Chapter 01: Number Systems and the Invention of Positional Notation*

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This chapter discusses the differences between counting systems of notation for numbers and positional systems and introduces the real numbers. The invention of positional notation was the first profound mathematical advance. It made accurate and efficient calculations possible. Positional notations with respect to a general base are described. The Egyptian and Greek nonpositional notations are contrasted with the much more powerful Babylonian positional notation which uses the base 60.

1

The tally notches incised 30,000 years ago in the wolf’s bone described in the introduction are examples of symbols used to denote the cardinal numbers 1, 2, 3, . . . ; each notch corresponds to one unit. A similar primitive system is in common use today for counting a number of objects; strokes are grouped in fives like this: |||||

Repetition of a fundamental symbol to record a tally is cumbersome and space consuming. The number represented cannot be read off easily; it must be counted off, and it does not permit the development of arithmetic. For instance, in order to add tallies, say ||||||| and ||||, it is necessary only to group both sets together; thus |||||||||||. But what is the sum? Each complex civilization invented a notation or symbolism to represent large numbers and developed methods of performing elementary arithmetic operations with their symbols, although different civilizations attained varying degrees of effectiveness and sophistication. A well-known example of a notational system based on counting is the Roman, employed extensively in Western civilization until about 1400. Their symbols I, V, X, L, C, D stand for 1, 5, 10, 50, 100 and 500, respectively, and are now used to denote chapter numbers and positions of the hours on clock faces and for other decorative purposes. This system was supplanted by the present system of Arabic numerals, which is superior in many ways, as we shall see.

Measurement of distances immediately demands that the system of notation for numbers permits the expression of fractions corresponding to distances shorter than the basic unit of measure. For instance, if that unit is the meter, there must be a means of expressing centimeters; if it is the centimeter, there must be a means of expressing millimeters, and so on. However small the unit chosen, there will always be distances to be measured that are not integral multiples of it and that will therefore have to be represented

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as a fractional part. Systems of notation based on tally counting are obviously poorly suited to measuring distance and similar quantities.

Some of the systems of notation developed by early civilizations had a few special symbols for common fractions lumped together with a counting system notation; others, particularly the one introduced by the Akkadians, were almost as complete as our own decimal system but somewhat less efficient and convenient.

2

We discuss the methods used for representing numbers by the early Egyptian, Akkadian, Greek, and Roman civilizations in order to show how well adapted each was to the measurement of geometrical quantities. First consider the problem of representing whole numbers.

Consider a positive integer, the number 36,521, for instance. Recalling that its rightmost digit corresponds to the number of units, the next digit to tens, the third to hundreds, and so on with higher powers of the base 10, we see that 36,521 is shorthand notation for

\[ 3 \times 10^4 + 6 \times 10^3 + 5 \times 10^2 + 2 \times 10^1 + 1 \times 10^0 \]

where \( 10^4 = 10,000 \), \( 10^3 = 1,000 \), \( 10^2 = 100 \), \( 10^1 = 10 \), and \( 10^0 = 1 \). More generally, we know today that any positive whole number \( n \) can be represented in decimal notation as

\[ n = a_r a_{r-1} \ldots a_0 \]

where \( a_0, \ldots, a_r \) are symbols, each of which stands for one of the numbers 0, 1, \ldots, 9. This decimal shorthand stands for the positional representation

\[ n = a_r \cdot 10^r + a_{r-1} \cdot 10^{r-1} + \ldots + a_0 \]

Moreover, we can take any positive integer \( g > 1 \), write

\[ n = b_s b_{s-1} \ldots b_0 \]

with each of \( b_0, \ldots, b_s \) standing for one of the numbers 0, 1, \ldots, \( g - 1 \) and call the sequence of symbols \( b_s b_{s-1} \ldots b_0 \) the positional representation for \( n \) in the base \( g \). The position of a symbol determines the power of \( g \) it multiplies, hence its value.

Various positional number systems of different bases have been used by past civilizations, although the base 10 has been dominant. For instance, 5, 12, 20, and 60 have been used as bases; our present-day clock shows the influence of both base 12 and base 60. There is no universally best base, but for different purposes different bases will serve best. Historically there were various reasons for each system's use.

Modern computers employ combinations of the binary system (base 2), the octal system (base 8), the decimal system (base 10), the duodecimal system (base 12), and the hexadecimal system (base 16). The binary system is used to simplify computation, the hexadecimal to save storage space (the number of places needed to represent a given number is smaller, the larger the base, but correspondingly the number of symbols needed is larger), and the decimal to communicate with people. Hexadecimal digits are often expressed by pairs of decimal digits, but sometimes symbols drawn from the alphabet are used.

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Among the ancient cultures the only ones that used a positional system to represent numbers were the Babylonians and their predecessors, the Sumerians (of whom no written mathematical works survive). The Egyptians, Greeks, and Romans had more primitive, only partly positional, decimal systems that utilized counting by powers of 10 but required the introduction of a new symbol for each power of 10 rather than a new position. For example, the Egyptians used the hieroglyphic symbols shown in Figure 1 and expressed numbers by their juxtaposition (representing addition of the values represented by the symbols). Thus

The analogy with Roman numerals is clear; the Romans used additional symbols to represent 5, 50, etc, (V, L . . . ) so that there would be less writing involved, but the principle is the same.

The Greeks used their alphabetic characters to represent numbers in a manner similar to the Egyptians and the Romans but with an enormous saving of space (Table 1: Greek numerals.).
Note that three letters (, , ) in this table are not in the usual Greek alphabet. According to Heath’s Greek Mathematics¹, a multitude of Greek alphabets was derived from the earlier Phoenician syllabary, each with its own variations (cp. the variation in the modern Russian and Ukrainian alphabets). The first two extra letters were kept in their original places for use as numerals, even though they had fallen out of literary use. The last letter already discarded, was tacked on at the end, since it no longer had a natural place at the time of the invention of the numeration system (about -700).

