# Table of Contents

1 Preface .......................................................................................................................... 1

1 Introduction: The Nature of Science and Physics ....................................................... 5
   Physics: An Introduction .................................................................................................. 6
   Physical Quantities and Units ....................................................................................... 13
   Accuracy, Precision, and Significant Figures .............................................................. 20
   Approximation ............................................................................................................. 24

2 Kinematics .................................................................................................................... 31
   Displacement ............................................................................................................... 32
   Vectors, Scalars, and Coordinate Systems ..................................................................... 35
   Time, Velocity, and Speed ............................................................................................. 36
   Acceleration ................................................................................................................. 40
   Motion Equations for Constant Acceleration in One Dimension ............................... 49
   Problem-Solving Basics for One-Dimensional Kinematics ........................................ 59
   Falling Objects ........................................................................................................... 61
   Graphical Analysis of One-Dimensional Motion ....................................................... 69

3 Two-Dimensional Kinematics ....................................................................................... 89
   Kinematics in Two Dimensions: An Introduction ....................................................... 90
   Vector Addition and Subtraction: Graphical Methods ................................................ 92
   Vector Addition and Subtraction: Analytical Methods ............................................... 99
   Projectile Motion ......................................................................................................... 106
   Addition of Velocities ................................................................................................. 113

4 Dynamics: Force and Newton’s Laws of Motion ......................................................... 133
   Development of Force Concept .................................................................................... 134
   Newton’s First Law of Motion: Inertia ......................................................................... 135
   Newton’s Second Law of Motion: Concept of a System ............................................. 136
   Newton’s Third Law of Motion: Symmetry in Forces ................................................. 142
   Normal, Tension, and Other Examples of Forces ....................................................... 146
   Problem-Solving Strategies ......................................................................................... 153
   Further Applications of Newton’s Laws of Motion ..................................................... 155
   Extended Topic: The Four Basic Forces—An Introduction ......................................... 161

5 Further Applications of Newton’s Laws: Friction, Drag, and Elasticity .................... 177
   Friction ........................................................................................................................ 178
   Drag Forces ................................................................................................................. 180
   Elasticity: Stress and Strain ......................................................................................... 187

6 Uniform Circular Motion and Gravitation .................................................................. 203
   Rotation Angle and Angular Velocity ......................................................................... 204
   Centripetal Acceleration .............................................................................................. 207
   Centripetal Force ......................................................................................................... 210
   Fictitious Forces and Non-inertial Frames: The Coriolis Force ................................. 214
   Newton’s Universal Law of Gravitation ...................................................................... 217
   Satellites and Kepler’s Laws: An Argument for Simplicity ........................................ 224

7 Work, Energy, and Energy Resources ...................................................................... 241
   Work: The Scientific Definition ................................................................................... 242
   Kinetic Energy and the Work-Energy Theorem ............................................................ 244
   Gravitational Potential Energy .................................................................................... 249
   Conservative Forces and Potential Energy .................................................................. 254
   Nonconservative Forces ............................................................................................. 257
   Conservation of Energy .............................................................................................. 262
   Power .......................................................................................................................... 266
   Work, Energy, and Power in Humans ......................................................................... 270
   World Energy Use ....................................................................................................... 272

8 Linear Momentum and Collisions ............................................................................... 287
   Linear Momentum and Force ....................................................................................... 288
   Impulse ........................................................................................................................ 290
   Conservation of Momentum ....................................................................................... 292
   Elastic Collisions in One Dimension .......................................................................... 295
   Inelastic Collisions in One Dimension ........................................................................ 296
   Collisions of Point Masses in Two Dimensions ........................................................ 301
   Introduction to Rocket Propulsion ............................................................................. 305

9 Statics and Torque ....................................................................................................... 317
   The First Condition for Equilibrium ........................................................................... 318
   The Second Condition for Equilibrium ....................................................................... 319
   Stability ....................................................................................................................... 324
   Applications of Statics, Including Problem-Solving Strategies ............................... 327
   Simple Machines ........................................................................................................ 330
   Forces and Torques in Muscles and Joints ................................................................. 334
## Rotational Motion and Angular Momentum

- Angular Acceleration
- Kinematics of Rotational Motion
- Dynamics of Rotational Motion: Rotational Inertia
- Rotational Kinetic Energy: Work and Energy Revisited
- Angular Momentum and Its Conservation
- Collisions of Extended Bodies in Two Dimensions
- Gyroscopic Effects: Vector Aspects of Angular Momentum

## Oscillatory Motion and Waves

- Hooke's Law: Stress and Strain Revisited
- Period and Frequency in Oscillations
- Simple Harmonic Motion: A Special Periodic Motion
- The Simple Pendulum
- Energy and the Simple Harmonic Oscillator
- Uniform Circular Motion and Simple Harmonic Motion
- Damped Harmonic Motion
- Forced Oscillations and Resonance
- Waves
- Superposition and Interference
- Energy in Waves: Intensity

## Electric Charge and Electric Field

- Static Electricity and Charge: Conservation of Charge
- Conductors and Insulators
- Coulomb's Law
- Electric Field: Concept of a Field Revisited
- Electric Field Lines: Multiple Charges
- Electric Forces in Biology
- Conductors and Electric Fields in Static Equilibrium
- Applications of Electrostatics

## Electric Potential and Electric Field

- Electric Potential Energy: Potential Difference
- Electric Potential in a Uniform Electric Field
- Electrical Potential Due to a Point Charge
- Equipotential Lines
- Capacitors and Dielectrics
- Capacitors in Series and Parallel
- Energy Stored in Capacitors

## Electromagnetic Waves

- Maxwell's Equations: Electromagnetic Waves Predicted and Observed
- Production of Electromagnetic Waves
- The Electromagnetic Spectrum
- Energy in Electromagnetic Waves

## Wave Optics

- The Wave Aspect of Light: Interference
- Huygens's Principle: Diffraction
- Young's Double Slit Experiment
- Multiple Slit Diffraction
- Single Slit Diffraction
- Limits of Resolution: The Rayleigh Criterion
- Thin Film Interference
- Polarization

*Extended Topic* Microscopy Enhanced by the Wave Characteristics of Light

## Appendix A: Atomic Masses

## Appendix B: Useful Information

## Appendix C: Glossary of Key Symbols and Notation

## Index
Welcome to College Physics, an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

About OpenStax

OpenStax is a nonprofit based at Rice University, and it's our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 20 books for college and AP courses used by hundreds of thousands of students. Our adaptive learning technology, designed to improve learning outcomes through personalized educational paths, is being piloted in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

About OpenStax Resources

Customization

College Physics is licensed under a Creative Commons Attribution 4.0 International (CC BY) license, which means that you can distribute, remix, and build upon the content, as long as you provide attribution to OpenStax and its content contributors. Because our books are openly licensed, you are free to use the entire book or pick and choose the sections that are most relevant to the needs of your course. Feel free to remix the content by assigning your students certain chapters and sections in your syllabus, in the order that you prefer. You can even provide a direct link in your syllabus to the sections in the web view of your book.

Instructors also have the option of creating a customized version of their OpenStax book. The custom version can be made available to students in low-cost print or digital form through their campus bookstore. Visit your book page on openstax.org for more information.

Errata

All OpenStax textbooks undergo a rigorous review process. However, like any professional-grade textbook, errors sometimes occur. Since our books are web based, we can make updates periodically when deemed pedagogically necessary. If you have a correction to suggest, submit it through the link on your book page on openstax.org. Subject matter experts review all errata suggestions. OpenStax is committed to remaining transparent about all updates, so you will also find a list of past errata changes on your book page on openstax.org.

Format

You can access this textbook for free in web view or PDF through openstax.org, and in low-cost print and iBooks editions.

About College Physics

College Physics meets standard scope and sequence requirements for a two-semester introductory algebra-based physics course. The text is grounded in real-world examples to help students grasp fundamental physics concepts. It requires knowledge of algebra and some trigonometry, but not calculus. College Physics includes learning objectives, concept questions, links to labs and simulations, and ample practice opportunities for traditional physics application problems.

Coverage and Scope

College Physics is organized such that topics are introduced conceptually with a steady progression to precise definitions and analytical applications. The analytical aspect (problem solving) is tied back to the conceptual before moving on to another topic. Each introductory chapter, for example, opens with an engaging photograph relevant to the subject of the chapter and interesting applications that are easy for most students to visualize.

