}, x_i]$. So, p(x) lies between the numbers $f(x_{i-1})$ and $f(x_i)$ for all *i*. Therefore,

$$|f(x) - p(x)| \le |f(x) - f(x_i)| + |f(x_i) - l(x)| \le |f(x) - f(x_i)| + |f(x_i) - f(x_{i-1})| < \varepsilon.$$
(2)

Since this inequality holds for all i, part (1) is proved.

Next, for $2 \le i \le n-1$, and for each $x \in (x_{i-1}, x_i)$, we have $p'(x) = (f(x_i) - f(x_{i-1})) / (x_i - x_{i-1})$, which, by the Mean Value Theorem, is equal to $f'(y_i)$ for some $y_i \in (x_{i-1}, x_i)$. So, for each such $x \in (x_{i-1}, x_i)$, we have $|f'(x) - p'(x)| = |f'(x) - f'(y_i)|$, and this is less than $\varepsilon/4$ (b - a), because $|x - y_i| < \alpha$. On the two outer intervals, p(x) is a constant, so that p'(x) = 0. Hence,

$$\int_{a}^{b} |f' - p'| = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_{i}} |f' - p'|
= \int_{a}^{x_{1}} |f'| + \sum_{i=2}^{n-1} |f' - p'| + \int_{x_{n-1}}^{b} |f'|
\leq \int_{a}^{a+\delta'} |f'| + \int_{b-\delta'}^{b} |f'| + \frac{\varepsilon}{4(b-a)} \int_{x_{1}}^{x_{n-1}} 1
< \varepsilon.$$
(3)

The proof is now complete.

$\mathbf{2}$:

REMARK It should be evident that the preceding theorem can easily be generalized to a piecewise smooth function f, i.e., a function that is continuous on [a, b], continuously differentiable on each subinterval (t_{i-1}, t_i) of a partition $\{t_0 < t_1 < ... < t_n\}$, and whose derivative f' is absolutely integrable on (a, b). Indeed, just apply the theorem to each of the subintervals (t_{i-1}, t_i) , and then carefully piece together the piecewise linear functions on those subintervals.

Now we are ready to define what a smooth curve is.

Definition 1:

By a smooth curve from a point z_1 to a different point z_2 in the plane, we mean a set $C \subseteq C$ that is the range of a 1-1, smooth, function $\phi : [a, b] \to C$, where [a, b] is a bounded closed interval in R, where $z_1 = \phi(a)$ and $z_2 = \phi(b)$, and satisfying $\phi'(t) \neq 0$ for all $t \in (a, b)$.

More generally, if $\phi : [a, b] \to R^2$ is 1-1 and piecewise smooth on [a, b], and if $\{t_0 < t_1 < ... < t_n\}$ is a partition of [a, b] such that $\phi'(t) \neq 0$ for all $t \in (t_{i-1}, t_i)$, then the range C of ϕ is called a piecewise smooth curve from $z_1 = \phi(a)$ to $z_2 = \phi(b)$.

In either of these cases, ϕ is called a *parameterization* of the curve C.

Note that we do not assume that $|\phi'|$ is improperly-integrable, though the preceding theorem might have made you think we would.

3:

REMARK Throughout this chapter we will be continually faced with the fact that a given curve can have many different parameterizations. Indeed, if $\phi_1 : [a, b] \to C$ is a parameterization, and if $g : [c, d] \to [a, b]$ is a smooth function having a nonzero derivative, then $\phi_2(s) = \phi_1(g(s))$ is another parameterization of C. Since our definitions and proofs about curves often involve a parameterization, we will frequently need to prove that the results we obtain are independent of the parameterization. The next theorem will help; it shows that any two parameterizations of C are connected exactly as above, i.e., there always is such a function g relating ϕ_1 and ϕ_2 .

Theorem 2:

Let $\phi_1 : [a, b] \to C$ and $\phi_2 : [c, d] \to C$ be two parameterizations of a piecewise smooth curve C joining z_1 to z_2 . Then there exists a piecewise smooth function $g : [c, d] \to [a, b]$ such that $\phi_2(s) = \phi_1(g(s))$ for all $s \in [c, d]$. Moreover, the derivative g' of g is nonzero for all but a finite number of points in [c, d].

