The RIP and the NSP*

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Abstract

This module describes the relationship between the restricted isometry property (RIP) and the null space property (NSP). Specifically, it is shown that a matrix which satisfies the RIP will also satisfy the NSP.

Next we will show that if a matrix satisfies the restricted isometry property[1] (RIP), then it also satisfies the null space property[2] (NSP). Thus, the RIP is strictly stronger than the NSP.

Theorem 1:
Suppose that Φ satisfies the RIP of order $2K$ with $\delta_{2K} < \sqrt{2} - 1$. Then Φ satisfies the NSP of order $2K$ with constant

$$C = \frac{\sqrt{2}\delta_{2K}}{1 - \left(1 + \sqrt{2}\right)\delta_{2K}}. \tag{1}$$

Proof:
The proof of this theorem involves two useful lemmas. The first of these follows directly from standard norm inequality by relating a K-sparse vector to a vector in $\mathbb{R}^K$. We include a simple proof for the sake of completeness.

Lemma 1:
Suppose $u \in \Sigma_K$. Then

$$\frac{\|u\|_1}{\sqrt{K}} \leq \|u\|_2 \leq \sqrt{K}\|u\|_\infty. \tag{2}$$

Proof:
For any u, $\|u\|_1 = |< u, \operatorname{sgn}(u) >|$. By applying the Cauchy-Schwarz inequality we obtain $\|u\|_1 \leq \|u\|_2 \|\operatorname{sgn}(u)\|_2$. The lower bound follows since $\operatorname{sgn}(u)$ has exactly K nonzero entries all equal to ± 1 (since $u \in \Sigma_K$) and thus $\|\operatorname{sgn}(u)\| = \sqrt{K}$. The upper bound is obtained by observing that each of the K nonzero entries of u can be upper bounded by $\|u\|_\infty$.

Below we state the second key lemma that we will need in order to prove Theorem 1, p. 1. This result is a general result which holds for arbitrary h, not just vectors $h \in \mathcal{N}(\Phi)$. It should be clear that when we do have $h \in \mathcal{N}(\Phi)$, the argument could be simplified considerably. However, this lemma will prove immensely useful when we turn to the problem of sparse recovery from noisy

*Version 1.5: Apr 14, 2011 6:58 pm -0500
†http://creativecommons.org/licenses/by/3.0/
[1]"The restricted isometry property" <http://cnx.org/content/m37171/latest/>
[2]"Null space conditions" <http://cnx.org/content/m37170/latest/>

measurements[3] later in this course[4], and thus we establish it now in its full generality. We state the lemma here, which is proven in "ℓ_1 minimization proof"[5].

Lemma 2:

Suppose that Φ satisfies the RIP of order $2K$, and let $h \in \mathbb{R}^N$, $h \neq 0$ be arbitrary. Let Λ_0 be any subset of $\{1, 2, ..., N\}$ such that $|\Lambda_0| \leq K$. Define Λ_1 as the index set corresponding to the K entries of $h_{\Lambda_0^c}$ with largest magnitude, and set $\Lambda = \Lambda_0 \cup \Lambda_1$. Then

$$\|h_\Lambda\|_2 \leq \alpha \frac{\|h_{\Lambda_0^c}\|_1}{\sqrt{K}} + \beta \frac{|< \Phi h_\Lambda, \Phi h >|}{\|h_\Lambda\|_2}, \tag{3}$$

where

$$\alpha = \frac{\sqrt{2}\delta_{2K}}{1 - \delta_{2K}}, \quad \beta = \frac{1}{1 - \delta_{2K}}. \tag{4}$$

Again, note that Lemma 2, p. 2 holds for arbitrary h. In order to prove Theorem 1, p. 1, we merely need to apply Lemma 2, p. 2 to the case where $h \in \mathcal{N}(\Phi)$.

Towards this end, suppose that $h \in \mathcal{N}(\Phi)$. It is sufficient to show that

$$\|h_\Lambda\|_2 \leq C \frac{\|h_{\Lambda^c}\|_1}{\sqrt{K}} \tag{5}$$

holds for the case where Λ is the index set corresponding to the $2K$ largest entries of h. Thus, we can take Λ_0 to be the index set corresponding to the K largest entries of h and apply Lemma 2, p. 2.

The second term in Lemma 2, p. 2 vanishes since $\Phi h = 0$, and thus we have

$$\|h_\Lambda\|_2 \leq \alpha \frac{\|h_{\Lambda_0^c}\|_1}{\sqrt{K}}. \tag{6}$$

Using Lemma 1, p. 1,

$$\|h_{\Lambda_0^c}\|_1 = \|h_{\Lambda_1}\|_1 + \|h_{\Lambda^c}\|_1 \leq \sqrt{K}\|h_{\Lambda_1}\|_2 + \|h_{\Lambda^c}\|_1 \tag{7}$$

resulting in

$$\|h_\Lambda\|_2 \leq \alpha \left(\|h_{\Lambda_1}\|_2 + \frac{\|h_{\Lambda^c}\|_1}{\sqrt{K}} \right). \tag{8}$$

Since $\|h_{\Lambda_1}\|_2 \leq \|h_\Lambda\|_2$, we have that

$$(1 - \alpha) \|h_\Lambda\|_2 \leq \alpha \frac{\|h_{\Lambda^c}\|_1}{\sqrt{K}}. \tag{9}$$

The assumption $\delta_{2K} < \sqrt{2} - 1$ ensures that $\alpha < 1$, and thus we may divide by $1 - \alpha$ without changing the direction of the inequality to establish (5) with constant

$$C = \frac{\alpha}{1 - \alpha} = \frac{\sqrt{2}\delta_{2K}}{1 - \left(1 + \sqrt{2}\right)\delta_{2K}}, \tag{10}$$

as desired.

[3] "Signal recovery in noise" <http://cnx.org/content/m37182/latest/>

[4] *An Introduction to Compressive Sensing* <http://cnx.org/content/col11133/latest/>

[5] "ℓ_1 minimization proof" <http://cnx.org/content/m37187/latest/>