Chapter 1: Introduction: The Nature of Science and Physics
Chapter 2: Kinematics
Chapter 3: Two-Dimensional Kinematics
Chapter 4: Dynamics: Force and Newton's Laws of Motion
Chapter 5: Further Applications of Newton's Laws: Friction, Drag, and Elasticity
Chapter 6: Uniform Circular Motion and Gravitation
Chapter 7: Work, Energy, and Energy Resources
Chapter 8: Linear Momentum and Collisions
Chapter 9: Statics and Torque
Chapter 10: Rotational Motion and Angular Momentum
Chapter 11: Fluid Statics
Chapter 12: Fluid Dynamics and Its Biological and Medical Applications
Chapter 13: Temperature, Kinetic Theory, and the Gas Laws
Chapter 14: Heat and Heat Transfer Methods
Chapter 15: Thermodynamics
Chapter 16: Oscillatory Motion and Waves
Chapter 17: Physics of Hearing
Chapter 18: Electric Charge and Electric Field
Chapter 19: Electric Potential and Electric Field
Chapter 20: Electric Current, Resistance, and Ohm's Law
Chapter 21: Circuits and DC Instruments
Chapter 22: Magnetism
Chapter 23: Electromagnetic Induction, AC Circuits, and Electrical Technologies
Chapter 24: Electromagnetic Waves
Chapter 25: Geometric Optics
Chapter 26: Vision and Optical Instruments
Chapter 27: Wave Optics
Chapter 28: Special Relativity
Chapter 29: Introduction to Quantum Physics
Chapter 30: Atomic Physics
Chapter 31: Radioactivity and Nuclear Physics
Chapter 32: Medical Applications of Nuclear Physics
Chapter 33: Particle Physics
Appendix A: Atomic Masses
Appendix B: Selected Radioactive Isotopes
Appendix C: Useful Information
Appendix D: Glossary of Key Symbols and Notation

Concepts and Calculations
The ability to calculate does not guarantee conceptual understanding. In order to unify conceptual, analytical, and calculation skills within the learning process, we have integrated Strategies and Discussions throughout the text.

Modern Perspective
The chapters on modern physics are more complete than many other texts on the market, with an entire chapter devoted to medical applications of nuclear physics and another to particle physics. The final chapter of the text, "Frontiers of Physics," is devoted to the most exciting endeavors in physics. It ends with a module titled “Some Questions We Know to Ask.”

Key Features
Modularity
This textbook is organized as a collection of modules that can be rearranged and modified to suit the needs of a particular professor or class. That being said, modules often contain references to content in other modules, as most topics in physics cannot be discussed in isolation.

Learning Objectives
Every module begins with a set of learning objectives. These objectives are designed to guide the instructor in deciding what content to include or assign, and to guide the student with respect to what he or she can expect to learn. After completing the module and end-of-module exercises, students should be able to demonstrate mastery of the learning objectives.

Call-Outs
Key definitions, concepts, and equations are called out with a special design treatment. Call-outs are designed to catch readers’ attention, to make it clear that a specific term, concept, or equation is particularly important, and to provide easy reference for a student reviewing content.

Key Terms
Key terms are in bold and are followed by a definition in context. Definitions of key terms are also listed in the Glossary, which appears at the end of the module.

Worked Examples
Worked examples have four distinct parts to promote both analytical and conceptual skills. Worked examples are introduced in words, always using some application that should be of interest. This is followed by a Strategy section that emphasizes the concepts involved and how solving the problem relates to those concepts. This is followed by the mathematical Solution and Discussion.

Many worked examples contain multiple-part problems to help the students learn how to approach normal situations, in which problems tend to have multiple parts. Finally, worked examples employ the techniques of the problem-solving strategies so that students can see how those strategies succeed in practice as well as in theory.

Problem-Solving Strategies
Problem-solving strategies are first presented in a special section and subsequently appear at crucial points in the text where students can benefit most from them. Problem-solving strategies have a logical structure that is reinforced in the worked examples and supported in certain places by line drawings that illustrate various steps.
Misconception Alerts
Students come to physics with preconceptions from everyday experiences and from previous courses. Some of these preconceptions are misconceptions, and many are very common among students and the general public. Some are inadvertently picked up through misunderstandings of lectures and texts. The Misconception Alerts feature is designed to point these out and correct them explicitly.

Take-Home Investigations
Take Home Investigations provide the opportunity for students to apply or explore what they have learned with a hands-on activity.

Things Great and Small
In these special topic essays, macroscopic phenomena (such as air pressure) are explained with submicroscopic phenomena (such as atoms bouncing off walls). These essays support the modern perspective by describing aspects of modern physics before they are formally treated in later chapters. Connections are also made between apparently disparate phenomena.

Simulations
Where applicable, students are directed to the interactive PHeT physics simulations developed by the University of Colorado. There they can further explore the physics concepts they have learned about in the module.

Summary
Module summaries are thorough and functional and present all important definitions and equations. Students are able to find the definitions of all terms and symbols as well as their physical relationships. The structure of the summary makes plain the fundamental principles of the module or collection and serves as a useful study guide.

Glossary
At the end of every module or chapter is a Glossary containing definitions of all of the key terms in the module or chapter.

End-of-Module Problems
At the end of every chapter is a set of Conceptual Questions and/or skills-based Problems & Exercises. Conceptual Questions challenge students’ ability to explain what they have learned conceptually, independent of the mathematical details. Problems & Exercises challenge students to apply both concepts and skills to solve mathematical physics problems.

In addition to traditional skills-based problems, there are three special types of end-of-module problems: Integrated Concept Problems, Unreasonable Results Problems, and Construct Your Own Problems. All of these problems are indicated with a subtitle preceding the problem.

Integrated Concept Problems
In Integrated Concept Problems, students are asked to apply what they have learned about two or more concepts to arrive at a solution to a problem. These problems require a higher level of thinking because, before solving a problem, students have to recognize the combination of strategies required to solve it.

Unreasonable Results
In Unreasonable Results Problems, students are challenged to not only apply concepts and skills to solve a problem, but also to analyze the answer with respect to how likely or realistic it really is. These problems contain a premise that produces an unreasonable answer and are designed to further emphasize that properly applied physics must describe nature accurately and is not simply the process of solving equations.

Construct Your Own Problem
These problems require students to construct the details of a problem, justify their starting assumptions, show specific steps in the problem’s solution, and finally discuss the meaning of the result. These types of problems relate well to both conceptual and analytical aspects of physics, emphasizing that physics must describe nature. Often they involve an integration of topics from more than one chapter. Unlike other problems, solutions are not provided since there is no single correct answer. Instructors should feel free to direct students regarding the level and scope of their considerations. Whether the problem is solved and described correctly will depend on initial assumptions.

Additional Resources

Student and Instructor Resources
We’ve compiled additional resources for both students and instructors, including Getting Started Guides, an instructor solution manual, and PowerPoint slides. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

Partner Resources
OpenStax Partners are our allies in the mission to make high-quality learning materials affordable and accessible to students and instructors everywhere. Their tools integrate seamlessly with our OpenStax titles at a low cost. To access the partner resources for your text, visit your book page on openstax.org.
About the Authors

Senior Contributing Authors
Paul Peter Urone, Professor Emeritus at California State University, Sacramento
Roger Hinrichs, State University of New York, College at Oswego

Contributing Authors
Kim Dirks, University of Auckland
Manjula Sharma, University of Sydney

Reviewers
Matthew Adams, Crafton Hills College, San Bernardino Community College District
Erik Christensen, South Florida Community College
Douglas Ingram, Texas Christian University
Eric Kincanon, Gonzaga University
Lee H. LaRue, Paris Junior College
Chuck Pearson, Virginia Intermont College
Marc Sher, College of William and Mary
Ulrich Zurcher, Cleveland State University
INTRODUCTION: THE NATURE OF SCIENCE AND PHYSICS

Figure 1.1 Galaxies are as immense as atoms are small. Yet the same laws of physics describe both, and all the rest of nature—an indication of the underlying unity in the universe. The laws of physics are surprisingly few in number, implying an underlying simplicity to nature’s apparent complexity. (credit: NASA, JPL-Caltech, P. Barmby, Harvard-Smithsonian Center for Astrophysics)

Chapter Outline

1.1. Physics: An Introduction
  • Explain the difference between a principle and a law.
  • Explain the difference between a model and a theory.

1.2. Physical Quantities and Units
  • Perform unit conversions both in the SI and English units.
  • Explain the most common prefixes in the SI units and be able to write them in scientific notation.

1.3. Accuracy, Precision, and Significant Figures
  • Determine the appropriate number of significant figures in both addition and subtraction, as well as multiplication and division calculations.
  • Calculate the percent uncertainty of a measurement.

1.4. Approximation
  • Make reasonable approximations based on given data.

Introduction to Science and the Realm of Physics, Physical Quantities, and Units

What is your first reaction when you hear the word “physics”? Did you imagine working through difficult equations or memorizing formulas that seem to have no real use in life outside the physics classroom? Many people come to the subject of physics with a bit of fear. But as you begin your exploration of this broad-ranging subject, you may soon come to realize that physics plays a much larger role in your life than you first thought, no matter your life goals or career choice.