Proof:

Because both ϕ_1 and ϕ_2 are continuous and 1-1, it follows from here⁴ that the function $g = \phi_1^{-1} \circ \phi_2$ is continuous and 1-1 from [c, d] onto [a, b]. Moreover, from here⁵, it must also be that g is strictly increasing or strictly decreasing. Write $\phi_1(t) = u_1(t) + iv_1(t) \equiv (u_1(t), v_1(t))$, and $\phi_2(s) = u_2(s) + iv_2(s) \equiv (u_2(s), v_2(s))$. Let $\{x_0 < x_1 < \ldots < x_p\}$ be a partition of [a, b] for which ϕ'_1 is continuous and nonzero on the subintervals (x_{j-1}, x_j) , and let $\{y_0 < y_1 < \ldots < y_q\}$ be a partition of [c, d] for which ϕ'_2 is continuous and nonzero on the subintervals (y_{k-1}, y_k) . Then let $\{s_0 < s_1 < \ldots < s_n\}$ be the partition of [c, d] determined by the finitely many points $\{y_k\} \cup \{g^{-1}(x_j)\}$. We will show that g is continuously differentiable at each point s in the subintervals (s_{i-1}, s_i) . Fix an s in one of the intervals (s_{i-1}, s_i) , and let $t = \phi_1^{-1}(\phi_2(s)) = g(s)$. Of course this means

Fix an s in one of the intervals (s_{i-1}, s_i) , and let $t = \phi_1^{-1}(\phi_2(s)) = g(s)$. Of course this means that $\phi_1(t) = \phi_2(s)$, or $u_1(t) = u_2(s)$ and $v_1(t) = v_2(s)$. Then t is in some one of the intervals (x_{j-1}, x_j) , so that we know that $\phi_1(t) \neq 0$. Therefore, we must have that at least one of $u_1(t)$ or $v_1(t)$ is nonzero. Suppose it is $v_1(t)$ that is nonzero. The argument, in case it is $u_1(t)$ that is nonzero, is completely analogous. Now, because v_1 is continuous at t and $v_1(t) \neq 0$, it follows that v_1 is strictly monotonic in some neighborhood $(t - \delta, t + \delta)$ of t and therefore is 1-1 on that interval. Then v_1^{-1} is continuous by here⁶, and is differentiable at the point $v_1(t)$ by the Inverse Function Theorem. We will show that on this small interval $g = v_1^{-1} \circ v_2$, and this will prove that g is continuously differentiable at s.

Note first that if $\phi_2(\sigma) = x + iy$ is a point on the curve *C*, then $v_2(\phi_2^{-1}(x+iy)) = y$. Then, for any $\tau \in [a, b]$, we have

$$v_{1}^{-1} \left(v_{2} \left(g^{-1} \left(\tau \right) \right) \right) = v_{1}^{-1} \left(v_{2} \left(\phi_{2}^{-1} \left(\phi_{1} \left(\tau \right) \right) \right) \right) \\ = v_{1}^{-1} \left(v_{2} \left(\phi_{2}^{-1} \left(u_{1} \left(\tau \right) + i v_{1} \left(\tau \right) \right) \right) \right) \\ = v_{1}^{-1} \left(v_{1} \left(\tau \right) \right) \\ = \tau,$$

$$(4)$$

showing that $v_1^{-1} \circ v_2 = g^{-1^{-1}} = g$. Hence g is continuously differentiable at every point s in the subintervals (s_{i-1}, s_i) . Indeed $g'(\sigma) = v_1^{-1'}(v_2(\sigma))v_2'(\sigma)$ for all σ near s, and hence g is piecewise smooth.

Obviously, $\phi_2(s) = \phi_1(g(s))$ for all s, implying that $\phi'_2(s) = \phi'_1(g(s))g'(s)$. Since $\phi'_2(s) \neq 0$ for all but a finite number of points s, it follows that $g'(s) \neq 0$ for all but a finite number of points, and the theorem is proved.

