Functions and graphs: Introduction and key concepts*

Free High School Science Texts Project

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1 Introduction to Functions and Graphs

Functions are mathematical building blocks for designing machines, predicting natural disasters, curing diseases, understanding world economies and for keeping aeroplanes in the air. Functions can take input from many variables, but always give the same answer, unique to that function. It is the fact that you always get the same answer from a set of inputs that makes functions special.

A major advantage of functions is that they allow us to *visualise* equations in terms of a *graph*. A graph is an accurate drawing of a function and is much easier to read than lists of numbers. In this chapter we will learn how to understand and create real valued functions, how to read graphs and how to draw them.

In addition to their use in the problems facing humanity, functions also appear on a day-to-day level, so they are worth learning about. A function is always *dependent* on one or more variables, like time, distance or a more abstract quantity.

2 Functions and Graphs in the Real-World

Some typical examples of functions you may already have met include:-

- how much money you have, as a function of time. You never have more than one amount of money at any time because you can always add everything to give one number. By understanding how your money changes over time, you can plan to spend your money sensibly. Businesses find it very useful to *plot the graph* of their money over time so that they can see when they are spending too much. Such observations are not always obvious from looking at the numbers alone.
- the temperature is a very complicated function because it has so many inputs, including; the time of day, the season, the amount of clouds in the sky, the strength of the wind, where you are and many more. But the important thing is that there is only one temperature when you measure it in a specific place. By understanding how the temperature is effected by these things, you can plan for the day.
- where you are is a function of time, because you cannot be in two places at once! If you were to *plot the graphs* of where two people are as a function of time, if the lines cross it means that the two people meet each other at that time. This idea is used in *logistics*, an area of mathematics that tries to plan where people and items are for businesses.
- your weight is a function of how much you eat and how much exercise you do, but everybody has a different function so that is why people are all different sizes.

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3 Recap

The following should be familiar.

3.1 Variables and Constants

In Review of past work, we were introduced to variables and constants. To recap, a *variable* can take any value in some set of numbers, so long as the equation is consistent. Most often, a variable will be written as a letter.

A *constant* has a fixed value. The number 1 is a constant. Sometimes letters are used to represent constants, as they are easier to work with.

3.1.1 Investigation : Variables and Constants

In the following expressions, identify the variables and the constants:

1. $2x^2 = 1$
2. $3x + 4y = 7$
3. $y = \frac{-5}{x}$
4. $y = 7^x - 2$

3.2 Relations and Functions

In earlier grades, you saw that variables can be *related* to each other. For example, Alan is two years older than Nathan. Therefore the relationship between the ages of Alan and Nathan can be written as $A = N + 2$, where A is Alan's age and N is Nathan's age.

In general, a *relation* is an equation which relates two variables. For example, $y = 5x$ and $y^2 + x^2 = 5$ are relations. In both examples x and y are variables and 5 is a constant, but for a given value of x the value of y will be very different in each relation.

Besides writing relations as equations, they can also be represented as words, tables and graphs. Instead of writing $y = 5x$, we could also say "y is always five times as big as x". We could also give the following table:

x	$y = 5x$
2	10
6	30
8	40
13	65
15	75

Table 1

3.2.1 Investigation : Relations and Functions

Complete the following table for the given functions:

x	$y = x$	$y = 2x$	$y = x + 2$
1			
2			
3			
50			
100			

Table 2

3.3 The Cartesian Plane

When working with real valued functions, our major tool is drawing graphs. In the first place, if we have two real variables, x and y, then we can assign values to them simultaneously. That is, we can say "let x be 5 and y be 3". Just as we write "let $x = 5$" for "let x be 5", we have the shorthand notation "let $(x, y) = (5, 3)$" for "let x be 5 and y be 3". We usually think of the real numbers as an infinitely long line, and picking a number as putting a dot on that line. If we want to pick *two* numbers at the same time, we can do something similar, but now we must use two dimensions. What we do is use two lines, one for x and one for y, and rotate the one for y, as in Figure 1. We call this the *Cartesian plane*.