For example, take a look at the image above. This image is of the Andromeda Galaxy, which contains billions of individual stars, huge clouds of gas, and dust. Two smaller galaxies are also visible as bright blue spots in the background. At a staggering 2.5 million light years from the Earth, this galaxy is the nearest one to our own galaxy (which is called the Milky Way). The stars and planets that make up Andromeda might seem to be the furthest thing from most people’s regular, everyday lives. But Andromeda is a great starting point to think about the forces that hold together the universe. The forces that cause Andromeda to act as it does are the same forces we contend with here on Earth, whether we are planning to send a rocket into space or simply raise the walls for a new home. The same gravity that causes the stars of Andromeda to rotate and revolve also causes water to flow over hydroelectric dams here on Earth. Tonight, take a moment to look up at the stars. The forces out there are the same as the ones here on Earth. Through a study of physics, you may gain a greater understanding of the interconnectedness of everything we can see and know in this universe.

Think now about all of the technological devices that you use on a regular basis. Computers, smart phones, GPS systems, MP3 players, and satellite radio might come to mind. Next, think about the most exciting modern technologies that you have heard about in the news, such as trains that levitate above tracks, “invisibility cloaks” that bend light around them, and microscopic robots that fight cancer cells in our bodies. All of these groundbreaking advancements, commonplace or unbelievable, rely on the principles of physics. Aside from playing a significant role in technology, professionals such as engineers, pilots, physicians, physical therapists, electricians, and computer programmers apply physics concepts in their daily work. For example, a pilot must understand how wind forces affect a flight path and a physical therapist must understand how the muscles in the body experience forces as they move and bend. As you will learn in this text, physics principles are propelling new, exciting technologies, and these principles are applied in a wide range of careers.
In this text, you will begin to explore the history of the formal study of physics, beginning with natural philosophy and the ancient Greeks, and leading up through a review of Sir Isaac Newton and the laws of physics that bear his name. You will also be introduced to the standards scientists use when they study physical quantities and the interrelated system of measurements most of the scientific community uses to communicate in a single mathematical language. Finally, you will study the limits of our ability to be accurate and precise, and the reasons scientists go to painstaking lengths to be as clear as possible regarding their own limitations.

1.1 Physics: An Introduction

The physical universe is enormously complex in its detail. Every day, each of us observes a great variety of objects and phenomena. Over the centuries, the curiosity of the human race has led us collectively to explore and catalog a tremendous wealth of information. From the flight of birds to the colors of flowers, from lightning to gravity, from quarks to clusters of galaxies, from the flow of time to the mystery of the creation of the universe, we have asked questions and assembled huge arrays of facts. In the face of all these details, we have discovered that a surprisingly small and unified set of physical laws can explain what we observe. As humans, we make generalizations and seek order. We have found that nature is remarkably cooperative—it exhibits the underlying order and simplicity we so value.

It is the underlying order of nature that makes science in general, and physics in particular, so enjoyable to study. For example, what do a bag of chips and a car battery have in common? Both contain energy that can be converted to other forms. The law of conservation of energy (which says that energy can change form but is never lost) ties together such topics as food calories, batteries, heat, light, and watch springs. Understanding this law makes it easier to learn about the various forms energy takes and how they relate to one another. Apparently unrelated topics are connected through broadly applicable physical laws, permitting an understanding beyond just the memorization of lists of facts.

The unifying aspect of physical laws and the basic simplicity of nature form the underlying themes of this text. In learning to apply these laws, you will, of course, study the most important topics in physics. More importantly, you will gain analytical abilities that will enable you to apply these laws far beyond the scope of what can be included in a single book. These analytical skills will help you to excel academically, and they will also help you to think critically in any professional career you choose to pursue. This module discusses the realm of physics (to define what physics is), some applications of physics (to illustrate its relevance to other disciplines), and more precisely what constitutes a physical law (to illuminate the importance of experimentation to theory).

Science and the Realm of Physics

Science consists of the theories and laws that are the general truths of nature as well as the body of knowledge they encompass. Scientists are continually trying to expand this body of knowledge and to perfect the expression of the laws that describe it. Physics is concerned with describing the interactions of energy, matter, space, and time, and it is especially interested in what fundamental mechanisms underlie every phenomenon. The concern for describing the basic phenomena in nature essentially defines the realm of physics.

Physics aims to describe the function of everything around us, from the movement of tiny charged particles to the motion of people, cars, and spaceships. In fact, almost everything around you can be described quite accurately by the laws of physics. Consider a smart phone (Figure 1.3). Physics describes how electricity interacts with the various circuits inside the device. This knowledge helps engineers select the appropriate materials and circuit layout when building the smart phone. Next, consider a GPS system. Physics describes the relationship between the speed of an object, the distance over which it travels, and the time it takes to travel that distance. When you use a GPS device in a vehicle, it utilizes these physics equations to determine the travel time from one location to another.
Applications of Physics

You need not be a scientist to use physics. On the contrary, knowledge of physics is useful in everyday situations as well as in nonscientific professions. It can help you understand how microwave ovens work, why metals should not be put into them, and why they might affect pacemakers. (See Figure 1.4 and Figure 1.5.) Physics allows you to understand the hazards of radiation and rationally evaluate these hazards more easily. Physics also explains the reason why a black car radiator helps remove heat in a car engine, and it explains why a white roof helps keep the inside of a house cool. Similarly, the operation of a car’s ignition system as well as the transmission of electrical signals through our body’s nervous system are much easier to understand when you think about them in terms of basic physics.

Physics is the foundation of many important disciplines and contributes directly to others. Chemistry, for example—since it deals with the interactions of atoms and molecules—is rooted in atomic and molecular physics. Most branches of engineering are applied physics. In architecture, physics is at the heart of structural stability, and is involved in the acoustics, heating, lighting, and cooling of buildings. Parts of geology rely heavily on physics, such as radioactive dating of rocks, earthquake analysis, and heat transfer in the Earth. Some disciplines, such as biophysics and geophysics, are hybrids of physics and other disciplines.

Physics has many applications in the biological sciences. On the microscopic level, it helps describe the properties of cell walls and cell membranes (Figure 1.6 and Figure 1.7). On the macroscopic level, it can explain the heat, work, and power associated with the human body. Physics is involved in medical diagnostics, such as x-rays, magnetic resonance imaging (MRI), and ultrasonic blood flow measurements. Medical therapy sometimes directly involves physics; for example, cancer radiotherapy uses ionizing radiation. Physics can also explain sensory phenomena, such as how musical instruments make sound, how the eye detects color, and how lasers can transmit information.

It is not necessary to formally study all applications of physics. What is most useful is knowledge of the basic laws of physics and a skill in the analytical methods for applying them. The study of physics also can improve your problem-solving skills. Furthermore, physics has retained the most basic aspects of science, so it is used by all of the sciences, and the study of physics makes other sciences easier to understand.
These two applications of physics have more in common than meets the eye. Microwave ovens use electromagnetic waves to heat food. Magnetic resonance imaging (MRI) also uses electromagnetic waves to yield an image of the brain, from which the exact location of tumors can be determined. (credit: Rashmi Chawla, Daniel Smith, and Paul E. Marik)

Physics, chemistry, and biology help describe the properties of cell walls in plant cells, such as the onion cells seen here. (credit: Umberto Salvagnin)

An artist's rendition of the structure of a cell membrane. Membranes form the boundaries of animal cells and are complex in structure and function. Many of the most fundamental properties of life, such as the firing of nerve cells, are related to membranes. The disciplines of biology, chemistry, and physics all help us understand the membranes of animal cells. (credit: Mariana Ruiz)

Models, Theories, and Laws; The Role of Experimentation

The laws of nature are concise descriptions of the universe around us; they are human statements of the underlying laws or rules that all natural processes follow. Such laws are intrinsic to the universe; humans did not create them and so cannot change them. We can only discover and understand them. Their discovery is a very human endeavor, with all the elements of mystery, imagination, struggle, triumph, and disappointment inherent in any creative effort. (See Figure 1.8 and Figure 1.9.) The cornerstone of discovering natural laws is observation; science must describe the universe as it is, not as we may imagine it to be.
Figure 1.8 Isaac Newton (1642–1727) was very reluctant to publish his revolutionary work and had to be convinced to do so. In his later years, he stepped down from his academic post and became exchequer of the Royal Mint. He took this post seriously, inventing reeding (or creating ridges) on the edge of coins to prevent unscrupulous people from trimming the silver off of them before using them as currency. (credit: Arthur Shuster and Arthur E. Shipley: Britain’s Heritage of Science. London, 1917.)

Figure 1.9 Marie Curie (1867–1934) sacrificed monetary assets to help finance her early research and damaged her physical well-being with radiation exposure. She is the only person to win Nobel prizes in both physics and chemistry. One of her daughters also won a Nobel Prize. (credit: Wikimedia Commons)

We all are curious to some extent. We look around, make generalizations, and try to understand what we see—for example, we look up and wonder whether one type of cloud signals an oncoming storm. As we become serious about exploring nature, we become more organized and formal in collecting and analyzing data. We attempt greater precision, perform controlled experiments (if we can), and write down ideas about how the data may be organized and unified. We then formulate models, theories, and laws based on the data we have collected and analyzed to generalize and communicate the results of these experiments.

