TRIGONOMETRY: GRAPHS OF TRIG
FUNCTIONS (GRADE 11)*

Free High School Science Texts Project

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1 History of Trigonometry

Work in pairs or groups and investigate the history of the development of trigonometry. Describe the various stages of development and how different cultures used trigonometry to improve their lives.

The works of the following people or cultures can be investigated:

1. Cultures
   a. Ancient Egyptians
   b. Mesopotamians
   c. Ancient Indians of the Indus Valley

2. People
   a. Lagadha (circa 1350-1200 BC)
   b. Hipparchus (circa 150 BC)
   c. Ptolemy (circa 100)
   d. Aryabhata (circa 499)
   e. Omar Khayyam (1048-1131)
   f. Bhaskara (circa 1150)
   g. Nasir al-Din (13th century)
   h. al-Kashi and Ulugh Beg (14th century)
   i. Bartholemaeus Pitiscus (1595)

2 Graphs of Trigonometric Functions

2.1 Functions of the form \( y = \sin (k \theta) \)

In the equation, \( y = \sin (k \theta) \), \( k \) is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 1 for the function \( f (\theta) = \sin (2\theta) \).

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2.1.1 Functions of the form \( y = \sin (k\theta) \)

On the same set of axes, plot the following graphs:

1. \( a(\theta) = \sin 0, 5\theta \)
2. \( b(\theta) = \sin 1\theta \)
3. \( c(\theta) = \sin 1, 5\theta \)
4. \( d(\theta) = \sin 2\theta \)
5. \( e(\theta) = \sin 2, 5\theta \)

Use your results to deduce the effect of \( k \).

You should have found that the value of \( k \) affects the period or frequency of the graph. Notice that in the case of the sine graph, the period (length of one wave) is given by \( \frac{360^\circ}{k} \).

These different properties are summarised in Table 1.

<table>
<thead>
<tr>
<th>( k &gt; 0 )</th>
<th>( k &lt; 0 )</th>
</tr>
</thead>
</table>

Table 1: Table summarising general shapes and positions of graphs of functions of the form \( y = \sin (k\theta) \).

The curve \( y = \sin (x) \) is shown as a dotted line.

2.1.2 Domain and Range

For \( f(\theta) = \sin (k\theta) \), the domain is \( \{ \theta : \theta \in \mathbb{R} \} \) because there is no value of \( \theta \in \mathbb{R} \) for which \( f(\theta) \) is undefined.

The range of \( f(\theta) = \sin (k\theta) \) is \( \{ f(\theta) : f(\theta) \in [-1, 1] \} \).

2.1.3 Intercepts

For functions of the form, \( y = \sin (k\theta) \), the details of calculating the intercepts with the \( y \) axis are given.

There are many \( x \)-intercepts.
The y-intercept is calculated by setting \( \theta = 0 \):

\[
\begin{align*}
    y &= \sin(k\theta) \\
    y_{int} &= \sin(0) \\
    &= 0
\end{align*}
\]

(3)

2.2 Functions of the form \( y = \cos(k\theta) \)

In the equation, \( y = \cos(k\theta) \), \( k \) is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 4 for the function \( f(\theta) = \cos(2\theta) \).

![Image not finished]

Figure 4: Graph of \( f(\theta) = \cos(2\theta) \) (solid line) and the graph of \( g(\theta) = \cos(\theta) \) (dotted line).

2.2.1 Functions of the form \( y = \cos(k\theta) \)

On the same set of axes, plot the following graphs:

1. \( a(\theta) = \cos(0,5\theta) \)
2. \( b(\theta) = \cos(1\theta) \)
3. \( c(\theta) = \cos(1,5\theta) \)
4. \( d(\theta) = \cos(2\theta) \)
5. \( e(\theta) = \cos(2,5\theta) \)

Use your results to deduce the effect of \( k \).

You should have found that the value of \( k \) affects the period or frequency of the graph. The period of the cosine graph is given by \( \frac{360^\circ}{k} \).

