Appendix A*

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This appendix contains outline proofs and derivations for the theorems and formulas given in early part of Chapter . They are not intended to be complete or formal, but they should be sufficient to understand the ideas behind why a result is true and to give some insight into its interpretation as well as to indicate assumptions and restrictions.

Proof 1 *The conditions given by and can be derived by integrating both sides of*

$$\phi(x) = \sum_n h(n)\ \sqrt{M}\,\phi(M\,x - n) \tag{1}$$

and making the change of variables $y = Mx$

$$\int \phi(x)\ dx = \sum_n h(n) \int \sqrt{M}\,\phi(Mx - n)\ dx \tag{2}$$

and noting the integral is independent of translation which gives

$$= \sum_n h(n)\ \sqrt{M}\ \int \phi(y)\,\frac{1}{M}\,dy. \tag{3}$$

With no further requirements other than $\phi \in L^1$ *to allow the sum and integral interchange and* $\int \phi(x)\ dx \neq 0$*, this gives as*

$$\sum_n h(n) = \sqrt{M} \tag{4}$$

and for $M = 2$ *gives . Note this does not assume orthogonality nor any specific normalization of* $\phi(t)$ *and does not even assume* M *is an integer.*

This is the most basic necessary condition for the existence of $\phi(t)$ and it has the fewest assumptions or restrictions.

Proof 2 *The conditions in and are a down-sampled orthogonality of translates by* M *of the coefficients which results from the orthogonality of translates of the scaling function given by*

$$\int \phi(x)\ \phi(x - m)\ dx = E\,\delta(m) \tag{5}$$

in . The basic scaling equation (1) is substituted for both functions in (5) giving

$$\int \left[\sum_n h(n)\ \sqrt{M}\,\phi(Mx - n)\right]\left[\sum_k h(k)\ \sqrt{M}\,\phi(Mx - Mm - k)\right] dx = E\,\delta(m) \tag{6}$$

*Version 1.1: Nov 6, 2012 11:23 am +0000 *which, after reordering and a change of variable $y = M\,x$, gives*

$$\sum_n \sum_k h\left(n\right)\; h\left(k\right) \int \phi\left(y-n\right)\; \phi\left(y-Mm-k\right)\; dy = E\,\delta\left(m\right). \tag{7}$$

Using the orthogonality in (5) gives our result

$$\sum_n h\left(n\right)\; h\left(n-Mm\right) \;\; = \;\; \delta\left(m\right) \tag{8}$$

in and . This result requires the orthogonality condition (5), M must be an integer, and any non-zero normalization E may be used.

Proof 3 (Corollary 2) *The result that*

$$\sum_n h\left(2n\right) = \sum_n h\left(2n+1\right) = 1/\sqrt{2} \tag{9}$$

in or, more generally

$$\sum_n h\left(M\,n\right) = \sum_n h\left(M\,n+k\right) = 1/\sqrt{M} \tag{10}$$

is obtained by breaking (4) for $M = 2$ into the sum of the even and odd coefficients.

$$\sum_n h\left(n\right) = \sum_k h\left(2k\right) + \sum_k h\left(2k+1\right) = K_0 + K_1 = \sqrt{2}. \tag{11}$$

Next we use (8) and sum over n to give

$$\sum_n \sum_k h\left(k+2n\right) h\left(k\right) = 1 \tag{12}$$

which we then split into even and odd sums and reorder to give:

$$\sum_n \left[\sum_k h\left(2k+2n\right) h\left(2k\right) + \sum_k h\left(2k+1+2n\right) h\left(2k+1\right)\right] \tag{13}$$

$$= \sum_k \left[\sum_n h\left(2k+2n\right)\right] h\left(2k\right) + \sum_k \left[\sum_n h\left(2k+1+2n\right)\right] h\left(2k+1\right) \tag{14}$$

$$= \sum_k K_0 h\left(2k\right) + \sum_k K_1 h\left(2k+1\right) = K_0^2 + K_1^2 = 1. \tag{15}$$

Solving (11) and (15) simultaneously gives $K_0 = K_1 = 1/\sqrt{2}$ and our result or (9) for $M = 2$.

If the same approach is taken with and for $M = 3$, we have

$$\sum_n x\left(n\right) = \sum_n x\left(3n\right) + \sum_n x\left(3n+1\right) + \sum_n x\left(3n+2\right) = \sqrt{3} \tag{16}$$

which, in terms of the partial sums K_i, is

$$\sum_n x\left(n\right) = K_0 + K_1 + K_2 = \sqrt{3}. \tag{17}$$

Using the orthogonality condition (8) as was done in (12) and (15) gives

$$K_0^2 + K_1^2 + K_2^2 = 1. \tag{18}$$

Equation (17) and (18) are simultaneously true if and only if $K_0 = K_1 = K_2 = 1/\sqrt{3}$. *This process is valid for any integer* M *and any non-zero normalization.*

Proof 3 *If the support of* $\phi(x)$ *is* $[0, N-1]$, *from the basic recursion equation with support of* $h(n)$ *assumed as* $[N_1, N_2]$ *we have*

$$\phi(x) = \sum_{n=N_1}^{N_2} h(n)\ \sqrt{2}\,\phi(2x-n) \tag{19}$$

where the support of the right hand side of (19) is $[N_1/2, (N-1+N_2)/2)$. *Since the support of both sides of (19) must be the same, the limits on the sum, or, the limits on the indices of the non zero* $h(n)$ *are such that* $N_1 = 0$ *and* $N_2 = N$, *therefore, the support of* $h(n)$ *is* $[0, N-1]$.