A **model** is a representation of something that is often too difficult (or impossible) to display directly. While a model is justified with experimental proof, it is only accurate under limited situations. An example is the planetary model of the atom in which electrons are pictured as orbiting the nucleus, analogous to the way planets orbit the Sun. (See Figure 1.10.) We cannot observe electron orbits directly, but the mental image helps explain the observations we can make, such as the emission of light from hot gases (atomic spectra). Physicists use models for a variety of purposes. For example, models can help physicists analyze a scenario and perform a calculation, or they can be used to represent a situation in the form of a computer simulation. A **theory** is an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers. Some theories include models to help visualize phenomena, whereas others do not. Newton’s theory of gravity, for example, does not require a model or mental image, because we can observe the objects directly with our own senses. The kinetic theory of gases, on the other hand, is a model in which a gas is viewed as being composed of atoms and molecules. Atoms and molecules are too small to be observed directly with our senses—thus, we picture them mentally to understand what our instruments tell us about the behavior of gases.

A **law** uses concise language to describe a generalized pattern in nature that is supported by scientific evidence and repeated experiments. Often, a law can be expressed in the form of a single mathematical equation. Laws and theories are similar in that they are both scientific statements that result from a tested hypothesis and are supported by scientific evidence. However, the
designation *law* is reserved for a concise and very general statement that describes phenomena in nature, such as the law that energy is conserved during any process, or Newton’s second law of motion, which relates force, mass, and acceleration by the simple equation \( F = ma \). A theory, in contrast, is a less concise statement of observed phenomena. For example, the Theory of Evolution and the Theory of Relativity cannot be expressed concisely enough to be considered a law. The biggest difference between a law and a theory is that a theory is much more complex and dynamic. A law describes a single action, whereas a theory explains an entire group of related phenomena. And, whereas a law is a postulate that forms the foundation of the scientific method, a theory is the end result of that process.

Less broadly applicable statements are usually called principles (such as Pascal’s principle, which is applicable only in fluids), but the distinction between laws and principles often is not carefully made.

![Figure 1.10](https://example.com/figure1.10.png)

**Figure 1.10** What is a model? This planetary model of the atom shows electrons orbiting the nucleus. It is a drawing that we use to form a mental image of the atom that we cannot see directly with our eyes because it is too small.

---

**Models, Theories, and Laws**

Models, theories, and laws are used to help scientists analyze the data they have already collected. However, often after a model, theory, or law has been developed, it points scientists toward new discoveries they would not otherwise have made.

The models, theories, and laws we devise sometimes *imply the existence of objects or phenomena as yet unobserved*. These predictions are remarkable triumphs and tributes to the power of science. It is the underlying order in the universe that enables scientists to make such spectacular predictions. However, if experiment does not verify our predictions, then the theory or law is wrong, no matter how elegant or convenient it is. Laws can never be known with absolute certainty because it is impossible to perform every imaginable experiment in order to confirm a law in every possible scenario. Physicists operate under the assumption that all scientific laws and theories are valid until a counterexample is observed. If a good-quality, verifiable experiment contradicts a well-established law, then the law must be modified or overthrown completely.

The study of science in general and physics in particular is an adventure much like the exploration of uncharted ocean. Discoveries are made; models, theories, and laws are formulated; and the beauty of the physical universe is made more sublime for the insights gained.

---

**The Scientific Method**

As scientists inquire and gather information about the world, they follow a process called the *scientific method*. This process typically begins with an observation and question that the scientist will research. Next, the scientist typically performs some research about the topic and then devises a hypothesis. Then, the scientist will test the hypothesis by performing an experiment. Finally, the scientist analyzes the results of the experiment and draws a conclusion. Note that the scientific method can be applied to many situations that are not limited to science, and this method can be modified to suit the situation.

Consider an example. Let us say that you try to turn on your car, but it will not start. You undoubtedly wonder: Why will the car not start? You can follow a scientific method to answer this question. First off, you may perform some research to determine a variety of reasons why the car will not start. Next, you will state a hypothesis. For example, you may believe that the car is not starting because it has no engine oil. To test this, you open the hood of the car and examine the oil level. You observe that the oil is at an acceptable level, and you thus conclude that the oil level is not contributing to your car issue. To troubleshoot the issue further, you may devise a new hypothesis to test and then repeat the process again.

---

**The Evolution of Natural Philosophy into Modern Physics**

Physics was not always a separate and distinct discipline. It remains connected to other sciences to this day. The word *physics* comes from Greek, meaning nature. The study of nature came to be called “natural philosophy.” From ancient times through the Renaissance, natural philosophy encompassed many fields, including astronomy, biology, chemistry, physics, mathematics, and medicine. Over the last few centuries, the growth of knowledge has resulted in ever-increasing specialization and branching of natural philosophy into separate fields, with physics retaining the most basic facets. (See Figure 1.11, Figure 1.12, and Figure 1.13.) Physics as it developed from the Renaissance to the end of the 19th century is called *classical physics*. It was transformed into modern physics by revolutionary discoveries made starting at the beginning of the 20th century.

---

This OpenStax book is available for free at https://legacy.cnx.org/content/col11951/1.1
Over the centuries, natural philosophy has evolved into more specialized disciplines, as illustrated by the contributions of some of the greatest minds in history. The Greek philosopher Aristotle (384–322 B.C.) wrote on a broad range of topics including physics, animals, the soul, politics, and poetry. (credit: Jastrow (2006)/Ludovisi Collection)

Galileo Galilei (1564–1642) laid the foundation of modern experimentation and made contributions in mathematics, physics, and astronomy. (credit: Domenico Tintoretto)

Niels Bohr (1885–1962) made fundamental contributions to the development of quantum mechanics, one part of modern physics. (credit: United States Library of Congress Prints and Photographs Division)

Classical physics is not an exact description of the universe, but it is an excellent approximation under the following conditions: Matter must be moving at speeds less than about 1% of the speed of light, the objects dealt with must be large enough to be seen with a microscope, and only weak gravitational fields, such as the field generated by the Earth, can be involved. Because humans live under such circumstances, classical physics seems intuitively reasonable, while many aspects of modern physics seem bizarre. This is why models are so useful in modern physics—they let us conceptualize phenomena we do not ordinarily experience. We can relate to models in human terms and visualize what happens when objects move at high speeds or imagine what objects too small to observe with our senses might be like. For example, we can understand an atom’s properties because we can picture it in our minds, although we have never seen an atom with our eyes. New tools, of course, allow us to better
picture phenomena we cannot see. In fact, new instrumentation has allowed us in recent years to actually “picture” the atom.

### Limits on the Laws of Classical Physics

For the laws of classical physics to apply, the following criteria must be met: Matter must be moving at speeds less than about 1% of the speed of light, the objects dealt with must be large enough to be seen with a microscope, and only weak gravitational fields (such as the field generated by the Earth) can be involved.

*Figure 1.14* Using a scanning tunneling microscope (STM), scientists can see the individual atoms that compose this sheet of gold. (credit: Erwinrossen)

Some of the most spectacular advances in science have been made in modern physics. Many of the laws of classical physics have been modified or rejected, and revolutionary changes in technology, society, and our view of the universe have resulted. Like science fiction, modern physics is filled with fascinating objects beyond our normal experiences, but it has the advantage over science fiction of being very real. Why, then, is the majority of this text devoted to topics of classical physics? There are two main reasons: Classical physics gives an extremely accurate description of the universe under a wide range of everyday circumstances, and knowledge of classical physics is necessary to understand modern physics.

Modern physics itself consists of the two revolutionary theories, relativity and quantum mechanics. These theories deal with the very fast and the very small, respectively. Relativity must be used whenever an object is traveling at greater than about 1% of the speed of light or experiences a strong gravitational field such as that near the Sun. Quantum mechanics must be used for objects smaller than can be seen with a microscope. The combination of these two theories is relativistic quantum mechanics, and it describes the behavior of small objects traveling at high speeds or experiencing a strong gravitational field. Relativistic quantum mechanics is the best universally applicable theory we have. Because of its mathematical complexity, it is used only when necessary, and the other theories are used whenever they will produce sufficiently accurate results. We will find, however, that we can do a great deal of modern physics with the algebra and trigonometry used in this text.

### Check Your Understanding

A friend tells you he has learned about a new law of nature. What can you know about the information even before your friend describes the law? How would the information be different if your friend told you he had learned about a scientific theory rather than a law?

**Solution**

Without knowing the details of the law, you can still infer that the information your friend has learned conforms to the requirements of all laws of nature: it will be a concise description of the universe around us; a statement of the underlying rules that all natural processes follow. If the information had been a theory, you would be able to infer that the information will be a large-scale, broadly applicable generalization.

### Equation Grapher

Learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (e.g. \( y = bx \)) to see how they add to generate the polynomial curve.