Corollary 1:

Let ϕ_1 and ϕ_2 be as in the theorem. Then, for all but a finite number of points $z = \phi_1(t) = \phi_2(s)$ on the curve C, we have

$$\frac{\phi_1'(t)}{|\phi_1'(t)|} = \frac{\phi_2'(s)}{|\phi_2'(s)|}.$$
(5)

Proof:

From the theorem we have that

$$\phi_{2}^{'}(s) = \phi_{1}^{'}(g(s))g^{'}(s) = \phi_{1}^{'}(t)g^{'}(s)$$
(6)

 $^{^4}$ "Functions and Continuity: Deeper Analytic Properties of Continuous Functions", Theorem 5<http://cnx.org/content/m36167/latest/#fs-id8649209>

 $^{^5&}quot;Functions and Continuity: Deeper Analytic Properties of Continuous Functions", Theorem 6 <math display="inline">< http://cnx.org/content/m36167/latest/\#fs-id1169172121798>$

 $^{^{6}}$ "Functions and Continuity: Deeper Analytic Properties of Continuous Functions", Theorem 5 < http://cnx.org/content/m36167/latest/#fs-id8649209>

for all but a finite number of points $s \in (c, d)$. Also, g is strictly increasing, so that $g'(s) \ge 0$ for all points s where g is differentiable. And in fact, $g'(s) \ne 0$ for all but a finite number of s's, because g'(s) is either $(v_1^{-1} \circ v_2)'(s)$ or $(u_1^{-1} \circ u_2)'(s)$, and these are nonzero except for a finite number of points. Now the corollary follows by direct substitution.

4:

REMARK If we think of $\phi'(t) = (x'(t), y'(t))$ as a vector in the plane \mathbb{R}^2 , then the corollary asserts that the direction of this vector is independent of the parameterization, at least at all but a finite number of points. This direction vector will come up again as the unit tangent of the curve.

The adjective "smooth" is meant to suggest that the curve is bending in some reasonable way, and specifically it should mean that the curve has a tangent, or tangential direction, at each point. We give the definition of tangential direction below, but we note that in the context of a moving particle, the tangential direction is that direction in which the particle would continue to move if the force that is keeping it on the curve were totally removed. If the derivative $\phi'(t) \neq 0$, then this vector is the velocity vector, and its direction is exactly what we should mean by the tangential direction.

The adjective "piecewise" will allow us to consider curves that have a finite number of points where there is no tangential direction, e.g., where there are "corners."

We are carefully orienting our curves at the moment. A curve C from z_1 to z_2 is being distinguished from the same curve from z_2 to z_1 , even though the set C is the same in both instances. Which way we traverse a curve will be of great importance at the end of this chapter, when we come to Green's Theorem.

Definition 2:

Let C, the range of $\phi : [a, b] \to C$, be a piecewise smooth curve, and let $z = (x, y) = \phi(c)$ be a point on the curve. We say that the curve C has a tangential direction at z, relative to the parameterization ϕ , if the following limit exists:

$$\lim_{t \to c} \frac{\phi\left(t\right) - z}{\left|\phi\left(t\right) - z\right|} = \lim_{t \to c} \frac{\phi\left(t\right) - \phi\left(c\right)}{\left|\phi\left(t\right) - \phi\left(c\right)\right|}.$$
(7)

If this limit exists, it is a vector of length 1 in \mathbb{R}^2 , and this unit vector is called the unit tangent (relative to the parameterization ϕ) to C at z.

The curve C has a unit tangent at the point z if there exists a parameterization ϕ for which the unit tangent at z relative to ϕ exists.

Exercise 1

- a. Restate the definition of tangential direction and unit tangent using the R^2 version of the plane instead of the C version. That is, restate the definition in terms of pairs (x, y) of real numbers instead of a complex number z.
- b. Suppose $\phi : [a, b] \to C$ is a parameterization of a piecewise smooth curve C, and that $t \in (a, b)$ is a point where ϕ is differentiable with $\phi'(t) \neq 0$. Show that the unit tangent (relative to the parameterization ϕ) to C at $z = \phi(t)$ exists and equals $\phi'(t) / |\phi'(t)|$. Conclude that, except possibly for a finite number of points, the unit tangent to C at z is independent of the parameterization.
- c. Let C be the graph of the function f(t) = |t| for $t \in [-1, 1]$. Is C a smooth curve? Is it a piecewise smooth curve? Does C have a unit tangent at every point?
- d. Let C be the graph of the function $f(t) = t^{2/3} = (t^{1/3})^2$ for $t \in [-1, 1]$. Is C a smooth curve? Is it a piecewise smooth curve? Does C have a unit tangent at every point?
- e. Consider the set C that is the right half of the unit circle in the plane. Let $\phi_1 : [-1,1] \to C$ be defined by

$$\phi_1(t) = \left(\cos\left(t\frac{\pi}{2}\right), \sin\left(t\frac{\pi}{2}\right)\right),\tag{8}$$

and let $\phi_2: [-1,1] \to C$ be defined by

$$\phi_2(t) = \left(\cos\left(t^3\frac{\pi}{2}\right), \sin\left(t^3\frac{\pi}{2}\right)\right). \tag{9}$$