These different properties are summarised in Table 2.

<table>
<thead>
<tr>
<th>( k &gt; 0 )</th>
<th>( k &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \text{Image not finished} ]</td>
<td>[ \text{Image not finished} ]</td>
</tr>
</tbody>
</table>

Figure 5

Table 2: Table summarising general shapes and positions of graphs of functions of the form \( y = \cos(kx) \).

The curve \( y = \cos(x) \) is plotted with a dotted line.
2.2.2 Domain and Range

For \( f(\theta) = \cos(k\theta) \), the domain is \( \{\theta : \theta \in \mathbb{R}\} \) because there is no value of \( \theta \in \mathbb{R} \) for which \( f(\theta) \) is undefined.

The range of \( f(\theta) = \cos(k\theta) \) is \( \{f(\theta) : f(\theta) \in [-1, 1]\} \).

2.2.3 Intercepts

For functions of the form, \( y = \cos(k\theta) \), the details of calculating the intercepts with the \( y \) axis are given.

The \( y \)-intercept is calculated as follows:

\[
\begin{align*}
y &= \cos(k\theta) \\
y_{\text{int}} &= \cos(0) \\
&= 1
\end{align*}
\]

2.3 Functions of the form \( y = \tan(k\theta) \)

In the equation, \( y = \tan(k\theta) \), \( k \) is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 7 for the function \( f(\theta) = \tan(2\theta) \).

![Image not finished](http://cnx.org/content/m38866/1.1/)

**Figure 7:** The graph of \( \tan(2\theta) \) (solid line) and the graph of \( g(\theta) = \tan(\theta) \) (dotted line). The asymptotes are shown as dashed lines.

\[
\begin{align*}
\text{2.3.1 Functions of the form } y &= \tan(k\theta) \\
\end{align*}
\]

On the same set of axes, plot the following graphs:

1. \( a(\theta) = \tan0, 5\theta \)
2. \( b(\theta) = \tan1\theta \)
3. \( c(\theta) = \tan1, 5\theta \)
4. \( d(\theta) = \tan2\theta \)
5. \( e(\theta) = \tan2, 5\theta \)

Use your results to deduce the effect of \( k \).

You should have found that, once again, the value of \( k \) affects the periodicity (i.e. frequency) of the graph. As \( k \) increases, the graph is more tightly packed. As \( k \) decreases, the graph is more spread out. The period of the tan graph is given by \( \frac{180^\circ}{k} \).

These different properties are summarised in Table 3.
Table 3: Table summarising general shapes and positions of graphs of functions of the form \( y = \tan(k \theta) \).

2.3.2 Domain and Range

For \( f(\theta) = \tan(k \theta) \), the domain of one branch is \( \{ \theta : \theta \in \left(-\frac{90^\circ}{k}, \frac{90^\circ}{k} \right) \} \) because the function is undefined for \( \theta = -\frac{90^\circ}{k} \) and \( \theta = \frac{90^\circ}{k} \).

The range of \( f(\theta) = \tan(k \theta) \) is \( \{ f(\theta) : f(\theta) \in (-\infty, \infty) \} \).

2.3.3 Intercepts

For functions of the form, \( y = \tan(k \theta) \), the details of calculating the intercepts with the \( x \) and \( y \) axis are given.

There are many \( x \)-intercepts; each one is halfway between the asymptotes.

The \( y \)-intercept is calculated as follows:

\[
\begin{align*}
y &= \tan(k \theta) \\
y_{int} &= \tan(0) \\
&= 0
\end{align*}
\]  

(9)

2.3.4 Asymptotes

The graph of \( \tan k \theta \) has asymptotes because as \( k \theta \) approaches \( 90^\circ \), \( \tan k \theta \) approaches infinity. In other words, there is no defined value of the function at the asymptote values.