*(This media type is not supported in this reader. Click to open media in browser.)* (http://legacy.cnx.org/content/m42092/1.15/#graphing_polynomials)
1.2 Physical Quantities and Units

Figure 1.16 The distance from Earth to the Moon may seem immense, but it is just a tiny fraction of the distances from Earth to other celestial bodies. (credit: NASA)

The range of objects and phenomena studied in physics is immense. From the incredibly short lifetime of a nucleus to the age of the Earth, from the tiny sizes of sub-nuclear particles to the vast distance to the edges of the known universe, from the force exerted by a jumping flea to the force between Earth and the Sun, there are enough factors of 10 to challenge the imagination of even the most experienced scientist. Giving numerical values for physical quantities and equations for physical principles allows us to understand nature much more deeply than does qualitative description alone. To comprehend these vast ranges, we must also have accepted units in which to express them. And we shall find that (even in the potentially mundane discussion of meters, kilograms, and seconds) a profound simplicity of nature appears—most physical quantities can be expressed as combinations of only four fundamental physical quantities: length, mass, time, and electric current.

We define a physical quantity either by specifying how it is measured or by stating how it is calculated from other measurements. For example, we define distance and time by specifying methods for measuring them, whereas we define average speed by stating that it is calculated as distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of units, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way. (See Figure 1.17.)

Figure 1.17 Distances given in unknown units are maddeningly useless.

There are two major systems of units used in the world: SI units (also known as the metric system) and English units (also known as the customary or imperial system). English units were historically used in nations once ruled by the British Empire and are still widely used in the United States. Virtually every other country in the world now uses SI units as the standard; the metric system is also the standard system agreed upon by scientists and mathematicians. The acronym “SI” is derived from the French Système International.

SI Units: Fundamental and Derived Units

Table 1.1 gives the fundamental SI units that are used throughout this textbook. This text uses non-SI units in a few applications where they are in very common use, such as the measurement of blood pressure in millimeters of mercury (mm Hg). Whenever non-SI units are discussed, they will be tied to SI units through conversions.

<table>
<thead>
<tr>
<th>Table 1.1 Fundamental SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
</tr>
<tr>
<td>meter (m)</td>
</tr>
</tbody>
</table>
It is an intriguing fact that some physical quantities are more fundamental than others and that the most fundamental physical quantities can be defined only in terms of the procedure used to measure them. The units in which they are measured are thus called fundamental units. In this textbook, the fundamental physical quantities are taken to be length, mass, time, and electric current. (Note that electric current will not be introduced until much later in this text.) All other physical quantities, such as force and electric charge, can be expressed as algebraic combinations of length, mass, time, and current (for example, speed is length divided by time); these units are called derived units.

Units of Time, Length, and Mass: The Second, Meter, and Kilogram

The Second

The SI unit for time, the second (abbreviated s), has a long history. For many years it was defined as 1/86,400 of a mean solar day. More recently, a new standard was adopted to gain greater accuracy and to define the second in terms of a non-varying, or constant, physical phenomenon (because the solar day is getting longer due to very gradual slowing of the Earth's rotation). Cesium atoms can be made to vibrate in a very steady way, and these vibrations can be readily observed and counted. In 1967 the second was redefined as the time required for 9,192,631,770 of these vibrations. (See Figure 1.18.) Accuracy in the fundamental units is essential, because all measurements are ultimately expressed in terms of fundamental units and can be no more accurate than are the fundamental units themselves.

![Figure 1.18](https://legacy.cnx.org/content/col11951/1.1/figure/1.18)

An atomic clock such as this one uses the vibrations of cesium atoms to keep time to a precision of better than a microsecond per year. The fundamental unit of time, the second, is based on such clocks. This image is looking down from the top of an atomic fountain nearly 30 feet tall! (credit: Steve Jurvetson/Flickr)

The Meter

The SI unit for length is the meter (abbreviated m); its definition has also changed over time to become more accurate and precise. The meter was first defined in 1791 as 1/10,000,000 of the distance from the equator to the North Pole. This measurement was improved in 1889 by redefining the meter to be the distance between two engraved lines on a platinum-iridium bar now kept near Paris. By 1960, it had become possible to define the meter even more accurately in terms of the wavelength of light, so it was again redefined as 1,650,763.73 wavelengths of orange light emitted by krypton atoms. In 1983, the meter was given its present definition (partly for greater accuracy) as the distance light travels in a vacuum in 1/299,792,458 of a second. (See Figure 1.19.) This change defines the speed of light to be exactly 299,792,458 meters per second. The length of the meter will change if the speed of light is someday measured with greater accuracy.

![Figure 1.19](https://legacy.cnx.org/content/col11951/1.1/figure/1.19)

The meter is defined to be the distance light travels in 1/299,792,458 of a second in a vacuum. Distance traveled is speed multiplied by time.

The Kilogram

The SI unit for mass is the kilogram (abbreviated kg); it was previously defined to be the mass of a platinum-iridium cylinder kept with the old meter standard at the International Bureau of Weights and Measures near Paris. Exact replicas of the previously defined kilogram are also kept at the United States' National Institute of Standards and Technology, or NIST, located in Gaithersburg, Maryland outside of Washington D.C., and at other locations around the world. The determination of all other masses could be ultimately traced to a comparison with the standard mass. Even though the platinum-iridium cylinder was resistant to corrosion, airborne contaminants were able to adhere to its surface, slightly changing its mass over time. In May 2019, the scientific community adopted a more stable definition of the kilogram. The kilogram is now defined in terms of the second, the meter, and Planck's constant, $h$ (a quantum mechanical value that relates a photon's energy to its frequency).

Electric current and its accompanying unit, the ampere, will be introduced in Introduction to Electric Current, Resistance, and Ohm's Law (https://legacy.cnx.org/content/m42339/latest/) when electricity and magnetism are covered. The initial modules
in this textbook are concerned with mechanics, fluids, heat, and waves. In these subjects all pertinent physical quantities can be expressed in terms of the fundamental units of length, mass, and time.

**Metric Prefixes**

SI units are part of the **metric system**. The metric system is convenient for scientific and engineering calculations because the units are categorized by factors of 10. Table 1.2 gives metric prefixes and symbols used to denote various factors of 10.

Metric systems have the advantage that conversions of units involve only powers of 10. There are 100 centimeters in a meter, 1000 meters in a kilometer, and so on. In nonmetric systems, such as the system of U.S. customary units, the relationships are not as simple—there are 12 inches in a foot, 5280 feet in a mile, and so on. Another advantage of the metric system is that the same unit can be used over extremely large ranges of values simply by using an appropriate metric prefix. For example, distances in meters are suitable in construction, while distances in kilometers are appropriate for air travel, and the tiny measure of nanometers are convenient in optical design. With the metric system there is no need to invent new units for particular applications.

The term **order of magnitude** refers to the scale of a value expressed in the metric system. Each power of 10 in the metric system represents a different order of magnitude. For example, $10^1$, $10^2$, $10^3$, and so forth are all different orders of magnitude. All quantities that can be expressed as a product of a specific power of 10 are said to be of the same order of magnitude. For example, the number 800 can be written as $8 \times 10^2$, and the number 450 can be written as $4.5 \times 10^2$. Thus, the numbers 800 and 450 are of the same order of magnitude: $10^2$. Order of magnitude can be thought of as a ballpark estimate for the scale of a value. The diameter of an atom is on the order of $10^{-10}$ m, while the diameter of the Sun is on the order of $10^9$ m.

---

**The Quest for Microscopic Standards for Basic Units**

The fundamental units described in this chapter are those that produce the greatest accuracy and precision in measurement. There is a sense among physicists that, because there is an underlying microscopic substructure to matter, it would be most satisfying to base our standards of measurement on microscopic objects and fundamental physical phenomena such as the speed of light. A microscopic standard has been accomplished for the standard of time, which is based on the oscillations of the cesium atom.

The standard for length was once based on the wavelength of light (a small-scale length) emitted by a certain type of atom, but it has been supplanted by the more precise measurement of the speed of light. If it becomes possible to measure the mass of atoms or a particular arrangement of atoms such as a silicon sphere to greater precision than the kilogram standard, it may become possible to base mass measurements on the small scale. There are also possibilities that electrical phenomena on the small scale may someday allow us to base a unit of charge on the charge of electrons and protons, but at present current and charge are related to large-scale currents and forces between wires.
**Table 1.2 Metric Prefixes for Powers of 10 and their Symbols**

