Bases and frames*

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Abstract

This module provides an overview of bases and frames in finite-dimensional Hilbert spaces.

A set $\Psi = \{\psi_i\}_{i \in \mathcal{I}}$ is called a basis for a finite-dimensional vector space[1] $\mathcal{V}$ if the vectors in the set span $\mathcal{V}$ and are linearly independent. This implies that each vector in the space can be represented as a linear combination of this (smaller, except in the trivial case) set of basis vectors in a unique fashion. Furthermore, the coefficients of this linear combination can be found by the inner product of the signal and a dual set of vectors. In discrete settings, we will only consider real finite-dimensional Hilbert spaces where $\mathcal{V} = \mathbb{R}^N$ and $\mathcal{I} = \{1, ..., N\}$.

Mathematically, any signal $x \in \mathbb{R}^N$ may be expressed as,

$$x = \sum_{i \in \mathcal{I}} a_i \tilde{\psi}_i, \tag{1}$$

where our coefficients are computed as $a_i = < x, \psi_i >$ and $\{\tilde{\psi}_i\}_{i \in I}$ are the vectors that constitute our dual basis. Another way to denote our basis and its dual is by how they operate on x. Here, we call our dual basis $\tilde{\Psi}$ our *synthesis basis* (used to reconstruct our signal by (1)) and Ψ is our *analysis basis*.

An orthonormal basis (ONB) is defined as a set of vectors $\Psi = \{\psi_i\}_{i \in \mathcal{I}}$ that form a basis and whose elements are orthogonal and unit norm. In other words, $< \psi_i, \psi_j >= 0$ if $i \neq j$ and one otherwise. In the case of an ONB, the *synthesis basis* is simply the Hermitian adjoint of *analysis basis* ($\tilde{\Psi} = \Psi^T$).

It is often useful to generalize the concept of a basis to allow for sets of possibly linearly dependent vectors, resulting in what is known as a *frame*. More formally, a frame is a set of vectors $\{\Psi_i\}_{i=1}^n$ in $\mathbb{R}^d$, $d < n$ corresponding to a matrix $\Psi \in \mathbb{R}^{d \times n}$, such that for all vectors $x \in \mathbb{R}^d$,

$$A\|x\|_2^2 \leq \|\Psi^T x\|_2^2 \leq B\|x\|_2^2 \tag{2}$$

with $0 < A \leq B < \infty$. Note that the condition $A > 0$ implies that the rows of Ψ must be linearly independent. When A is chosen as the largest possible value and B as the smallest for these inequalities to hold, then we call them the *(optimal) frame bounds*. If A and B can be chosen as $A = B$, then the frame is called *A-tight*, and if $A = B = 1$, then Ψ is a *Parseval frame*. A frame is called *equal-norm*, if there exists some $\lambda > 0$ such that $\|\Psi_i\|_2 = \lambda$ for all $i = 1, ..., N$, and it is *unit-norm* if $\lambda = 1$. Note also that while the concept of a frame is very general and can be defined in infinite-dimensional spaces, in the case where Ψ is a $d \times N$ matrix A and B simply correspond to the smallest and largest eigenvalues of $\Psi\Psi^T$, respectively.

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[1]"Introduction to vector spaces" <http://cnx.org/content/m37167/latest/>

Frames can provide richer representations of data due to their redundancy: for a given signal x, there exist infinitely many coefficient vectors α such that $x = \Psi\alpha$. In particular, each choice of a dual frame $\tilde{\Psi}$ provides a different choice of a coefficient vector α. More formally, any frame satisfying

$$\Psi\tilde{\Psi}^T = \tilde{\Psi}\Psi^T = I \quad (3)$$

is called an (alternate) dual frame. The particular choice $\tilde{\Psi} = \left(\Psi\Psi^T\right)^{-1}\Psi$ is referred to as the *canonical dual frame*. It is also known as the Moore-Penrose pseudoinverse. Note that since $A > 0$ requires Ψ to have linearly independent rows, we ensure that $\Psi\Psi^T$ is invertible, so that $\tilde{\Psi}$ is well-defined. Thus, one way to obtain a set of feasible coefficients is via

$$\alpha_d = \Psi^T\left(\Psi\Psi^T\right)^{-1}x. \quad (4)$$

One can show that this sequence is the smallest coefficient sequence in ℓ_2 norm, i.e., $\|\alpha_d\|_2 \leq \|\alpha\|_2$ for all α such that $x = \Psi\alpha$.

Finally, note that in the sparse approximation[2] literature, it is also common for a basis or frame to be referred to as a *dictionary* or *overcomplete dictionary* respectively, with the dictionary elements being called *atoms*.

[2]"Sparse representations" <http://cnx.org/content/m37168/latest/>