### Greek numerals.

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<td>80</td>
<td>π</td>
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<td>9</td>
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<td>theta</td>
<td>90</td>
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<td>α</td>
<td>1000,</td>
<td>2000,</td>
<td></td>
<td>etc.</td>
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| 100 | ρ | rho |
| 200 | σ | sigma |
| 300 | τ | tau |
| 400 | υ | upsilon |
| 500 | φ | phi |
| 600 | χ | chi |
| 700 | ψ | psi |
| 800 | ω | omega |
| 900 | sampi |

This system of numeration had the advantage over the Roman (and a previous Greek system, similar to the Roman, called Attic) that much less space was needed to represent a given number; for example,

\[
849 = \omega \mu \theta = \Gamma \Pi \Pi \Pi \Pi \Pi \Pi \Pi
\]

We can guess what the various Attic symbols must mean from the context. The new numbers had the political advantage that they could be stamped on coins. The great Greek mathematicians, such as Archimedes, developed remarkable skill in computing with the alphabetic numeration system, as we show in . The Attic system was primarily used to denote ordinal numbers, much as we use Roman numerals today to denote the number of a given chapter in a book, a volume in a serial publication, or an hour on the face of a clock.

For large numerals the Greeks wrote the symbol Μ (Attic for 10,000) with alphabetic numerals above it;

---

for example,

\[
\begin{align*}
\frac{1}{2} &= 0.5 \\
\frac{1}{3} &= 0.333...
\end{align*}
\]

\[
\begin{align*}
\frac{1}{4} &= 0.25 \\
\frac{1}{5} &= 0.2 \\
\frac{1}{6} &= 0.1666...
\end{align*}
\]

\[
\begin{align*}
\frac{1}{7} &= 0.142857142857... \\
\frac{1}{8} &= 0.125
\end{align*}
\]

(2)

The \( M \) serves as a place system on which a number system of base 10,000 could be built up by inventing symbols (or positions) for higher order powers of \( M = 10,000 \) (a myriad). Something similar was done by Archimedes in his *Sandrecker*, wherein he estimated the number of grains of sand in the universe by making certain "astronomical assumptions" about its size. He "computed" with numbers to base \( 10^8 \) (the *second myriad* \( = 10^4 \cdot 10^4 = M \cdot M \)).

In fact Archimedes considered all numbers from 1 to \( 10^8 \) to be of first order and took the last number \( 10^8 \) as the unit of numbers of the second order (\( 10^8 \) to \( 10^{16} \)), up to numbers of the \( 10^8 \)-th order [all numbers from \( 10^8(10^8-1) \) to \( 10^{8 \cdot 10^8} \)]. All numbers from 1 to the \( 10^8 \)-th order form the first period; that is, if \( P = (100,000,000)^{10^8} \), then the first period consists of the numbers between 1 and \( P \). \( P \) is the unit of the first order of the second period, that is, the numbers from \( P \) to \( 10^8 \cdot P \); continue in this way to construct \( 10^8 \cdot P \) to \( 10^{16} \cdot P \), etc. Archimedes ends with The Period, which is given by

\[
\text{The Period } = P^{10^8} = \left[ \left(10^8\right)^{10^8}\right]^{10^8}
\]

(2)

The next basic conceptual step, which occurred much earlier chronologically, was taken by the Babylonians who invented the sexagesimal system, that is, a base 60 positional number system. This system was used strictly for scientific purposes and in a very sophisticated manner. First we describe their notation and then employ a transliteration device invented by O. Neugebauer so that we can later analyze Babylonian mathematics in context. Basically they had two numerical symbols, \( \Upsilon \) and \( \kappa \), which correspond to the one and ten of a primitive decimal system, such as the Egyptian \( \text{I} \) and \( \text{n} \). Numbers smaller than 60 were formed in a straightforward fashion.

<table>
<thead>
<tr>
<th>3</th>
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<td>25</td>
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<td>49</td>
<td>( \kappa \kappa \kappa \kappa \kappa )</td>
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Table 2

which is as primitive and cumbersome as the Egyptian-Greco-Roman notations. Later on they introduced shorthand versions, for example,
The basic wedge (cuneiform) symbols \( \text{𒃹} \) and \( \text{𒌦} \) formed the characters in the Babylonian writing system as well as their numbers.

These symbols are difficult to write; they were originally impressed on soft clay by a wedge-shaped instrument, as shown schematically in Figure 3. This was an advance over earlier methods of Babylonian writing which corresponded to engraving (hieroglyphic style characters taken from daily life) on a clay surface with a sharp stylus. Gradually these forms developed into a syllabary for writing and keeping records, which principally concerned economic matters. There were two fortunate finds of libraries in which thousands of clay tablets were assembled, some of which correspond to the mathematics sections of our own libraries. An important selection of the collection has been deciphered, analyzed, and published (with pictures or sketches of the actual clay tablets) by Neugebauer and Sachs, Mathematical Cuneiform Texts\(^2\). Analyses of these and similar tablets are our principal and most reliable source of information about the state of mathematics and astronomy in Babylonia from about -3000 to -200.

\[ \begin{array}{c|c}
9 & \text{𒃹𒃹} \\
\end{array} \]

Table 3

With this system of writing, based on the symbols \( \text{𒃹} \) and \( \text{𒌦} \), it is clear that the Babylonians could easily record numbers up to, say, 100, without requiring any new symbols. Their remarkable achievement, one of the most distinguished in antiquity, was that with these two symbols they were able to systematically represent arbitrarily large numbers and arbitrarily small fractions. For instance, they wrote \( \text{𒃹𒃹} \) \( \text{𒌦} \) \( \text{𒃹} \) (in which the first \( \text{𒃹} \) has the value 60) and read it as \( 60 + 11 = 71 \). Similarly,

\[ \text{𒃹𒃹} \text{𒌦} = 2 \cdot 60 + 23 = 143 \]

\[ \text{𒌦 الكتابة} = 11 \cdot 60 + 10 = 670 \]

This process could be carried on to three or more places, and each time a new power of 60 would appear:


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This example shows how (and why!) an individual number such as 3 was written with vertices touching to distinguish the symbols from the notation for two distinct numbers, each with a different value. The analogy to our decimal system is quite clear; this system is called the sexagesimal number system.