2.4 Functions of the form \( y = \sin(\theta + p) \)

In the equation, \( y = \sin(\theta + p) \), \( p \) is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 10 for the function \( f(\theta) = \sin(\theta + 30^\circ) \).

Figure 10: Graph of \( f(\theta) = \sin(\theta + 30^\circ) \) (solid line) and the graph of \( g(\theta) = \sin(\theta) \) (dotted line).
2.4.1 Functions of the Form \( y = \sin(\theta + p) \)

On the same set of axes, plot the following graphs:

1. \( a(\theta) = \sin(\theta - 90^\circ) \)
2. \( b(\theta) = \sin(\theta - 60^\circ) \)
3. \( c(\theta) = \sin \theta \)
4. \( d(\theta) = \sin(\theta + 90^\circ) \)
5. \( e(\theta) = \sin(\theta + 180^\circ) \)

Use your results to deduce the effect of \( p \).

You should have found that the value of \( p \) affects the position of the graph along the \( y \)-axis (i.e. the \( y \)-intercept) and the position of the graph along the \( x \)-axis (i.e. the phase shift). The \( p \) value shifts the graph horizontally. If \( p \) is positive, the graph shifts left and if \( p \) is negative the graph shifts right.

These different properties are summarised in Table 4.

\[
\begin{array}{|c|c|}
\hline
p > 0 & p < 0 \\
\hline
\end{array}
\]

![Image not finished](http://cnx.org/content/m38866/1.1/)

Figure 11

<table>
<thead>
<tr>
<th>( p &gt; 0 )</th>
<th>( p &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Table 4: Table summarising general shapes and positions of graphs of functions of the form } \ y = \sin(\theta + p) ). The curve ( y = \sin(\theta) ) is plotted with a dotted line.</td>
<td></td>
</tr>
</tbody>
</table>

2.4.2 Domain and Range

For \( f(\theta) = \sin(\theta + p) \), the domain is \( \{\theta : \theta \in \mathbb{R} \} \) because there is no value of \( \theta \in \mathbb{R} \) for which \( f(\theta) \) is undefined.

The range of \( f(\theta) = \sin(\theta + p) \) is \( \{ f(\theta) : f(\theta) \in [-1,1] \} \).

2.4.3 Intercepts

For functions of the form, \( y = \sin(\theta + p) \), the details of calculating the intercept with the \( y \) axis are given.

The \( y \)-intercept is calculated as follows: set \( \theta = 0^\circ \)

\[
\begin{align*}
y &= \sin(\theta + p) \\
y_{\text{int}} &= \sin(0 + p) \\
&= \sin(p)
\end{align*}
\] (12)
2.5 Functions of the form $y = \cos(\theta + p)$

In the equation, $y = \cos(\theta + p)$, $p$ is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 13 for the function $f(\theta) = \cos(\theta + 30^\circ)$.

![Image not finished]

Figure 13: Graph of $f(\theta) = \cos(\theta + 30^\circ)$ (solid line) and the graph of $g(\theta) = \cos(\theta)$ (dotted line).

2.5.1 Functions of the Form $y = \cos(\theta + p)$

On the same set of axes, plot the following graphs:

1. $a(\theta) = \cos(\theta - 90^\circ)$
2. $b(\theta) = \cos(\theta - 60^\circ)$
3. $c(\theta) = \cos\theta$
4. $d(\theta) = \cos(\theta + 90^\circ)$
5. $e(\theta) = \cos(\theta + 180^\circ)$

Use your results to deduce the effect of $p$.

You should have found that the value of $p$ affects the $y$-intercept and phase shift of the graph. As in the case of the sine graph, positive values of $p$ shift the cosine graph left while negative $p$ values shift the graph right.

These different properties are summarised in Table 5.

<table>
<thead>
<tr>
<th>$p &gt; 0$</th>
<th>$p &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Image not finished]</td>
<td>![Image not finished]</td>
</tr>
</tbody>
</table>

Figure 14

Table 5: Table summarising general shapes and positions of graphs of functions of the form $y = \cos(\theta + p)$. The curve $y = \cos\theta$ is plotted with a dotted line.