Proof 4 *First define the autocorrelation function*

$$a(t) = \int \phi(x)\ \phi(x-t)\ dx \tag{20}$$

and the power spectrum

$$A(\omega) = \int a(t)\ e^{-j\omega t}\,dt = \int\int \phi(x)\ \phi(x-t)\ dx\ e^{-j\omega t}\,dt \tag{21}$$

which after changing variables, $y = x - t$, *and reordering operations gives*

$$A(\omega) = \int \phi(x)\ e^{-j\omega x}\,dx\ \int \phi(y)\ e^{j\omega y}\,dy \tag{22}$$

$$= \Phi(\omega)\ \Phi(-\omega) = |\Phi(\omega)|^2 \tag{23}$$

If we look at (20) as being the inverse Fourier transform of (23) and sample $a(t)$ *at* $t = k$, *we have*

$$a(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\Phi(\omega)|^2\, e^{j\omega k}\,d\omega \tag{24}$$

$$= \frac{1}{2\pi}\sum_{\ell}\int_0^{2\pi} |\Phi(\omega+2\pi\ell)|^2\, e^{j\omega k}\,d\omega = \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{\ell}|\Phi(\omega+2\pi\ell)|^2\right]\ e^{j\omega k}\,d\omega \tag{25}$$

but this integral is the form of an inverse discrete-time Fourier transform (DTFT) which means

$$\sum_{,} a(k)\ e^{j\omega k} = \sum_{\ell}|\Phi(\omega+2\pi\ell)|^2. \tag{26}$$

If the integer translates of $\phi(t)$ *are orthogonal,* $a(k) = \delta(k)$ *and we have our result*

$$\sum_{\ell}|\Phi(\omega+2\pi\ell)|^2 = 1. \tag{27}$$

If the scaling function is not normalized

$$\sum_{\ell}|\Phi(\omega+2\pi\ell)|^2 = \int |\phi(t)|^2\,dt \tag{28}$$

which is similar to Parseval's theorem relating the energy in the frequency domain to the energy in the time domain.

Proof 6 *Equation states a very interesting property of the frequency response of an FIR filter with the scaling coefficients as filter coefficients. This result can be derived in the frequency or time domain. We will* *show the frequency domain argument. The scaling equation (1) becomes in the frequency domain. Taking the squared magnitude of both sides of a scaled version of*

$$\Phi(\omega) = \frac{1}{\sqrt{2}} H(\omega/2)\ \Phi(\omega/2) \tag{29}$$

gives

$$|\Phi(2\omega)|^2 = \frac{1}{2}|H(\omega)|^2 |\Phi(\omega)|^2 \tag{30}$$

Add $k\pi$ to ω and sum over k to give for the left side of (30)

$$\sum_k |\Phi(2\omega + 2\pi k)|^2 = K = 1 \tag{31}$$

which is unity from . Summing the right side of (30) gives

$$\sum_k \frac{1}{2}|H(\omega + k\pi)|^2 |\Phi(\omega + k\pi)|^2 \tag{32}$$

Break this sum into a sum of the even and odd indexed terms.

$$\sum_k \frac{1}{2}|H(\omega + 2\pi k)|^2 |\Phi(\omega + 2\pi k)|^2 + \sum_k \frac{1}{2}|H(\omega + (2k+1)\pi)|^2 |\Phi(\omega + (2k+1)\pi)|^2 \tag{33}$$

$$= \frac{1}{2}|H(\omega)|^2 \sum_k |\Phi(\omega + 2\pi k)|^2 + \frac{1}{2}|H(\omega + \pi)|^2 \sum_k |\Phi(\omega + (2k+1)\pi)|^2 \tag{34}$$

which after using (31) gives

$$= \frac{1}{2}|H(\omega)|^2 + \frac{1}{2}|H(\omega + \pi)|^2 = 1 \tag{35}$$

which gives . This requires both the scaling and orthogonal relations but no specific normalization of $\phi(t)$. If viewed as an FIR filter, $h(n)$ is called a quadrature mirror filter (QMF) because of the symmetry of its frequency response about π.