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Symbol</th>
<th>Value</th>
<th>Example (some are approximate)</th>
</tr>
</thead>
<tbody>
<tr>
<td>exa</td>
<td>E</td>
<td>$10^{18}$</td>
<td>exameter $10^{18}$ m distance light travels in a century</td>
</tr>
<tr>
<td>peta</td>
<td>P</td>
<td>$10^{15}$</td>
<td>petasecond $10^{15}$ s 30 million years</td>
</tr>
<tr>
<td>tera</td>
<td>T</td>
<td>$10^{12}$</td>
<td>terawatt $10^{12}$ W powerful laser output</td>
</tr>
<tr>
<td>giga</td>
<td>G</td>
<td>$10^{9}$</td>
<td>gigahertz $10^{9}$ Hz a microwave frequency</td>
</tr>
<tr>
<td>mega</td>
<td>M</td>
<td>$10^{6}$</td>
<td>megacurie $10^{6}$ Ci high radioactivity</td>
</tr>
<tr>
<td>kilo</td>
<td>k</td>
<td>$10^{3}$</td>
<td>kilometer $10^{3}$ m about 6/10 mile</td>
</tr>
<tr>
<td>hecto</td>
<td>h</td>
<td>$10^{2}$</td>
<td>hectoliter $10^{2}$ L 26 gallons</td>
</tr>
<tr>
<td>deka</td>
<td>da</td>
<td>$10^{1}$</td>
<td>dekagram $10^{1}$ g teaspoon of butter</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>$10^{0}$</td>
<td>(=1)</td>
</tr>
<tr>
<td>deci</td>
<td>d</td>
<td>$10^{-1}$</td>
<td>deciliter $10^{-1}$ L less than half a soda</td>
</tr>
<tr>
<td>centi</td>
<td>c</td>
<td>$10^{-2}$</td>
<td>centimeter $10^{-2}$ m fingertip thickness</td>
</tr>
<tr>
<td>milli</td>
<td>m</td>
<td>$10^{-3}$</td>
<td>millimeter $10^{-3}$ m flea at its shoulders</td>
</tr>
<tr>
<td>micro</td>
<td>µ</td>
<td>$10^{-6}$</td>
<td>micrometer $10^{-6}$ m detail in microscope</td>
</tr>
<tr>
<td>nano</td>
<td>n</td>
<td>$10^{-9}$</td>
<td>nanogram $10^{-9}$ g small speck of dust</td>
</tr>
<tr>
<td>pico</td>
<td>p</td>
<td>$10^{-12}$</td>
<td>picofarad $10^{-12}$ F small capacitor in radio</td>
</tr>
<tr>
<td>femto</td>
<td>f</td>
<td>$10^{-15}$</td>
<td>femtometer $10^{-15}$ m size of a proton</td>
</tr>
<tr>
<td>atto</td>
<td>a</td>
<td>$10^{-18}$</td>
<td>attosecond $10^{-18}$ s time light crosses an atom</td>
</tr>
</tbody>
</table>

**Known Ranges of Length, Mass, and Time**

The vastness of the universe and the breadth over which physics applies are illustrated by the wide range of examples of known lengths, masses, and times in Table 1.3. Examination of this table will give you some feeling for the range of possible topics and numerical values. (See Figure 1.20 and Figure 1.21.)

![Figure 1.20 Tiny phytoplankton swims among crystals of ice in the Antarctic Sea. They range from a few micrometers to as much as 2 millimeters in length. (credit: Prof. Gordon T. Taylor, Stony Brook University; NOAA Corps Collections)]
Unit Conversion and Dimensional Analysis

It is often necessary to convert from one type of unit to another. For example, if you are reading a European cookbook, some quantities may be expressed in units of liters and you need to convert them to cups. Or, perhaps you are reading walking directions from one location to another and you are interested in how many miles you will be walking. In this case, you will need to convert units of feet to miles.

Let us consider a simple example of how to convert units. Let us say that we want to convert 80 meters (m) to kilometers (km).

The first thing to do is to list the units that you have and the units that you want to convert to. In this case, we have units in meters and we want to convert to kilometers.

Next, we need to determine a conversion factor relating meters to kilometers. A conversion factor is a ratio expressing how many of one unit are equal to another unit. For example, there are 12 inches in 1 foot, 100 centimeters in 1 meter, 60 seconds in 1 minute, and so on. In this case, we know that there are 1,000 meters in 1 kilometer.

Now we can set up our unit conversion. We will write the units that we have and then multiply them by the conversion factor so that the units cancel out, as shown:

$$80\text{m} \times \frac{1\text{km}}{1000\text{m}} = 0.080\text{km}.$$ (1.1)

Note that the unwanted m unit cancels, leaving only the desired km unit. You can use this method to convert between any types of unit.

Click Appendix C for a more complete list of conversion factors.
### Example 1.1 Unit Conversions: A Short Drive Home

Suppose that you drive the 10.0 km from your university to home in 20.0 min. Calculate your average speed (a) in kilometers per hour (km/h) and (b) in meters per second (m/s). (Note: Average speed is distance traveled divided by time of travel.)

**Strategy**

1. First we calculate the average speed using the given units. Then we can get the average speed into the desired units by picking the correct conversion factor and multiplying by it. The correct conversion factor is the one that cancels the unwanted unit and leaves the desired unit in its place.

**Solution for (a)**

1. Calculate average speed. Average speed is distance traveled divided by time of travel. (Take this definition as a given for now—average speed and other motion concepts will be covered in a later module.) In equation form,

   \[
   \text{average speed} = \frac{\text{distance}}{\text{time}}
   \]

   (1.2)

2. Substitute the given values for distance and time.
average speed $= \frac{10.0 \text{ km}}{20.0 \text{ min}} = 0.500 \frac{\text{km}}{\text{min}}$ \hfill (1.3)

(3) Convert km/min to km/h: multiply by the conversion factor that will cancel minutes and leave hours. That conversion factor is 60 min/hr. Thus,

$$\text{average speed} = 0.500 \frac{\text{km}}{\text{min}} \times \frac{60 \text{ min}}{1 \text{ h}} = 30.0 \frac{\text{km}}{\text{h}}$$ \hfill (1.4)

**Discussion for (a)**

To check your answer, consider the following:

(1) Be sure that you have properly cancelled the units in the unit conversion. If you have written the unit conversion factor upside down, the units will not cancel properly in the equation. If you accidentally get the ratio upside down, then the units will not cancel; rather, they will give you the wrong units as follows:

$$\text{km} \times \frac{1 \text{ hr}}{60 \text{ min}} = \frac{1 \text{ km}}{60 \text{ min}^2},$$ \hfill (1.5)

which are obviously not the desired units of km/h.

(2) Check that the units of the final answer are the desired units. The problem asked us to solve for average speed in units of km/h and we have indeed obtained these units.

(3) Check the significant figures. Because each of the values given in the problem has three significant figures, the answer should also have three significant figures. The answer 30.0 km/hr does indeed have three significant figures, so this is appropriate. Note that the significant figures in the conversion factor are not relevant because an hour is defined to be 60 minutes, so the precision of the conversion factor is perfect.

(4) Next, check whether the answer is reasonable. Let us consider some information from the problem—if you travel 10 km in a third of an hour (20 min), you would travel three times that far in an hour. The answer does seem reasonable.

**Solution for (b)**

There are several ways to convert the average speed into meters per second.

(1) Start with the answer to (a) and convert km/h to m/s. Two conversion factors are needed—one to convert hours to seconds, and another to convert kilometers to meters.

(2) Multiplying by these yields

$$\text{Average speed} = 30.0 \frac{\text{km}}{\text{h}} \times \frac{1 \text{ h}}{3,600 \text{ s}} \times \frac{1,000 \text{ m}}{1 \text{ km}},$$ \hfill (1.6)

$$\text{Average speed} = 8.33 \frac{\text{m}}{\text{s}}.$$ \hfill (1.7)

**Discussion for (b)**

If we had started with 0.500 km/min, we would have needed different conversion factors, but the answer would have been the same: 8.33 m/s.

You may have noted that the answers in the worked example just covered were given to three digits. Why? When do you need to be concerned about the number of digits in something you calculate? Why not write down all the digits your calculator produces? The module *Accuracy, Precision, and Significant Figures* will help you answer these questions.

---

**Nonstandard Units**

While there are numerous types of units that we are all familiar with, there are others that are much more obscure. For example, a *firkin* is a unit of volume that was once used to measure beer. One firkin equals about 34 liters. To learn more about nonstandard units, use a dictionary or encyclopedia to research different “weights and measures.” Take note of any unusual units, such as a barleycorn, that are not listed in the text. Think about how the unit is defined and state its relationship to SI units.

---

**Check Your Understanding**

Some hummingbirds beat their wings more than 50 times per second. A scientist is measuring the time it takes for a hummingbird to beat its wings once. Which fundamental unit should the scientist use to describe the measurement? Which factor of 10 is the scientist likely to use to describe the motion precisely? Identify the metric prefix that corresponds to this factor of 10.

**Solution**

The scientist will measure the time between each movement using the fundamental unit of seconds. Because the wings beat so fast, the scientist will probably need to measure in milliseconds, or $10^{-3}$ seconds. (50 beats per second corresponds to...
20 milliseconds per beat.

Check Your Understanding

One cubic centimeter is equal to one milliliter. What does this tell you about the different units in the SI metric system?

Solution

The fundamental unit of length (meter) is probably used to create the derived unit of volume (liter). The measure of a milliliter is dependent on the measure of a centimeter.