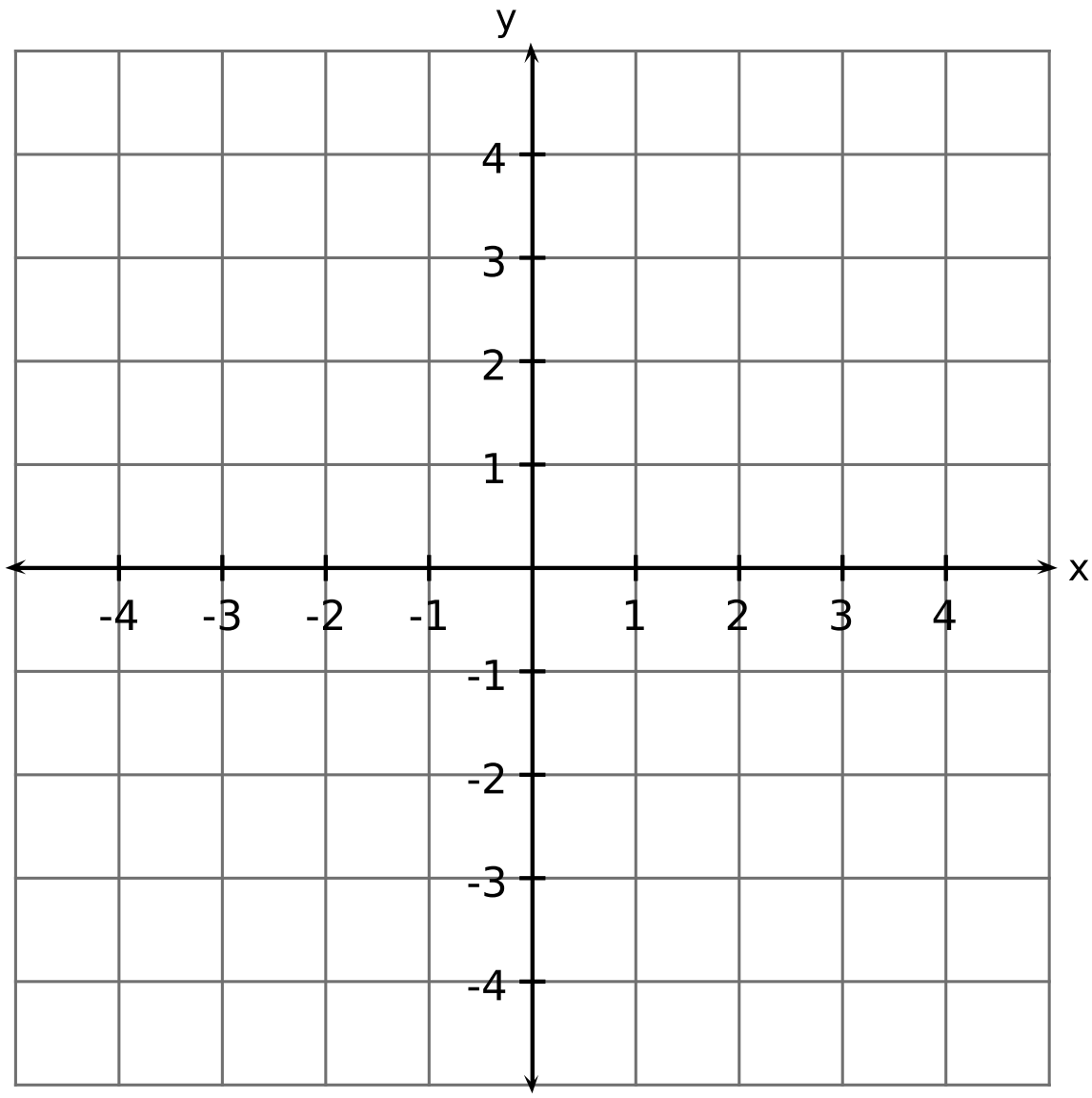

Figure 1: The Cartesian plane is made up of an $x-$axis (horizontal) and a $y-$axis (vertical).