As used by the early Babylonians, the sexagesimal number system lacked two basic features of our modern decimal system. There was no zero and there was no "sexagesimal point." Later on the special symbol was introduced to denote an unfilled position, as in

$$\gamma \gamma \gamma = 1 \cdot 60^2 + 0 \cdot 60 + 1$$

$$= 3601$$

but initially there was no set way to tell which power of 60 a given \( \gamma \) represented, except from the context and the fact that the powers decreased to the right. The spacing between the basic numerals was not uniform. Presumably most of their practical work was done in context, and there was no confusion, just as when we ask "how much is a loaf of bread" and the answer given is "thirty," it is clear that we do not mean $30.

Because their system lacked a decimal point, the symbol \( \gamma \) could also mean 3600, 1/60, or 1/3600, depending on the context, but as we know from our system, as far as computations are concerned, we need only make sure that we add (or subtract) the corresponding coefficients of a given power of 60, whereas for multiplication and division we operate with the coefficients and place the decimal point correctly after the computation. In practice, we often know ahead of time where the decimal point will finally be and only need to carry out the computation on the various place entries. This was, as we shall see, the secret of Babylonian computational ability and why their astronomy and algebra were far superior to that of their Egyptian contemporaries.

Following Neugebauer, we represent a Babylonian sexagesimal number by the following scheme:

$$\ldots a, b, c; d, e, \ldots$$

(3)

where \( a, b, c, \ldots \) are (decimal) numbers between 0 and 59 and the semicolon represents the sexagesimal point, which separates the integral part of the number from the nonintegral part just as the decimal point does in the decimal system. For instance, 12, 20; 21 represents the number

$$12 \cdot 60 + 20 + 21 \cdot \frac{1}{60} = 740 \frac{7}{20}$$

(3)

A system of numerical notations must be capable of expressing parts of a whole as well as whole (integral) quantities. All positional notations have this capability. The following table, which shows how reciprocals
of small integers are represented in Neugebauer's sexagesimal notation, is easy to construct:

\[
\begin{align*}
\frac{1}{7} &= 0; 30 \\
\frac{1}{3} &= 0; 20 \\
\frac{1}{2} &= 0; 15 \\
\frac{1}{5} &= 0; 12 \\
\frac{1}{6} &= 0; 10 \\
\frac{1}{7} &= 0; 8, 34, 17, 8, 34, 17, 8, 34, 17, ... \\
\frac{1}{8} &= 0; 7, 30 \\
\end{align*}
\]

(3)

The decimal (base 10) representations of these fractions are familiar:

\[
\begin{align*}
\frac{1}{2} &= 0.5 \\
\frac{1}{3} &= 0.333... \\
\frac{1}{4} &= 0.25 \\
\frac{1}{5} &= 0.2 \\
\frac{1}{6} &= 0.16666... \\
\frac{1}{7} &= 0.142857142857... \\
\frac{1}{8} &= 0.125 \\
\end{align*}
\]

(3)

In the binary system the same numbers appear in a different guise:

\[
\begin{align*}
\frac{1}{2} &= (0.1)_2 \\
\frac{1}{3} &= (0.010101...)_2 \\
\frac{1}{4} &= (0.01)_2 \\
\end{align*}
\]

(3)

where \((\ )_2\) denotes the base 2 representation.

By examining these expressions it is easy to convince ourselves that some fractions have finite expressions, whereas others appear to have unending but ultimately repetitive expressions. Moreover, no matter what base is chosen, some fractions will always have unending expressions relative to that base.

These observations lead us to inquire about the nature of numbers expressed by infinite expansions in, say, the decimal system of notation, and this in turn leads to a characterization of the real number system which underlies all the mathematical accomplishments described in this book.
4

Much of mathematics is, and was, concerned with properties of the real number system. The integers (the set of positive and negative whole numbers, together with 0: . . . , −4, −3, −2, −1, 0, 1, 2, 3 . . . ) are included amongst the real numbers. The rational numbers, that is, the ratios of integers (with denominator different from 0, of course), are also contained in the embracing collection of real numbers. This means that all numbers of the form p/q, where p and q are integers and q is different from 0, are real numbers. You already know that there are some "real" numbers that are not rational-these are called, unimaginatively enough, irrational numbers. Here are three examples: 1 + √3, √2, and π.

It is not hard to prove that 1 + √3, √2, and π and are irrational — we do so later on in this chapter but the proof that π (the ratio of the circumference of a circle to its diameter) is irrational is much more difficult. In any case, how often do numbers like 1 + √3 or √2 occur in the normal course of human affairs? And, although π is more popular — perhaps, you may think, because wheels are useful in daily life — it nevertheless seems to be a unique type of number, the sole example of its species. Can you think of another number as strange as π? Perhaps you have heard of one: e, the base for natural logarithms (1).

Your experience may lead you to believe that most real numbers must be rational or at least that all of the "important" numbers, with perhaps a few exceptions admitted to spice the pie, are rational and that the irrationals for all practical and most other purposes could well be ignored; that the "real numbers" are really the friendly fractions of old, dressed up in a new name. If you so believe, gentle reader, you're in for a surprise.

Consider, for instance, why it is that when we refer to a particular rational number, say "six-sevenths," it is described in a straightforward way, namely, 6/7; but the description of some of these irrational numbers recalls to mind the practice of certain cults which forbear the explicit mention of the name of a certain deed or god or edible. For instance, we write π or e, which conceals as much as it reveals. These letters from alphabets, foreign and domestic, are certainly poor substitutes for a concrete description of a real number, like 6/7. We want to understand why it is that some numbers must be expressed with help from outside the ordinary apparatus of arithmetic. It must already be clear to you that if there really are such numbers we would be wise to avoid using them if possible, especially in daily life, for otherwise what a muddle would result from the inexplicit nature of their descriptions. One of our state legislatures decreed that π = 3, but not even a village council will try to assert that 1 = 2. Why?