1.3 Accuracy, Precision, and Significant Figures

Figure 1.22 A double-pan mechanical balance is used to compare different masses. Usually an object with unknown mass is placed in one pan and objects of known mass are placed in the other pan. When the bar that connects the two pans is horizontal, then the masses in both pans are equal. The "known masses" are typically metal cylinders of standard mass such as 1 gram, 10 grams, and 100 grams. (credit: Serge Melki)

Figure 1.23 Many mechanical balances, such as double-pan balances, have been replaced by digital scales, which can typically measure the mass of an object more precisely. Whereas a mechanical balance may only read the mass of an object to the nearest tenth of a gram, many digital scales can measure the mass of an object up to the nearest thousandth of a gram. (credit: Karel Jakubec)

Accuracy and Precision of a Measurement

Science is based on observation and experiment—that is, on measurements. **Accuracy** is how close a measurement is to the correct value for that measurement. For example, let us say that you are measuring the length of standard computer paper. The packaging in which you purchased the paper states that it is 11.0 inches long. You measure the length of the paper three times and obtain the following measurements: 11.1 in., 11.2 in., and 10.9 in. These measurements are quite accurate because they are very close to the correct value of 11.0 inches. In contrast, if you had obtained a measurement of 12 inches, your measurement would not be very accurate.

The **precision** of a measurement system refers to how close the agreement is between repeated measurements (which are repeated under the same conditions). Consider the example of the paper measurements. The precision of the measurements refers to the spread of the measured values. One way to analyze the precision of the measurements would be to determine the range, or difference, between the lowest and the highest measured values. In that case, the lowest value was 10.9 in. and the highest value was 11.2 in. Thus, the measured values deviated from each other by at most 0.3 in. These measurements were relatively precise because they did not vary too much in value. However, if the measured values had been 10.9, 11.1, and 11.9,
then the measurements would not be very precise because there would be significant variation from one measurement to another.

The measurements in the paper example are both accurate and precise, but in some cases, measurements are accurate but not precise, or they are precise but not accurate. Let us consider an example of a GPS system that is attempting to locate the position of a restaurant in a city. Think of the restaurant location as existing at the center of a bull's-eye target, and think of each GPS attempt to locate the restaurant as a black dot. In Figure 1.24, you can see that the GPS measurements are spread out far apart from each other, but they are all relatively close to the actual location of the restaurant at the center of the target. This indicates a low precision, high accuracy measuring system. However, in Figure 1.25, the GPS measurements are concentrated quite closely to one another, but they are far away from the target location. This indicates a high precision, low accuracy measuring system.

![Figure 1.24 A GPS system attempts to locate a restaurant at the center of the bull's-eye. The black dots represent each attempt to pinpoint the location of the restaurant. The dots are spread out quite far apart from one another, indicating low precision, but they are each rather close to the actual location of the restaurant, indicating high accuracy. (credit: Dark Evil)](https://example.com/fig124)

![Figure 1.25 In this figure, the dots are concentrated rather closely to one another, indicating high precision, but they are rather far away from the actual location of the restaurant, indicating low accuracy. (credit: Dark Evil)](https://example.com/fig125)

### Accuracy, Precision, and Uncertainty

The degree of accuracy and precision of a measuring system are related to the uncertainty in the measurements. Uncertainty is a quantitative measure of how much your measured values deviate from a standard or expected value. If your measurements are not very accurate or precise, then the uncertainty of your values will be very high. In more general terms, uncertainty can be thought of as a disclaimer for your measured values. For example, if someone asked you to provide the mileage on your car, you might say that it is 45,000 miles, plus or minus 500 miles. The plus or minus amount is the uncertainty in your value. That is, you are indicating that the actual mileage of your car might be as low as 44,500 miles or as high as 45,500 miles, or anywhere in between. All measurements contain some amount of uncertainty. In our example of measuring the length of the paper, we might say that the length of the paper is 11 in., plus or minus 0.2 in. The uncertainty in a measurement, \( \Delta A \), is often denoted as \( \delta A \) (“delta A”), so the measurement result would be recorded as \( A \pm \Delta A \). In our paper example, the length of the paper could be expressed as 11 in. \( \pm \) 0.2.

The factors contributing to uncertainty in a measurement include:

1. Limitations of the measuring device,
2. The skill of the person making the measurement,
3. Irregularities in the object being measured,
4. Any other factors that affect the outcome (highly dependent on the situation).

In our example, such factors contributing to the uncertainty could be the following: the smallest division on the ruler is 0.1 in., the person using the ruler has bad eyesight, or one side of the paper is slightly longer than the other. At any rate, the uncertainty in a measurement must be based on a careful consideration of all the factors that might contribute and their possible effects.
Making Connections: Real-World Connections – Fevers or Chills?

Uncertainty is a critical piece of information, both in physics and in many other real-world applications. Imagine you are caring for a sick child. You suspect the child has a fever, so you check his or her temperature with a thermometer. What if the uncertainty of the thermometer were $3.0^\circ C$? If the child’s temperature reading was $37.0^\circ C$ (which is normal body temperature), the “true” temperature could be anywhere from a hypothermic $34.0^\circ C$ to a dangerously high $40.0^\circ C$. A thermometer with an uncertainty of $3.0^\circ C$ would be useless.

Percent Uncertainty

One method of expressing uncertainty is as a percent of the measured value. If a measurement $A$ is expressed with uncertainty, $\delta A$, the percent uncertainty ($%unc$) is defined to be

$$% unc = \frac{\delta A}{A} \times 100\%.$$  

(1.8)

Example 1.2 Calculating Percent Uncertainty: A Bag of Apples

A grocery store sells 5-lb bags of apples. You purchase four bags over the course of a month and weigh the apples each time. You obtain the following measurements:

- Week 1 weight: 4.8 lb
- Week 2 weight: 5.3 lb
- Week 3 weight: 4.9 lb
- Week 4 weight: 5.4 lb

You determine that the weight of the 5-lb bag has an uncertainty of $\pm 0.4$ lb. What is the percent uncertainty of the bag’s weight?

Strategy

First, observe that the expected value of the bag’s weight, $A$, is 5 lb. The uncertainty in this value, $\delta A$, is 0.4 lb. We can use the following equation to determine the percent uncertainty of the weight:

$$% unc = \frac{\delta A}{A} \times 100\%.$$  

(1.9)

Solution

Plug the known values into the equation:

$$% unc = \frac{0.4 \text{ lb}}{5 \text{ lb}} \times 100\% = 8\%.$$  

(1.10)

Discussion

We can conclude that the weight of the apple bag is 5 lb $\pm 8\%$. Consider how this percent uncertainty would change if the bag of apples were half as heavy, but the uncertainty in the weight remained the same. Hint for future calculations: when calculating percent uncertainty, always remember that you must multiply the fraction by 100%. If you do not do this, you will have a decimal quantity, not a percent value.

Uncertainties in Calculations

There is an uncertainty in anything calculated from measured quantities. For example, the area of a floor calculated from measurements of its length and width has an uncertainty because the length and width have uncertainties. How big is the uncertainty in something you calculate by multiplication or division? If the measurements going into the calculation have small uncertainties (a few percent or less), then the method of adding percents can be used for multiplication or division. This method says that the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation. For example, if a floor has a length of $4.00 \text{ m}$ and a width of $3.00 \text{ m}$, with uncertainties of 2% and 1%, respectively, then the area of the floor is $12.0 \text{ m}^2$ and has an uncertainty of 3%. (Expressed as an area this is $0.36 \text{ m}^2$, which we round to $0.4 \text{ m}^2$ since the area of the floor is given to a tenth of a square meter.)

Check Your Understanding

A high school track coach has just purchased a new stopwatch. The stopwatch manual states that the stopwatch has an uncertainty of $\pm 0.05 \text{ s}$. The team’s top sprinter clocked a 100 meter sprint at 12.04 seconds last week and at 11.96 seconds
this week. Can we conclude that this week’s time was faster?

Solution
No, the uncertainty in the stopwatch is too great to effectively differentiate between the sprint times.

Precision of Measuring Tools and Significant Figures

An important factor in the accuracy and precision of measurements involves the precision of the measuring tool. In general, a precise measuring tool is one that can measure values in very small increments. For example, a standard ruler can measure length to the nearest millimeter, while a caliper can measure length to the nearest 0.01 millimeter. The caliper is a more precise measuring tool because it can measure extremely small differences in length. The more precise the measuring tool, the more precise and accurate the measurements can be.

When we express measured values, we can only list as many digits as we initially measured with our measuring tool. For example, if you use a standard ruler to measure the length of a stick, you may measure it to be \(36.7\) cm. You could not express this value as \(36.71\) cm because your measuring tool was not precise enough to measure a hundredth of a centimeter. It should be noted that the last digit in a measured value has been estimated in some way by the person performing the measurement. For example, the person measuring the length of a stick with a ruler notices that the stick length seems to be somewhere in between \(36.6\) cm and \(36.7\) cm, and he or she must estimate the value of the last digit. Using the method of significant figures, the rule is that the last digit written down in a measurement is the first digit with some uncertainty. In order to determine the number of significant digits in a value, start with the first measured value at the left and count the number of digits through the last digit written on the right. For example, the measured value \(36.7\) cm has three digits, or significant figures. Significant figures indicate the precision of a measuring tool that was used to measure a value.