3.4 Drawing Graphs

In order to draw the graph of a function, we need to calculate a few points. Then we plot the points on the Cartesian Plane and join the points with a smooth line.

Assume that we were investigating the properties of the function $f(x) = 2x$. We could then consider all the points $(x; y)$ such that $y = f(x)$, i.e. $y = 2x$. For example, $(1; 2), (2,5; 5)$, and $(3; 6)$ would all be such points, whereas $(3; 5)$ would not since $5 \neq 2 \times 3$. If we put a dot at each of those points, and then at every

similar one for all possible values of x, we would obtain the graph shown in Figure 2

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Figure 2: Graph of $f(x) = 2x$

The form of this graph is very pleasing – it is a simple straight line through the middle of the plane. The technique of "plotting", which we have followed here, is *the* key element in understanding functions.

3.4.1 Investigation : Drawing Graphs and the Cartesian Plane

Plot the following points and draw a smooth line through them. (-6; -8),(-2; 0), (2; 8), (6; 16)

3.5 Notation used for Functions

Thus far you would have seen that we can use $y = 2x$ to represent a function. This notation however gets confusing when you are working with more than one function. A more general form of writing a function is to write the function as $f(x)$, where f is the function name and x is the independent variable. For example, $f(x) = 2x$ and $g(t) = 2t + 1$ are two functions.

Both notations will be used in this book.

Exercise 1: Function notation ***(Solution on p. 11.)***

If $f(n) = n^2 - 6n + 9$, find $f(k-1)$ in terms of k.

Exercise 2: Function notation ***(Solution on p. 11.)***

If $f(x) = x^2 - 4$, calculate b if $f(b) = 45$.

3.5.1 Recap

1. Guess the function in the form $y = ...$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	1	2	3	40	50	600	700	800	900	1000

Table 3

Click here for the solution[1]

2. Guess the function in the form $y = ...$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	2	4	6	80	100	1200	1400	1600	1800	2000

Table 4

[1] http://www.fhsst.org/lxO

Click here for the solution[2]

3. Guess the function in the form $y = ...$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	10	20	30	400	500	6000	7000	8000	9000	10000

Table 5

Click here for the solution[3]

4. On a Cartesian plane, plot the following points: (1;2), (2;4), (3;6), (4;8), (5;10). Join the points. Do you get a straight line?
Click here for the solution[4]
5. If $f(x) = x + x^2$, write out:
   a. $f(t)$
   b. $f(a)$
   c. $f(1)$
   d. $f(3)$

   Click here for the solution[5]
6. If $g(x) = x$ and $f(x) = 2x$, write out:
   a. $f(t) + g(t)$
   b. $f(a) - g(a)$
   c. $f(1) + g(2)$
   d. $f(3) + g(s)$

   Click here for the solution[6]
7. A car drives by you on a straight highway. The car is travelling 10 m every second. Complete the table below by filling in how far the car has travelled away from you after 5, 10 and 20 seconds.

Time (s)	0	1	2	5	10	20
Distance (m)	0	10	20			

Table 6

Use the values in the table and draw a graph of distance on the y-axis and time on the x-axis. Click here for the solution[7]

4 Characteristics of Functions - All Grades

There are many characteristics of graphs that help describe the graph of any function. These properties will be described in this chapter and are:

1. dependent and independent variables
2. domain and range

[2] http://www.fhsst.org/lxc
[3] http://www.fhsst.org/lxx
[4] http://www.fhsst.org/lxa
[5] http://www.fhsst.org/lxC
[6] http://www.fhsst.org/lx1
[7] http://www.fhsst.org/lxr

3. intercepts with axes
4. turning points
5. asymptotes
6. lines of symmetry
7. intervals on which the function increases/decreases
8. continuous nature of the function

Some of these words may be unfamiliar to you, but each will be clearly described. Examples of these properties are shown in Figure 3.