2.5.2 Domain and Range

For $f(\theta) = \cos(\theta + p)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \cos(\theta + p)$ is $\{f(\theta) : f(\theta) \in [-1, 1]\}$.
2.5.3 Intercepts

For functions of the form, \( y = \cos(\theta + p) \), the details of calculating the intercept with the \( y \) axis are given.

The \( y \)-intercept is calculated as follows: set \( \theta = 0^\circ \)

\[
\begin{align*}
y &= \cos(\theta + p) \\
y_{\text{int}} &= \cos(0 + p) \\
&= \cos(p)
\end{align*}
\] (15)

2.6 Functions of the form \( y = \tan(\theta + p) \)

In the equation, \( y = \tan(\theta + p) \), \( p \) is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 16 for the function \( f(\theta) = \tan(\theta + 30^\circ) \).

![Image not finished]

Figure 16: The graph of \( \tan(\theta + 30^\circ) \) (solid lines) and the graph of \( g(\theta) = \tan(\theta) \) (dotted lines).

2.6.1 Functions of the Form \( y = \tan(\theta + p) \)

On the same set of axes, plot the following graphs:

1. \( a(\theta) = \tan(\theta - 90^\circ) \)
2. \( b(\theta) = \tan(\theta - 60^\circ) \)
3. \( c(\theta) = \tan\theta \)
4. \( d(\theta) = \tan(\theta + 60^\circ) \)
5. \( e(\theta) = \tan(\theta + 180^\circ) \)

Use your results to deduce the effect of \( p \).

You should have found that the value of \( p \) once again affects the \( y \)-intercept and phase shift of the graph. There is a horizontal shift to the left if \( p \) is positive and to the right if \( p \) is negative.

These different properties are summarised in Table 6.

<table>
<thead>
<tr>
<th>( k &gt; 0 )</th>
<th>( k &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Image not finished] (Figure 17)</td>
<td></td>
</tr>
</tbody>
</table>
Table 6: Table summarising general shapes and positions of graphs of functions of the form \( y = \tan (\theta + p) \). The curve \( y = \tan (\theta) \) is plotted with a dotted line.

2.6.2 Domain and Range

For \( f (\theta) = \tan (\theta + p) \), the domain for one branch is \( \{ \theta : \theta \in (-90^\circ - p, 90^\circ - p) \} \) because the function is undefined for \( \theta = -90^\circ - p \) and \( \theta = 90^\circ - p \).

The range of \( f (\theta) = \tan (\theta + p) \) is \( \{ f (\theta) : f (\theta) \in (-\infty, \infty) \} \).

2.6.3 Intercepts

For functions of the form, \( y = \tan (\theta + p) \), the details of calculating the intercepts with the \( y \) axis are given.

The \( y \)-intercept is calculated as follows: set \( \theta = 0^\circ \)

\[
\begin{align*}
  y &= \tan (\theta + p) \\
  y_{\text{int}} &= \tan (p)
\end{align*}
\]

(18)

2.6.4 Asymptotes

The graph of \( \tan (\theta + p) \) has asymptotes because as \( \theta + p \) approaches \( 90^\circ \), \( \tan (\theta + p) \) approaches infinity. Thus, there is no defined value of the function at the asymptote values.

2.6.4.1 Functions of various form

Using your knowledge of the effects of \( p \) and \( k \) draw a rough sketch of the following graphs without a table of values.

1. \( y = \sin 3x \)
2. \( y = -\cos 2x \)
3. \( y = \tan \frac{1}{2}x \)
4. \( y = \sin (x - 45^\circ) \)
5. \( y = \cos (x + 45^\circ) \)
6. \( y = \tan (x - 45^\circ) \)
7. \( y = 2\sin 2x \)
8. \( y = \sin (x + 30^\circ) + 1 \)