What is a real number? We can give you one perfectly good definition which has the advantage of using a body of knowledge that you already possess. A real number is a number expressed by a decimal expansion. You are all familiar with decimal expansions. A number expressed in this form can be written as

\[ x = a_n a_{n-1} a_{n-2} \cdots a_1 a_0 \cdot a_{-1} a_{-2} a_{-3} a_{-4} \cdots \]  

(3)

where each of the numbers ai is a nonnegative integer less than 10 and the overemphasized dot "." denotes the decimal point. The sequence of a's to the right of the decimal point may or may not terminate, and here of course is the difficult and important point; when the decimal expansion does not terminate, we are no longer considering a simple finite process. Intuition is easily led astray, and notions that in normal finite circumstances are of the most elementary character must in the infinite situation be redefined and used with utmost care.

The sequence of integers a−1, · · · a−1 a0 is called the integral part of \( x \); it may be 0. The sequence of integers to the right of the decimal point is sometimes mis-named the fractional part of \( x \); we shall see that it cannot always be presented as a fraction.

Here are a few examples of real numbers to illustrate what the three dots to the right of \( a_{-1} \) mean in the above expression. A real number is given when each term of the expansion ((3)) is well defined. Consider
the expressions

(a) 0.50000...
(b) 0.6666...
(c) 0.857142857142...
(d) 0.2121121121112...
(e) 3.141596...
(f) 1.414213...
(g) 0.156231...

Which of these represent real numbers? This is a matter of interpretation. In (a) and (b) the obvious rule is that 0 in the first expression and 6 in the second should be indefinitely repeated; these expansions then represent the real numbers $\frac{1}{2}$ and $\frac{2}{3}$, respectively. Similarly, in the next expression (c) we see that the block of six numbers “857142” is to be repeated, and this represents a rule for determining the value of any term arbitrarily far out in the expansion. (This expansion represents the fraction $\frac{2}{7}$) For (d) we see that the rule is clear from the context: a “2n” followed by one more “1” than the preceding time around. This rule will allow anyone to write out the decimal expansion to as many decimal places as desired; hence it is well determined and well defined. In (e), (f), and (g) we have expressions in which the rule for forming the next terms in the sequence is not clear from what is written down. Therefore with only this information the last three expressions given do not represent real numbers unless we give more information. We can specify (e) by requiring that the expansion for (e) be the ratio of the circumference to the diameter of any circle (usually denoted by $\pi$). This will uniquely specify the remainder of the expansion, not an easy fact to understand completely. In (f) we require that the expansion represent a positive number whose square is 2 (the expansion should represent $\sqrt{2}$). Again this uniquely specifies some positive real number and will uniquely determine the rest of the terms; that is, we can find an algorithm that will give increasingly accurate approximations to the decimal expansion (see ). In (g) we have a sequence of six numbers which has been written down with no apparent rule in mind. Hence the remaining terms (...) are not well determined; for example, one person might write 0.1562313 and another might write 0.1562314444..., both of which are now well determined and obviously distinct real numbers.

We have given examples of real numbers expressed as decimal expansions. Since we want to add, subtract, multiply, and divide such numbers, we may well ask how to proceed. Let us illustrate the simple answer by some examples. Suppose we wanted to add the expressions in (b) and (c). We could proceed as follows: (b) is $\frac{2}{3}$, (c) is $\frac{6}{7}$, so $(b) + (c) = \frac{2}{3} + \frac{6}{7} = \frac{22}{21}$, and by long division we can see that $\frac{2}{3}$ has the decimal expansion 1.52380... Thus we have added the two expansions in (b) and (c). This procedure, however, really avoids the real problem because we used the rules for adding fractions, which are not applicable to those decimal expansions that do not correspond to fractions. Suppose we try to add (c) to (d). What do we do? Since (d) is not a fraction, none of the usual rules applies. The correct idea is to construct a sequence of finite decimal expansions which approaches, that is, approximates the value of, the desired sum:

\[
\begin{align*}
0.8 + 0.2 &= 1.0 \\
0.85 + 0.21 &= 1.06 \\
0.857 + 0.212 &= 1.069 \\
0.8571 + 0.2121 &= 1.0692 \\
0.85714 + 0.21211 &= 1.06925
\end{align*}
\]

Thus the sum (c) + (d) is given by 1.06925..., and the rule for determining the rest of the terms is clear.

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Similarly, for (b) + (c), we have

\[
0.6 + 0.8 = 1.4 \\
0.66 + 0.85 = 1.51 \\
0.666 + 0.857 = 1.523 \\
0.6666 + 0.8571 = 1.5237 \\
0.66666 + 0.85714 = 1.52380
\] (3)

We note three things here:
1. A finite approximating expansion may differ in its last digit from the real expansion at that same level but nowhere else; e.g. 1.51 but 1.523. The rule is still well defined.
2. We get precisely the same expansion as we did above by the long division \( \frac{32}{21} \).
3. It is not necessary to know that the sum represents the fraction \( \frac{32}{21} \) to be able to compute it.

In a similar manner we can easily subtract, multiply, and divide decimal expansions of numbers.

Is it ever the case that two decimal expansions represent the same number? What should we mean by "same"? Let us agree that two real numbers are the same (although possibly expressed in different external forms) if their difference is smaller than any number we might choose. Then you will see immediately (if you do not, stick with it for a few moments anyway) that the two decimal expansions

\[
y = b_n \cdot \cdot b_0.\cdot b_{-1} \cdot \cdot \cdot \cdot b_{-k}000000.\ldots \tag{3}
\]

and

\[
x = b_n \cdot \cdot b_0.\cdot b_{-1} \cdot \cdot \cdot \cdot (b_{-k} - 1)9999.\ldots \tag{3}
\]

represent the same number if \( b_{-k} \) is greater than 0; for example,

\[
y = 2.5630000.\ldots \tag{3}
\]

and

\[
x = 2.5629999.\ldots \tag{3}
\]

represent the same real number.

If one decimal expansion continues with all 9's from some point on, it represents the same number as the corresponding expansion with the 9's replaced by 0 and the digit preceding the 9's increased by 1. This is the most general case possible; in every other instance different decimal expansions represent different real numbers.