Proof 10 *The multiresolution assumptions in require the scaling function and wavelet satisfy and*

$$\phi(t) = \sum_n h(n)\ \sqrt{2}\,\phi(2t - n), \qquad \psi(t) = \sum_n h_1(n)\ \sqrt{2}\,\phi(2t - n) \tag{36}$$

and orthonormality requires

$$\int \phi(t)\ \phi(t-k)\ dt = \delta(k) \tag{37}$$

and

$$\int \psi(t)\ \phi(t-k)\ dt = 0 \tag{38}$$

for all $k \in \mathbf{Z}$. Substituting (36) into (38) gives

$$\int \sum_n h_1(n)\ \sqrt{2}\,\phi(2t - n)\ \sum_\ell h(\ell)\ \sqrt{2}\,\phi(2t - 2k - \ell)\ dt = 0 \tag{39}$$

Rearranging and making a change of variables gives

$$\sum_{n,\ell} h_1(n)\ h(\ell)\ \frac{1}{2}\int \phi(y-n)\ \phi(y-2k-\ell)\ dy = 0 \tag{40}$$

Using (37) gives

$$\sum_{n,\ell} h_1(n)\ h(\ell)\ \delta(n-2k-\ell) = 0 \tag{41}$$

for all $k \in \mathbf{Z}$. Summing over ℓ gives

$$\sum_{n} h_1(n)\ h(n-2k) = 0 \tag{42}$$

Separating (42) into even and odd indices gives

$$\sum_{m} h_1(2m)\ h(2m-2k) + \sum_{\ell} h_1(2\ell+1)\ h(2\ell+1-2k) = 0 \tag{43}$$

which must be true for all integer k. Defining $h_e(n) = h(2n)$, $h_o(n) = h(2n+1)$ and $\tilde{g}(n) = g(-n)$ for any sequence g, this becomes

$$h_e \,\text{[U+2606]}\, \tilde{h}_{1e} + h_o \,\text{[U+2606]}\, \tilde{h}_{1o} = 0. \tag{44}$$

From the orthonormality of the translates of ϕ and ψ one can similarly obtain the following:

$$h_e \,\text{[U+2606]}\, \tilde{h}_e + h_o \,\text{[U+2606]}\, \tilde{h}_o = \delta. \tag{45}$$

$$h_{1e} \,\text{[U+2606]}\, \tilde{h}_{1e} + h_{1o} \,\text{[U+2606]}\, \tilde{h}_{1o} = \delta. \tag{46}$$

This can be compactly represented as

$$\begin{bmatrix} h_e & h_o \\ h_{1e} & h_{1o} \end{bmatrix} \text{[U+2606]} \begin{bmatrix} \tilde{h}_e & \tilde{h}_{1e} \\ \tilde{h}_o & \tilde{h}_{1o} \end{bmatrix} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}. \tag{47}$$

Assuming the sequences are finite length (47) can be used to show that

$$h_e \,\text{[U+2606]}\, h_{1o} - h_o \,\text{[U+2606]}\, h_{1e} = \pm\delta_k, \tag{48}$$

where $\delta_k(n) = \delta(n-k)$. Indeed, taking the Z-transform of (47) we get using the notation of Chapter $H_p(z)\, H_p^T(z^{-1}) = I$. Because, the filters are FIR $H_p(z)$ is a (Laurent) polynomial matrix with a polynomial matrix inverse. Therefore the determinant of $H_p(z)$ is of the form $\pm z^k$ for some integer k. This is equivalent to (48). Now, convolving both sides of (48) by $\tilde{h}_e$ we get

$$\begin{aligned} \pm\tilde{h}_e \,\text{[U+2606]}\, \delta_k &= \left[h_e \,\text{[U+2606]}\, h_{1o} - h_o \,\text{[U+2606]}\, h_{1e}\right] \text{[U+2606]}\, \tilde{h}_e \\ &= \left[h_e \,\text{[U+2606]}\, \tilde{h}_e \,\text{[U+2606]}\, h_{1o} - h_{1e} \,\text{[U+2606]}\, \tilde{h}_e \,\text{[U+2606]}\, h_o\right] \\ &= \left[h_e \,\text{[U+2606]}\, \tilde{h}_e \,\text{[U+2606]}\, h_{1o} + h_{1o} \,\text{[U+2606]}\, \tilde{h}_o \,\text{[U+2606]}\, h_o\right] \\ &= \left[h_e \,\text{[U+2606]}\, \tilde{h}_e + h_o \,\text{[U+2606]}\, \tilde{h}_o\right] \text{[U+2606]}\, h_{1o} \\ &= h_{1o}. \end{aligned} \tag{49}$$

Similarly by convolving both sides of (48) by $\tilde{h}_o$ *we get*

$$\mp\tilde{h}_o\text{[U+2606]}\delta_k = h_{1e}. \tag{50}$$

Combining (49) and (50) gives the result

$$h_1(n) = \pm(-1)^n h(-n+1-2k). \tag{51}$$

Proof 11 *We show the integral of the wavelet is zero by integrating both sides of ((36)b) gives*

$$\int \psi(t)\, dt = \sum_n h_1(n) \int \sqrt{2}\,\phi(2t-n)\, dt \tag{52}$$

But the integral on the right hand side is A_0*, usually normalized to one and from or (9) and (51) we know that*

$$\sum_n h_1(n) \;=\; 0 \tag{53}$$

and, therefore, from (52), the integral of the wavelet is zero.

The fact that multiplying in the time domain by $(-1)^n$ *is equivalent to shifting in the frequency domain by* π *gives* $H_1(\omega) = H(\omega+\pi)$.

References