Compressible signals*

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Abstract

This module describes compressible signals, i.e., signals that can be well-approximated by sparse signals.

1 Compressibility and K-term approximation

An important assumption used in the context of compressive sensing (CS) is that signals exhibit a degree of structure. So far the only structure we have considered is sparsity[1], i.e., the number of non-zero values the signal has when representation in an orthonormal basis[2] Ψ. The signal is considered sparse if it has only a few nonzero values in comparison with its overall length.

Few structured signals are truly sparse; rather they are compressible. A signal is *compressible* if its sorted coefficient magnitudes in Ψ decay rapidly. To consider this mathematically, let x be a signal which is compressible in the basis Ψ:

$$x = \Psi\alpha, \tag{1}$$

where α are the coefficients of x in the basis Ψ. If x is compressible, then the magnitudes of the sorted coefficients α_s observe a power law decay:

$$|\alpha_s| \leq C_1 s^{-q}, s = 1, 2, ... \tag{2}$$

We define a signal as being compressible if it obeys this power law decay. The larger q is, the faster the magnitudes decay, and the more compressible a signal is. Figure 1 shows images that are compressible in different bases.

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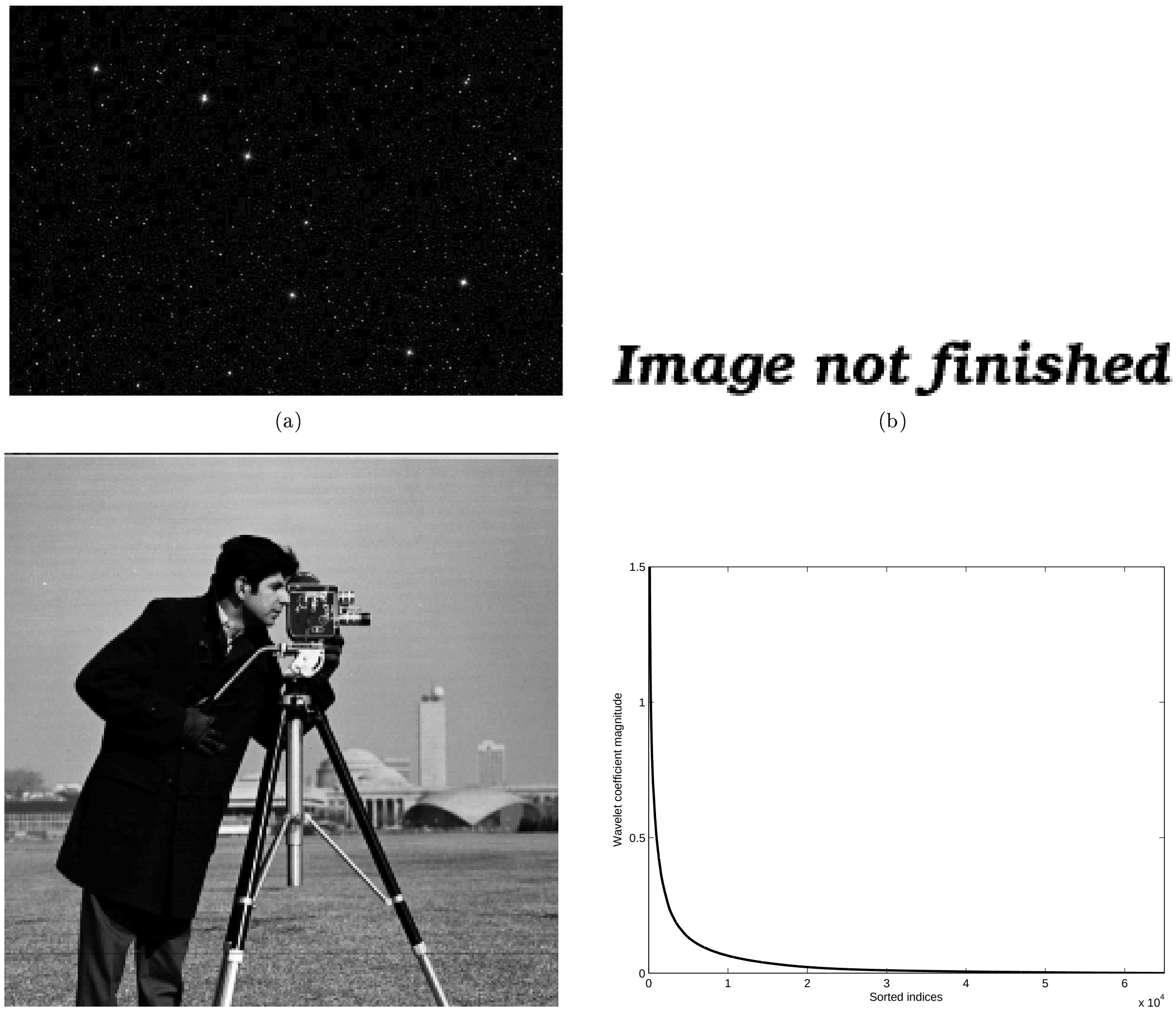
†http://creativecommons.org/licenses/by/3.0/
[1]"Sparse representations" <http://cnx.org/content/m37168/latest/>
[2]"Bases and frames" <http://cnx.org/content/m37165/latest/>