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Figure 3: (a) Example graphs showing the characteristics of a function. (b) Example graph showing asymptotes of a function. The asymptotes are shown as dashed lines.

4.1 Dependent and Independent Variables

Thus far, all the graphs you have drawn have needed two values, an x-value and a y-value. The y-value is usually determined from some relation based on a given or chosen x-value. These values are given special names in mathematics. The given or chosen x-value is known as the *independent* variable, because its value can be chosen freely. The calculated y-value is known as the *dependent* variable, because its value depends on the chosen x-value.

4.2 Domain and Range

The *domain* of a relation is the set of all the x values for which there exists at least one y value according to that relation. The *range* is the set of all the y values, which can be obtained using at least one x value.

If the relation is of height to people, then the domain is all living people, while the range would be about 0,1 to 3 metres — no living person can have a height of 0m, and while strictly it's not impossible to be taller than 3 metres, no one alive is. An important aspect of this range is that it does not contain *all* the numbers between 0,1 and 3, but at most six billion of them (as many as there are people).

As another example, suppose x and y are real valued variables, and we have the relation $y = 2^x$. Then for *any* value of x, there is a value of y, so the domain of this relation is the whole set of real numbers. However, we know that no matter what value of x we choose, 2^x can never be less than or equal to 0. Hence the range of this function is all the real numbers strictly greater than zero.

These are two ways of writing the domain and range of a function, *set notation* and *interval notation*. Both notations are used in mathematics, so you should be familiar with each.

4.2.1 Set Notation

A set of certain x values has the following form:

$$x : \text{conditions, more conditions} \tag{1}$$

We read this notation as "the set of all x values where all the conditions are satisfied". For example, the set of all positive real numbers can be written as $\{x : x \in \mathbb{R}, x > 0\}$ which reads as "the set of all x values where x is a real number and is greater than zero".

4.2.2 Interval Notation

Here we write an interval in the form '*lower bracket, lower number, comma, upper number, upper bracket*'. We can use two types of brackets, square ones [;] or round ones (;). A square bracket means including the number at the end of the interval whereas a round bracket means excluding the number at the end of the interval. It is important to note that this notation can only be used for all real numbers in an interval. It cannot be used to describe integers in an interval or rational numbers in an interval.

So if x is a real number greater than 2 and less than or equal to 8, then x is any number in the interval

$$(2;8] \tag{2}$$

It is obvious that 2 is the lower number and 8 the upper number. The round bracket means 'excluding 2', since x is greater than 2, and the square bracket means 'including 8' as x is less than or equal to 8.

4.3 Intercepts with the Axes

The *intercept* is the point at which a graph intersects an axis. The x-intercepts are the points at which the graph cuts the x-axis and the y-intercepts are the points at which the graph cuts the y-axis.

In Figure 3(a), the A is the y-intercept and B, C and F are x-intercepts.

You **will** usually need to calculate the intercepts. The two most important things to remember is that at the x-intercept, $y = 0$ and at the y-intercept, $x = 0$.

For example, calculate the intercepts of $y = 3x + 5$. For the y-intercept, $x = 0$. Therefore the y-intercept is $y_{int} = 3(0) + 5 = 5$. For the x-intercept, $y = 0$. Therefore the x-intercept is found from $0 = 3x_{int} + 5$, giving $x_{int} = -\frac{5}{3}$.

4.4 Turning Points

Turning points only occur for graphs of functions whose highest power is greater than 1. For example, graphs of the following functions will have turning points.

$$\begin{aligned} f(x) &= 2x^2 - 2 \\ g(x) &= x^3 - 2x^2 + x - 2 \\ h(x) &= \tfrac{2}{3}x^4 - 2 \end{aligned} \tag{3}$$

There are two types of turning points: a minimal turning point and a maximal turning point. A minimal turning point is a point on the graph where the graph stops decreasing in value and starts increasing in value and a maximal turning point is a point on the graph where the graph stops increasing in value and starts decreasing. These are shown in Figure 4.