Prove that ϕ_1 and ϕ_2 are both parameterizations of *C*. Discuss the existence of a unit tangent at the point $(1,0) = \phi_1(0) = \phi_2(0)$ relative to these two parameterizations.

f. Suppose $\phi : [a, b] \to C$ is a parameterization of a curve C from z_1 to z_2 . Define ψ on [a, b] by $\psi(t) = \phi(a + b - t)$. Show that ψ is a parameterization of a curve from z_2 to z_1 .

Exercise 2

- a. Suppose f is a smooth, real-valued function defined on the closed interval [a, b], and let $C \subseteq R^2$ be the graph of f. Show that C is a smooth curve, and find a "natural" parameterization $\phi : [a, b] \to C$ of C. What is the unit tangent to C at the point (t, f(t))?
- b. Let z_1 and z_2 be two distinct points in C, and define $\phi : [0,1] \to c$ by $\phi(t) = (1-t) z_1 + t z_2$. Show that ϕ is a parameterization of the straight line from the point z_1 to the point z_2 . Consequently, a straight line is a smooth curve. (Indeed, what is the definition of a straight line?)
- c. Define a function $\phi : [-r, r] \to R^2$ by $\phi(t) = (t, \sqrt{r^2 t^2})$. Show that the range C of ϕ is a smooth curve, and that ϕ is a parameterization of C.
- d. Define ϕ on $[0, \pi/2)$ by $\phi(t) = e^{it}$. For what curve is ϕ a parametrization?
- e. Let $z_1, z_2, ..., z_n$ be *n* distinct points in the plane, and suppose that the polygonal line joing these points in order never crosses itself. Construct a parameterization of that polygonal line.
- f. Let S be a piecewise smooth geometric set determined by the interval [a, b] and the two piecewise smooth bounding functions u and l. Suppose z_1 and z_2 are two points in the interior S^0 of S. Show that there exists a piecewise smooth curve C joining z_1 to z_2 , i.e., a piecewise smooth function $\phi : \begin{bmatrix} a \\ b \end{bmatrix} \to C$ with $\phi \begin{pmatrix} a \\ c \end{pmatrix} = z_1$ and $\phi \begin{pmatrix} b \\ b \end{pmatrix} = z_2$, that lies entirely in S^0 .
- g. Let C be a piecewise smooth curve, and suppose $\phi : [a, b] \to C$ is a parameterization of C. Let [c, d] be a subinterval of [a, b]. Show that the range of the restriction of ϕ to [c, d] is a smooth curve.

Exercise 3

Suppose C is a smooth curve, parameterized by $\phi = u + iv : [a, b] \to C$.

- a. Suppose that $u'(t) \neq 0$ for all $t \in (a, b)$. Prove that there exists a smooth, real-valued function f on some closed interval [a', b'] such that C coincides with the graph of f. HINT: f should be something like $v \circ u^{-1}$.
- b. What if $v'(t) \neq 0$ for all $t \in (a, b)$?

Exercise 4

Let C be the curve that is the range of the function $\phi: [-1,1] \to C$, where $\phi(t) = t^3 + t^6 i$.

- a. Is C a piecewise smooth curve? Is it a smooth curve? What points z_1 and z_2 does it join?
- b. Is ϕ a parameterization of C?
- c. Find a parameterization for C by a function $\psi : [3, 4] \to C$.
- d. Find the unit tangent to C and the point 0 + 0i.

Exercise 5

Let C be the curve parameterized by $\phi: [-\pi, \pi - \varepsilon] \to C$ defined by $\phi(t) = e^{it} = \cos(t) + i\sin(t)$.

- a. What curve does ϕ parameterize?
- b. Find another parameterization of this curve, but base on the interval $[0, 1-\varepsilon]$.