Now let us see what the decimal expansions that correspond to rational numbers look like. Examples (a) through (c) above exhibit decimal expansions of rational numbers; they each ultimately repeat some sequence of digits indefinitely. In general, if \( x = p/q \) is a rational number, its decimal expansion can be found by long division. Let us show that there is an integer \( t \) such that \( p = tq + r \), where the remainder \( r \) is less than \( q \). (It can always be assumed that \( r \) and \( q \) are not negative; if \( x \) is negative, let \( p \) be negative.) It is always possible to find just one \( t \) with this property. Indeed, consider the integral multiples \( q : q, \ 2q, \ 3q, \ldots , \). After some time we will find a first (that is, smallest) integer \( t \) such that \((1 + t)q \) is larger than \( p \) if \( p \) is positive (if \( p \) is negative, read "larger than \(-p\). This is the \( t \) we want. Then

\[
p = \frac{tq + r}{q} = t + \frac{r}{q} \tag{3}
\]

Since \( r \) is less than \( q \), \( t \) is the integral part of the decimal expansion of \( x = p/q \). Now the process of dividing \( r \) by \( q \) to determine the "fractional" part of the expansion is a repetitive one involving division of certain
remainders by $q$. Since there are only $q$ possible remainders, namely $0, 1, 2, 3, \ldots, q - 1$, it is obvious that after we have divided $(q + 1)$ times in the repetitive process at least one of the remainders will have appeared twice. Once a remainder appears the second time the division process becomes a true and exact repetition of what has already occurred since the first appearance of that remainder, and therefore the sequence of digits in the quotient must repeat periodically; for example,

\[
\begin{array}{c c c c c c c}
\hline
\text{10} & \text{2} & \text{6} & \text{2} & \text{6} & \text{6} & \text{6} & \ldots \\
\hline
\end{array}
\]

Remainders are circled. Notice that once the remainder 6 is attained the calculation must be a copy of what went before and the answer can be written out with no further effort. Find the expansions of $\frac{1}{11}$ and $\frac{3}{17}$ yourself for comparison. Similar repetitions will occur when a rational number is expressed relative to any base.

We have just proved the important fact that the decimal expansion of a rational number has the form

\[ x = a_n a_{n-1} a_{n-2} \ldots a_{-k} c_1 c_2 \ldots c_m c_1 c_2 \ldots \]  \hspace{1cm} (3)

where the block $c_1 \ldots c_m$ is repeated indefinitely. We now ask the question: does every real number (decimal expansion) of the form ((3)) represent a rational number? In other words, given ((3)), can we find integers $p$ and $q$ so that $p/q$ will give ((3)) when we carry out the long division?

In order to determine that the answer is yes, we shall look at an example. Consider the simple repeating decimal

\[ r = 0.726666\ldots \]  \hspace{1cm} (3)

How do we write $r$ as the quotient of two integers? Observe that

\[ 10r = 7.266666\ldots \]

\[ r = 0.726666\ldots \]  \hspace{1cm} (3)

On subtracting $r$ from $10r$ the repeating parts to the right of the vertical bar will cancel and

\[ 10r - r = 7.26 - 0.72 = 6.54 = \frac{654}{10^2} \]  \hspace{1cm} (3)

This gives $9r = 654/100$ whence $r$ is the rational number

http://cnx.org/content/m50487/1.2/
Now apply the same procedure to the general ultimately repeating decimal in (3):

\[ x = a_n \cdot \cdot \cdot a_1 a_0 \cdot a_{-1} \cdot \cdot \cdot a_{-k} c_1 c_2 \cdot \cdot \cdot c_{m} \cdot \cdot \cdot \]  

(3)

in which the repeating part is \( c_1 c_2 \cdot \cdot \cdot c_m \). Multiply \( x \) by \( 10^m \); this is the same as shifting the decimal point \( m \) places to the right. Thus

\[ 10^m x = a_n \cdot \cdot \cdot a_1 a_0 \cdot \cdot \cdot a_{-m} a_{-m-1} \cdot \cdot \cdot a_{-k} c_1 c_2 \cdot \cdot \cdot c_{m} \cdot \cdot \cdot \]  

(3)

On subtraction the repeating parts to the right of the vertical bar will cancel and

\[ 10^m x - x = (10^m - 1) x \]  

(3)

is a number of the form, say, \( b_n \cdot \cdot \cdot b_0 , b_{-1} \cdot \cdot \cdot b_{-k} \) which is rational and equal to \( (b_n \cdot \cdot \cdot b_0 , b_{-1} \cdot \cdot \cdot b_{-k}) / 10^k \); so \( x \) must also be rational:

\[ x = \frac{(b_n \cdot \cdot \cdot b_0 , b_{-1} \cdot \cdot \cdot b_{-k})}{10^k (10^m - 1)} \]  

(3)

An alternative approach to the same problem is given by the use of geometric series, which we introduce here as an example of an infinite summation process that appears in later chapters. Consider, for instance, the expression

\[ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdot \cdot \cdot \]  

(3)

This is an example of a geometric series. It has the partial sums

\[ 1 \quad 1 + \frac{1}{2} \quad 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 \quad 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 \]  

(3)

It is not hard to convince ourselves geometrically that these partial sums get closer and closer to the number 2 as we take more and more terms in them. The limiting value of such partial sums is what is meant by “summing an infinite series.”

In general, a geometric series is of the form

\[ a + ar + ar^2 + ar^3 + \cdot \cdot \cdot \]  

(3)

and can be summed if \(-1 < r < 1\); in this case the sum is given by

\[ a + ar + ar^2 + ar^3 + \cdot \cdot \cdot = \frac{a}{1 - r} \]  

(3)

In the example above, \( a = 1 \) and \( r = \frac{1}{2} \). This formula is in a simple sense the limit of similar finite formulas (cp. ). Note that

\[ (1 + r + \cdot \cdot \cdot + r^{r-1}) (1 - r) = 1 - r^n \]  

(3)

Indeed, the left side equals

\[ (1 + r + \cdot \cdot \cdot + r^{r-1}) (1 - (1 + r + \cdot \cdot \cdot + r^{r-1}) r = (1 + r + \cdot \cdot \cdot + r^{r-1}) - (r + r^2 + \cdot \cdot \cdot + r^n) \]  