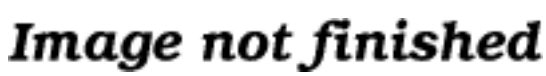

Figure 4: (a) Maximal turning point. (b) Minimal turning point.

In Figure 3(a), E is a maximal turning point and D is a minimal turning point.

4.5 Asymptotes

An asymptote is a straight or curved line, which the graph of a function will approach, but never touch.

In Figure 3(b), the y-axis and line h are both asymptotes as the graph approaches both these lines, but never touches them.

4.6 Lines of Symmetry

Graphs look the same on either side of lines of symmetry. These lines may include the x- and y- axes. For example, in Figure 5 is symmetric about the y-axis. This is described as the axis of symmetry. Not every graph will have a line of symmetry.

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Figure 5: Demonstration of axis of symmetry. The y-axis is an axis of symmetry, because the graph looks the same on both sides of the y-axis.

4.7 Intervals on which the Function Increases/Decreases

In the discussion of turning points, we saw that the graph of a function can start or stop increasing or decreasing at a turning point. If the graph in Figure 3(a) is examined, we find that the values of the graph increase and decrease over different intervals. We see that the graph increases (i.e. that the y-values increase) from -∞ to point E, then it decreases (i.e. the y-values decrease) from point E to point D and then it increases from point D to $+\infty$.

4.8 Discrete or Continuous Nature of the Graph

A graph is said to be continuous if there are no breaks in the graph. For example, the graph in Figure 3(a) can be described as a continuous graph, while the graph in Figure 3(b) has a break around the asymptotes which means that it is not continuous. In Figure 3(b), it is clear that the graph does have a break in it around the asymptote.

4.8.1 Domain and Range

1. The domain of the function $f(x) = 2x + 5$ is -3; -3; -3; 0. Determine the range of f. Click here for the solution[8]
2. If $g(x) = -x^2 + 5$ and x is between - 3 and 3, determine:
   a. the domain of $g(x)$
   b. the range of $g(x)$

   Click here for the solution[9]
3. On the following graph label:
   a. the x-intercept(s)
   b. the y-intercept(s)

[8] http://www.fhsst.org/lxY
[9] http://www.fhsst.org/lxg

c. regions where the graph is increasing
d. regions where the graph is decreasing

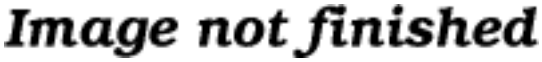

Figure 6

Click here for the solution[10]

4. On the following graph label:
   a. the x-intercept(s)
   b. the y-intercept(s)
   c. regions where the graph is increasing
   d. regions where the graph is decreasing

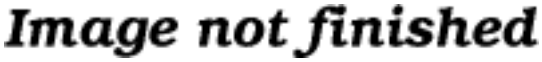

Figure 7

Click here for the solution[11]

[10] http://www.fhsst.org/lx4
[11] http://www.fhsst.org/lx2

Solutions to Exercises in this Module

Solution to Exercise (p. 5)

Step 1.

$$\begin{aligned} f(n) &= n^2 - 6n + 9 \\ f(k-1) &= (k-1)^2 - 6(k-1) + 9 \end{aligned} \tag{4}$$

Step 2.

$$\begin{aligned} &= k^2 - 2k + 1 - 6k + 6 + 9 \\ &= k^2 - 8k + 16 \end{aligned} \tag{5}$$

We have now simplified the function in terms of k.

Solution to Exercise (p. 5)

Step 1.

$$\begin{aligned} f(b) &= b^2 - 4 \\ \approx\approx f(b) &= 45 \end{aligned} \tag{6}$$

Step 2.

$$\begin{aligned} b^2 - 4 &= 45 \\ b^2 - 49 &= 0 \\ b &= +7 \quad \rtimes\setminus \quad -7 \end{aligned} \tag{7}$$