(a) (b) (c) (d)

Figure 1: *The image in the upper left is a signal that is compressible in space. When the pixel values are sorted from largest to smallest, there is a sharp descent. The image in the lower left is not compressible in space, but it is compressible in wavelets since its wavelet coefficients exhibit a power law decay.*

Because the magnitudes of their coefficients decay so rapidly, compressible signals can be represented well by $K \ll N$ coefficients. The best K-term approximation of a signal is the one in which the K largest coefficients are kept, with the rest being zero. The error between the true signal and its K term approximation is denoted the K-term approximation error $\sigma_K(x)$, defined as

$$\sigma_K(x) = arg \min_{\alpha \in \Sigma_K} \parallel x - \Psi\alpha \parallel_2. \tag{3}$$

For compressible signals, we can establish a bound with power law decay as follows:

$$\sigma_K(x) \leq C_2 K^{1/2-s}. \tag{4}$$

In fact, one can show that $\sigma_K(x)_2$ will decay as K^{-r} if and only if the sorted coefficients α_i decay as $i^{-r+1/2}$ [1]. Figure 2 shows an image and its K-term approximation.

(a)

(b)

Figure 2: Sparse approximation of a natural image. (a) Original image. (b) Approximation of image obtained by keeping only the largest 10% of the wavelet coefficients. Because natural images are compressible in a wavelet domain, approximating this image it in terms of its largest wavelet coefficients maintains good fidelity.

2 Compressibility and ℓ_p spaces

A signal's compressibility is related to the ℓ_p space to which the signal belongs. An infinite sequence $x(n)$ is an element of an ℓ_p space for a particular value of p if and only if its ℓ_p norm is finite:

$$\| x \|_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}} < \infty. \tag{5}$$

The smaller p is, the faster the sequence's values must decay in order to converge so that the norm is bounded. In the limiting case of $p = 0$, the "norm" is actually a pseudo-norm and counts the number of non-zero values. As p decreases, the size of its corresponding ℓ_p space also decreases. Figure 3 shows various ℓ_p unit balls (all sequences whose ℓ_p norm is 1) in 3 dimensions.

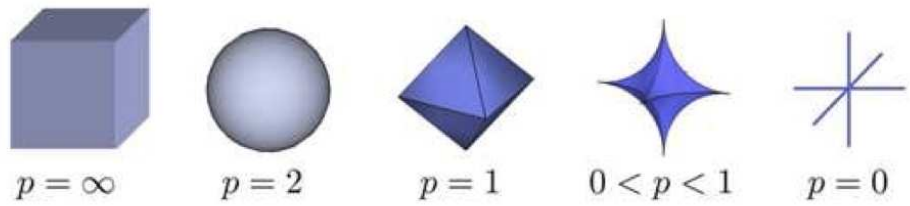


Figure 3: *As the value of p decreases, the size of the corresponding ℓ_p space also decreases. This can be seen visually when comparing the the size of the spaces of signals, in three dimensions, for which the ℓ_p norm is less than or equal to one. The volume of these ℓ_p "balls" decreases with p.*

Suppose that a signal is sampled infinitely finely, and call it $x[n]$. In order for this sequence to have a bounded ℓ_p norm, its coefficients must have a power-law rate of decay with $q > 1/p$. Therefore a signal which is in an ℓ_p space with $p \leq 1$ obeys a power law decay, and is therefore compressible.

References

[1] R. DeVore. Nonlinear approximation. *Acta Numerica*, 7:518211;150, 1998.