(3)
and all terms cancel except \(1 - r^n\); for any \(r\) and any positive integer \(n\) the middle terms in the multiplication cancel. If \(-1 < r < 1\), we see that \(r^n\) will tend to zero as \(n\) gets very large; for example,

\[
\left( \frac{1}{2} \right)^n = \frac{1}{2^n}
\]

which clearly becomes as close to zero as we want if \(n\) is taken large enough. Indeed,

\[
\begin{align*}
\frac{1}{2} &= \frac{1}{2} = 0.50000 \\
\left( \frac{1}{2} \right)^2 &= \frac{1}{4} = 0.25000 \\
\left( \frac{1}{2} \right)^3 &= \frac{1}{8} = 0.12500 \\
\left( \frac{1}{2} \right)^4 &= \frac{1}{16} = 0.06250 \\
\left( \frac{1}{2} \right)^5 &= \frac{1}{32} = 0.03125
\end{align*}
\]

Repetition of this process evidently will produce as many zeros after the decimal point as desired; that is \((1/2)^n\) tends to zero. So by using \((3)\) we have the following sequence of formulas:

\[
a + ar = \frac{a - ar^2}{1 - r} = \frac{a}{1 - r} - \left( \frac{a}{1 - r} \right) \cdot r^2
\]
\[
a + ar + ar^2 = \frac{a - ar^3}{1 - r} = \frac{a}{1 - r} - \left( \frac{a}{1 - r} \right) \cdot r^3
\]
\[
\ddots
\]
\[
a + ar + ar^2 + \cdots + ar^n = \frac{a}{1 - r} - \left( \frac{a}{1 - r} \right) \cdot r^{n+1}
\]

On the right-hand side of each equation stands \(a/(1 - r)\), which does not depend on \(n\), added to a term multiplied by \(r^{n+1}\), which we know approaches zero as \(n\) gets large. Therefore we say that the infinite series on the left has the sum \(a/(1 - r)\), since the "error term" for any finite partial sum tends to zero as \(n\) gets large. Notice that not every geometric series can have a sum. For instance, if \(a + ar + ar^2 + \cdots\) had a sum with \(a > 0\) and \(r > 1\), then \(s = a/(1 - r)\) would be negative, which is absurd; for example, uncritical use of the formula suggests the silly result

\[
1 + 2 + 2^2 + 2^3 + 2^4 + \cdots = \frac{1}{1 - 2} = -1
\]

Similarly, if \(a \neq 0\) but \(r = 1\), the series has no sum. It has a sum, given by \((3)\), only if \(-1 < r < +1\).

Now we show that the expansion \((3)\) corresponds to a rational number by using the formula for the sum of a geometric series. The expression in \((3)\) is equal, by straightforward factorization, to the number

\[
\frac{(a_n a_{n-1} \cdots a_1 a_0)}{(1 + \frac{1}{10^m} + \frac{1}{10^{2m}} + \cdots)} + \left( \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots + \frac{a_k}{10^k} \right) + \left( \frac{c_1}{10^{m+1}} + \frac{c_2}{10^{m+2}} + \cdots + \frac{c_m}{10^{m+m}} \right) \times
\]

\[
(3)
\]

The infinite sum in the last set of parentheses

\[
\left( 1 + \frac{1}{10^m} + \frac{1}{10^{2m}} + \cdots \right)
\]

is a geometric series with \(r = 1/10^m\) and \(a = 1\); using \((3)\), it sums to

\[
\left( \frac{1}{1 - \frac{1}{10^m}} \right) = \left( \frac{10^m}{10^m - 1} \right)
\]

\((3)\)
which is a rational number. Therefore the original number \( x \) must also be rational, since it is the sum of the integer \( (a_n \cdots a_0) \) and the rational \((0.a_{-1}a_{-2} \cdots a_{-k})\) added to the product of the two rationals 
\[
\left( \frac{c_1}{10^{m+1}} + \cdots + \frac{c_m}{10^{m+k}} \right) \text{ and } \left( \frac{10^m}{10^{m+k}} \right)
\]
We have shown that every decimal expansion with a periodic (repeating) sequence of digits represents a rational number and that, conversely, every rational number can be expressed as a periodic decimal.

We have also just learned how to construct decimal expansions for an infinite number of different irrational numbers. All that is necessary is to ensure that the decimal expansion never can become periodic. So, for instance, the number

\[
0.212112111211112...
\]

in example (d) above, where each group of 1’s has one more member than the group of 1’s directly to its left, must be an irrational number. Construct several irrational numbers yourself.

5

Since we are back on the subject of irrationals let us prove that \( 1 + \sqrt{3} \) and \( \sqrt{2} \) are irrational. Note that \( 1 + \sqrt{3} \) and \( \sqrt{2} \) both have decimal expansions, hence "qualify" as real numbers according to our definition. What is needed is an algorithm (rule) for writing down approximating finite decimal expansions which converge in the same sense that the finite expressions that approximated the sum of two decimal expansions "converged" to the desired decimal expansion. The early Babylonians had such an algorithm for \( \sqrt{x} \) for any positive integer \( x \) (see ), and, similarly, we can find an algorithm for higher order roots of numbers. It is simpler, however, to work (algebraically, not computationally) with the formal expressions \( 1 + \sqrt{3} \) and \( \sqrt{2} \). What we shall do now is prove (algebraically) that it is impossible to find integers \( p \) and \( q \) so that

\[
\frac{p}{q} = 1 + \sqrt{3}
\]

The proof we give goes back to Euclid and is a "proof by contradiction" or indirect proof. The idea is this: we suppose \( 1 + \sqrt{3} \) to be rational and show that this logically must imply a contradiction. Since we agree that we do not want any contradictions in our mathematical system, the conclusion is that the assumption we made cannot be true, hence is false, which is the fact (that \( 1 + \sqrt{3} \) is not rational) that we wanted to prove.

