COMPRESSIBLE SIGNALS*

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Abstract

This module describes compressible signals, i.e., signals that can be well-approximated by sparse signals.

1 Compressibility and K-term approximation

An important assumption used in the context of compressive sensing (CS) is that signals exhibit a degree of structure. So far the only structure we have considered is sparsity, i.e., the number of non-zero values the signal has when representation in an orthonormal basis $\Psi$. The signal is considered sparse if it has only a few nonzero values in comparison with its overall length.

Few structured signals are truly sparse; rather they are compressible. A signal is compressible if its sorted coefficient magnitudes in $\Psi$ decay rapidly. To consider this mathematically, let $x$ be a signal which is compressible in the basis $\Psi$:

$$x = \Psi \alpha,$$

where $\alpha$ are the coefficients of $x$ in the basis $\Psi$. If $x$ is compressible, then the magnitudes of the sorted coefficients $\alpha_s$ observe a power law decay:

$$|\alpha_s| \leq C_1 s^{-q}, s = 1, 2, ...$$

We define a signal as being compressible if it obeys this power law decay. The larger $q$ is, the faster the magnitudes decay, and the more compressible a signal is. Figure 2 shows images that are compressible in different bases.

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Figure 2: The image in the upper left is a signal that is compressible in space. When the pixel values are sorted from largest to smallest, there is a sharp descent. The image in the lower left is not compressible in space, but it is compressible in wavelets since its wavelet coefficients exhibit a power law decay.

Because the magnitudes of their coefficients decay so rapidly, compressible signals can be represented well by $K \ll N$ coefficients. The best $K$-term approximation of a signal is the one in which the $K$ largest coefficients are kept, with the rest being zero. The error between the true signal and its $K$ term approximation is denoted the $K$-term approximation error $\sigma_K(x)$, defined as

$$\sigma_K(x) = \arg \min_{\alpha \in \Sigma_K} \| x - \Psi \alpha \|_2.$$  

(3)
For compressible signals, we can establish a bound with power law decay as follows:

\[ \sigma_K(x) \leq C_2 K^{1/2-s}. \]  

(4)

In fact, one can show that \( \sigma_K(x) \) will decay as \( K^{-r} \) if and only if the sorted coefficients \( \alpha_i \) decay as \( i^{-r+1/2} \) [1]. Figure 4 shows an image and its \( K \)-term approximation.

![Figure 4: Sparse approximation of a natural image. (a) Original image. (b) Approximation of image obtained by keeping only the largest 10% of the wavelet coefficients. Because natural images are compressible in a wavelet domain, approximating this image in terms of its largest wavelet coefficients maintains good fidelity.](http://cnx.org/content/m37166/1.5/)

**2 Compressibility and \( \ell_p \) spaces**

A signal's compressibility is related to the \( \ell_p \) space to which the signal belongs. An infinite sequence \( x(n) \) is an element of an \( \ell_p \) space for a particular value of \( p \) if and only if its \( \ell_p \) norm is finite:

\[ \| x \|_p = \left( \sum_i |x_i|^p \right)^{1/p} < \infty. \]  

(5)

The smaller \( p \) is, the faster the sequence’s values must decay in order to converge so that the norm is bounded. In the limiting case of \( p = 0 \), the “norm” is actually a pseudo-norm and counts the number of non-zero values. As \( p \) decreases, the size of its corresponding \( \ell_p \) space also decreases. Figure 5 shows various \( \ell_p \) unit balls (all sequences whose \( \ell_p \) norm is 1) in 3 dimensions.
Figure 5: As the value of $p$ decreases, the size of the corresponding $\ell_p$ space also decreases. This can be seen visually when comparing the size of the spaces of signals, in three dimensions, for which the $\ell_p$ norm is less than or equal to one. The volume of these $\ell_p$ “balls” decreases with $p$.

Suppose that a signal is sampled infinitely finely, and call it $x[n]$. In order for this sequence to have a bounded $\ell_p$ norm, its coefficients must have a power-law rate of decay with $q > 1/p$. Therefore a signal which is in an $\ell_p$ space with $p \leq 1$ obeys a power law decay, and is therefore compressible.

References