Carrying out this program, we first note that \( 1 + \sqrt{3} \) is rational if and only if \( \sqrt{3} \) is rational. So we shall prove that \( \sqrt{3} \) is irrational. Suppose \( \sqrt{3} \) were rational, that is, \( \sqrt{3} = m/n \), where \( m \) and \( n \) are integers and where we can also assume that the fraction \( m/n \) is in lowest terms; that is, \( m \) and \( n \) have no common factor other than 1. Squaring both sides shows that

\[
3n^2 = m^2
\]

so 3 divides \( m^2 \). Now either \( m \) is divisible by 3 or it leaves a remainder of 1 or of 2 on division by 3; that is, \( m = 3k \), \( m = 3k + 1 \) or \( m = 3k + 2 \) for some integer \( k \). Then

\[
m^2 = 9k^2 \quad m^2 = 9k^2 + 6k + 1 \quad \text{or} \quad m^2 = 9k^2 + 12k + 4
\]

Since only the first of these squares can be divided by 3, it follows that the divisibility of \( m^2 \) by 3 implies that 3 divides \( m \) itself. So \( m = 3k \) and therefore from \( 3n^2 = m^2 \) we find \( 3n^2 = 9k^2 \); that is,

\[
3k^2 = n^2
\]

Then 3 divides \( n^2 \) and by the same argument we have already used 3 divides \( n \). We have shown, on the assumption that \( \sqrt{3} \) is rational, that 3 divides both \( m \) and \( n \). This is a contradiction, since we assumed that \( m/n \) was in lowest terms. Hence cannot be rational; it must be irrational.
6

How many rationals are there? An infinite number, of course. Still, it would be nice to have some idea, be it intuitive or imprecise, of the rough proportion among the infinite collection of all real numbers that is filled out by the rationals. In other words, do the periodic decimal expansions account for most of the decimal expansions or for a small portion? To make some sense of these questions, it is imperative that we state clearly how we will decide when one infinite set has more members than another. You might be tempted to argue, as is so easy in the finite case: there are 713 men and 4 women in a box. Are there more men than women? Since 713 is greater than 4, there are more men than women. There is another way to answer the question, however, which extends to the case of infinite sets; pair the men and women and remove the paired people from the box. Then, if the box is empty, we say that there is an equal number of men and women; if the box is not empty, it will contain only men or only women. We say that there were more men than women if men are left in the box, more women than men if women are left in the box.

Pleasant though it may be to consider men and/or women in a box, it is clear that we are really merely pairing the elements of two sets in order to compare their size; the names of the elements or of the sets do not matter at all. This technique of pairing works nearly as well for infinite sets as it does for finite collections of people. It will lead to some startling results because you (probably) have a fixation on finite things and either have no intuition or wrong intuition about what is true of things infinite.

The first place in which intuition may let you down concerns subsets of a given set. If a set $S$ is finite, and the set $T$ is a proper subset of $S$ (that is, everything in $T$ is also in $S$ but there are things in $S$ that are not in $T$), then you can be sure that there are more things in $S$ than in $T$. This is not true for infinite sets. Indeed, let $N$ denote the set of positive integers 1, 2, 3, 4, 5,... and let $2N$ denote the set of even positive integers 2, 4, 6, 8,... Then $2N$ is a proper subset of $N$, but now pair each integer $n$ in $N$ with the even integer $2n$ in $2N$. It is clear that every integer in $2N$ belongs to just one of these pairs and the same is time of every integer in $N$. Therefore, according to our agreement about pairing, we must say that there are just as many integers in $N$ as in $2N$. Difficult to countenance? The choice is between this, and deciding that pairing of objects does not work for infinite sets. It has been agreed for some time (since the late 1890’s) that the pairing process leads to useful (as well as startling) results and that it represents perhaps a more elementary and fundamental mode of reasoning than that involved in rejecting the notion that $N$ and $2N$ have the same number of elements. There are certain ways of describing this process that may make it appear more reasonable to you; unfortunately, we have no time to explore these byways.

Once this method of pairing sets to determine whether they are the “same size” is accepted, it is easy to test some of the more common subsets of the real numbers to find out how large they are. We have already seen that $2N$ is as large as $N$; in the same way we can discover that the integers that are multiples of any fixed nonzero integer are as numerous as the entire set of integers.

A more striking observation is that the set of rational numbers is no larger than the set of integers. We will content ourselves with showing that the set of positive rationals is the same size as the set of positive integers, for by the remarks above the set of negative integers can be put in a one-to-one correspondence with the positive integers, the set of negative rationals with the set of positive rationals, and it is really enough to prove that these two sets of positive numbers are the same size. Consider the diagram of rational numbers in Figure 4. Every rational finds a place in this diagram; $p/q$ lies at the intersection of the $p$-th row and $q$-th column, but some rationals are repeated because $kp/kq = p/q$ for any positive integer $k$. Therefore strike out all multiples of rationals that are expressed in reduced form, numerator and denominator having

\[ \sqrt{2} \]

We can proceed with the proof that $\sqrt{2}$ is irrational in a similar way. Assume that $\sqrt{2} = m/n$. Then $2m^2 = n^2$ and $2$ divides $m^2$. Show that this implies that $2$ divides $m$; that is, $m = 2k$. Then $2n^2 = 8k^2$ and $2$ divides $n^2$. Then both $m$ and $n$ are even and $m/n$ is not in lowest terms, and so on.

You can understand that the proof of the irrationality of a particular number can be difficult despite the fact that we already know how to construct infinitely many different irrationals. This is an example of the sad but universal fact that quite different methods are generally needed to solve problems of the particular as opposed to problems of the general and not only in mathematics.
no common divisors. Then each rational occurs exactly once in the modified diagram. Now trace a path through the diagram as shown and number the rational stepping stones as they occur along the path.
The rational numbers are countable.

http://cnx.org/content/m50487/1.2/
No matter how large the integer \( n \), there is a rational number (stepping-stone) corresponding to it in the diagram and to every rational there corresponds an integer \( n \). Hence the rationals and the integers are equally numerous.

Let us say that a number \( x \) is algebraic if it is the root of a polynomial equation whose coefficients are integers, that is, if

\[
a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0
\]

where the \( a_i \) are integers. For instance, every rational number \( p/q \) is algebraic because it is the root of \( qx - p = 0 \); \( \sqrt{2} \) is irrational but algebraic because it is a root of \( x^2 - 2 = 0 \); \( \pi \) is not algebraic, but it is difficult to prove that this is so. Almost all numbers we ever come in contact with are algebraic, but an application of the same diagonal path procedure that we used above shows that there are just as many integers as there are algebraic numbers.

The wary reader may suspect that we are wasting his time by not simply stating that the set of real numbers is no larger than the set of integers, but this is false. There are more real numbers (that is, decimal expansions) than there are integers (or rationals or algebraic numbers).

Suppose that it were possible to put the integers (say positive integers for convenience) into correspondence with the decimal expansions. There would be a first decimal, a second, and so on, as indicated in Table 4 where \( a_k \) stands for the \( k \)-th digit after the decimal point in the expansion of the \( n \)-th number in the list. We have taken the liberty of not writing the integer parts of the decimal expansions because the notation is already cluttered; you can imagine that they are there.

<table>
<thead>
<tr>
<th>Ordinal Numbers</th>
<th>Decimal Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( .a_1^1a_2^1a_3^1 \cdots )</td>
</tr>
<tr>
<td>2</td>
<td>( .a_1^2a_2^2a_3^2 \cdots )</td>
</tr>
<tr>
<td>3</td>
<td>( .a_1^3a_2^3a_3^3 \cdots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( n )</td>
<td>( .a_1^n a_2^n a_3^n \cdots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>

Table 4

Now we show that such a list is impossible. Recall that the list is asserted to contain every positive real number; to show that such a list cannot exist all we must do is produce a number - a decimal expansion - that is not in the list. Here is one:

\[
a = .a_1 a_2 a_3 \cdots \]

where

- \( a_1 = 1 \) if \( a_1^1 \neq 1 \) but \( a_1 = 2 \) otherwise
- \( a_2 = 1 \) if \( a_2^2 \neq 1 \) but \( a_2 = 2 \) otherwise
- \( a_3 = 1 \) if \( a_3^3 \neq 1 \) but \( a_3 = 2 \) otherwise
- \( a_n = 1 \) if \( a_n^n \neq 1 \) but \( a_n = 2 \) otherwise, and so on.
It is clear that the real number \( a \) with the decimal expansion just defined is different from the first number in the table in the first decimal place, from the second number in the second decimal place, from the third in the third, and so on. In fact \( a \) is different from every number in the list, but it is itself a real number, that is, a decimal expansion, and so the list must not be complete. Therefore there cannot be a one-to-one correspondence between the integers and the real numbers. There will always be a real number (indeed many of them) left over, so the set of real numbers is larger than the set of integers or rationals or algebraic numbers. This elegant proof was created by the German mathematician Georg Cantor at the end of the nineteenth century and has revolutionized our understanding of the system of real numbers.

7

One of the principal reasons that the branch of mathematics known as analysis (which includes calculus; see ) is particularly difficult is that it is concerned with the properties of the real numbers rather than those of just the rational or algebraic numbers.

It is remarkable that man has been able to master the set of real numbers, to use it for his science and technology and sometimes for his art. when he himself is finite and has at his disposal only the integers and other "countable" sets of numbers and, as a tool only the finitary principle of mathematical induction. This is in large measure why mathematics is difficult; its difficulty should be expected, but man’s achievements in the face of the Uncountable Infinite are, we think, awesome and inspiring.

Since there are more real numbers than algebraic numbers and since the symbols that we must use to denote them are all drawn from a finite collection (including the integers, signs such as and "+", "−", "×", and "÷", letters from various alphabets), it follows that we can express only countably many numbers by using finite combinations of these symbols. This means that there must be numbers for which we (an have no general and systematic means of expression as a finite combination of symbols drawn from a fixed finite inventory. By using the symbols for the integers and for division we can express any rational number in a systematic way as \( p/q \); similarly, any algebraic number can be expressed in finite form, although it is not customary to do so. For instance, if the algebraic number \( x \) is a root of the polynomial

\[
a_0x^n + a_1x^{n-1} + \cdots + a_n = 0
\]  

(4)

in which \( a_i \) the are all integers, this equation is uniquely determined by the ordered sequence of integers

\[
(a_0, a_1, \ldots, a_n)
\]  

(4)

Since an algebraic equation of degree \( n \) has exactly \( n \) (complex) roots, which are all distinct if we assume that the polynomial has no repeated factors, the particular root \( x \) will be determined if we state which of the \( n \) roots it is. Therefore a finite symbol such as

\[
(a_0, a_1, \ldots, a_n|k)
\]  

(4)

completely specifies an algebraic number, where \( k \) is an integer specifying which root of the polynomial is meant. Since it is clear that each algebraic number can be associated with an expression of this kind, we have constructed a notation that enables us to express any algebraic number in a systematic way.

No such scheme can be constructed for expressing all real numbers, as we have already seen. This means that we will have to provide special and distinctive expressions for any transcendental numbers that crop up in the course of our pursuits (a transcendental number is a real number which is not algebraic—it "transcends" the realm of algebraic methods). When a transcendental number is important enough to require a name, we give it one. The ratio of the circumference of a circle to its diameter (in a rather awkward way that expression defines a number! Is it larger than 3? Not easy to tell from this description) is the most famous transcendental number in history; its conventional name is \( \pi \). Two other transcendents that have appeared often enough in mathematics to deserve a name along with verbal "descriptions" of what the names are intended to "remind" you, are

\( e \), base of the system of natural logarithms (see )
and

\[ \gamma, \text{ the Euler-Mascheroni constant} \]

Many of the other transcendental numbers that actually appear in a natural way are logarithms of algebraic numbers and algebraic powers of \( e \). Notice that none of these symbolic notations tells you anything at all about how to compute the value of the number represented and that this differs in an essential way from the usual notation for rational numbers or even from the notation for algebraic numbers introduced above; letter names for transcendental numbers are just *names*, whereas the *notations* for rational and algebraic numbers contain information about the nature of the number that is sufficient to tell you how to calculate its value. This is one of the main reasons why in normal life we deal primarily with rational numbers; we have a systematic way of writing them that dovetails with their mathematical properties and, most important of all, with their *size* (that is, *value*). Imagine the difficulties that would occur in everyday calculation of bills, football scores, odds on betting, and income taxes if each number had a special symbolic name entirely unrelated to its size or arithmetical properties. That is precisely the situation in which the ancient Egyptians, Greeks, and Romans found themselves.