Zeros
Special consideration is given to zeros when counting significant figures. The zeros in 0.053 are not significant, because they are only placekeepers that locate the decimal point. There are two significant figures in 0.053. The zeros in 10.053 are not placekeepers but are significant—this number has five significant figures. The zeros in 1300 may or may not be significant depending on the style of writing numbers. They could mean the number is known to the last digit, or they could be placekeepers. So 1300 could have two, three, or four significant figures. (To avoid this ambiguity, write 1300 in scientific notation.) Zeros are significant except when they serve only as placekeepers.

Check Your Understanding

Determine the number of significant figures in the following measurements:

a. \(0.0009\)

b. \(15,450.0\)

c. \(6 \times 10^3\)

d. \(87.990\)

e. \(30.42\)

Solution
(a) 1; the zeros in this number are placekeepers that indicate the decimal point
(b) 6; here, the zeros indicate that a measurement was made to the 0.1 decimal point, so the zeros are significant
(c) 1; the value \(10^3\) signifies the decimal place, not the number of measured values
(d) 5; the final zero indicates that a measurement was made to the 0.001 decimal point, so it is significant
(e) 4; any zeros located in between significant figures in a number are also significant

Significant Figures in Calculations

When combining measurements with different degrees of accuracy and precision, the number of significant digits in the final answer can be no greater than the number of significant digits in the least precise measured value. There are two different rules, one for multiplication and division and the other for addition and subtraction, as discussed below.

1. **For multiplication and division:** The result should have the same number of significant figures as the quantity having the least significant figures entering into the calculation. For example, the area of a circle can be calculated from its radius using \(A = \pi r^2\). Let us see how many significant figures the area has if the radius has only two—say, \(r = 1.2\) m. Then,

\[
A = \pi r^2 = (3.1415927...)(1.2 \text{ m})^2 = 4.5238934 \text{ m}^2
\]  

(1.11)

is what you would get using a calculator that has an eight-digit output. But because the radius has only two significant figures, it limits the calculated quantity to two significant figures or
\[ A = 4.5 \text{ m}^2, \quad (1.12) \]

even though \( \pi \) is good to at least eight digits.

2. For addition and subtraction: The answer can contain no more decimal places than the least precise measurement.

Suppose that you buy 7.56-kg of potatoes in a grocery store as measured with a scale with precision 0.01 kg. Then you drop off 6.052-kg of potatoes at your laboratory as measured by a scale with precision 0.001 kg. Finally, you go home and add 13.7 kg of potatoes as measured by a bathroom scale with precision 0.1 kg. How many kilograms of potatoes do you now have, and how many significant figures are appropriate in the answer? The mass is found by simple addition and subtraction:

\[ \begin{align*}
7.56 \text{ kg} & - 6.052 \text{ kg} \\
+13.7 \text{ kg} & = 15.208 \text{ kg}
\end{align*} \quad (1.13) \]

Next, we identify the least precise measurement: 13.7 kg. This measurement is expressed to the 0.1 decimal place, so our final answer must also be expressed to the 0.1 decimal place. Thus, the answer is rounded to the tenths place, giving us 15.2 kg.

**Significant Figures in this Text**

In this text, most numbers are assumed to have three significant figures. Furthermore, consistent numbers of significant figures are used in all worked examples. You will note that an answer given to three digits is based on input good to at least three digits, for example. If the input has fewer significant figures, the answer will also have fewer significant figures. Care is also taken that the number of significant figures is reasonable for the situation posed. In some topics, particularly in optics, more accurate numbers are needed and more than three significant figures will be used. Finally, if a number is exact, such as the two in the formula for the circumference of a circle, \( c = 2\pi r \), it does not affect the number of significant figures in a calculation.

**Check Your Understanding**

Perform the following calculations and express your answer using the correct number of significant digits.

(a) A woman has two bags weighing 13.5 pounds and one bag with a weight of 10.2 pounds. What is the total weight of the bags?

(b) The force \( F \) on an object is equal to its mass \( m \) multiplied by its acceleration \( a \). If a wagon with mass 55 kg accelerates at a rate of 0.0255 m/s\(^2\), what is the force on the wagon? (The unit of force is called the newton, and it is expressed with the symbol N.)

**Solution**

(a) 37.2 pounds; Because the number of bags is an exact value, it is not considered in the significant figures.

(b) 1.4 N; Because the value 55 kg has only two significant figures, the final value must also contain two significant figures.

**Estimation**

Explore size estimation in one, two, and three dimensions! Multiple levels of difficulty allow for progressive skill improvement.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42120/1.20/#EstimationSimulation)

**Figure 1.26**

## 1.4 Approximation

On many occasions, physicists, other scientists, and engineers need to make approximations or “guesstimates” for a particular quantity. What is the distance to a certain destination? What is the approximate density of a given item? About how large a current will there be in a circuit? Many approximate numbers are based on formulae in which the input quantities are known only to a limited accuracy. As you develop problem-solving skills (that can be applied to a variety of fields through a study of physics), you will also develop skills at approximating. You will develop these skills through thinking more quantitatively, and by being willing to take risks. As with any endeavor, experience helps, as well as familiarity with units. These approximations allow us to rule out certain scenarios or unrealistic numbers. Approximations also allow us to challenge others and guide us in our approaches to our scientific world. Let us do two examples to illustrate this concept.

**Example 1.3 Approximate the Height of a Building**

Can you approximate the height of one of the buildings on your campus, or in your neighborhood? Let us make an approximation based upon the height of a person. In this example, we will calculate the height of a 39-story building.

**Strategy**
Think about the average height of an adult male. We can approximate the height of the building by scaling up from the height of a person.

**Solution**

Based on information in the example, we know there are 39 stories in the building. If we use the fact that the height of one story is approximately equal to the length of two adult humans (each human is about 2-m tall), then we can estimate the total height of the building to be

\[
\frac{2 \text{ m}}{1 \text{ person}} \times \frac{2 \text{ person}}{1 \text{ story}} \times 39 \text{ stories} = 156 \text{ m}. \tag{1.14}
\]

**Discussion**

You can use known quantities to determine an approximate measurement of unknown quantities. If your hand measures 10 cm across, how many hand lengths equal the width of your desk? What other measurements can you approximate besides length?

---

**Example 1.4 Approximating Vast Numbers: a Trillion Dollars**

![Image of bank stacks of money]

**Figure 1.27** A bank stack contains one-hundred $100 bills, and is worth $10,000. How many bank stacks make up a trillion dollars? (credit: Andrew Magill)

The U.S. federal debt in the 2008 fiscal year was a little less than $10 trillion. Most of us do not have any concept of how much even one trillion actually is. Suppose that you were given a trillion dollars in $100 bills. If you made 100-bill stacks and used them to evenly cover a football field (between the end zones), make an approximation of how high the money pile would become. (We will use feet/inches rather than meters here because football fields are measured in yards.) One of your friends says 3 in., while another says 10 ft. What do you think?

**Strategy**

When you imagine the situation, you probably envision thousands of small stacks of 100 wrapped $100 bills, such as you might see in movies or at a bank. Since this is an easy-to-approximate quantity, let us start there. We can find the volume of a stack of 100 bills, find out how many stacks make up one trillion dollars, and then set this volume equal to the area of the football field multiplied by the unknown height.

**Solution**

1. Calculate the volume of a stack of 100 bills. The dimensions of a single bill are approximately 3 in. by 6 in. A stack of 100 of these is about 0.5 in. thick. So the total volume of a stack of 100 bills is:

\[
\text{volume of stack} = \text{length} \times \text{width} \times \text{height}, \tag{1.15}
\]

\[
\text{volume of stack} = 6 \text{ in.} \times 3 \text{ in.} \times 0.5 \text{ in.},
\]

\[
\text{volume of stack} = 9 \text{ in.}^3.
\]

2. Calculate the number of stacks. Note that a trillion dollars is equal to \(1 \times 10^{12}\), and a stack of one-hundred $100 bills is equal to $10,000, or \(1 \times 10^4\). The number of stacks you will have is:

\[
(1 \times 10^{12} \text{ (a trillion dollars)}/1 \times 10^4 \text{ per stack}) = 1 \times 10^8 \text{ stacks.} \tag{1.16}
\]

3. Calculate the area of a football field in square inches. The area of a football field is 100 yd \times 50 yd, which gives
5,000 yd\(^2\). Because we are working in inches, we need to convert square yards to square inches:

\[
\text{Area} = 5,000 \text{ yd}^2 \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{12 \text{ in.}}{1 \text{ ft}} \times \frac{12 \text{ in.}}{1 \text{ ft}} = 6,480,000 \text{ in.}^2,
\]

This conversion gives us \(6 \times 10^6\) in\(^2\) for the area of the field. (Note that we are using only one significant figure in these calculations.)

(4) Calculate the total volume of the bills. The volume of all the \$100\,-\text{bill stacks is}

\[
9 \text{ in.}^3 / \text{stack} \times 10^8 \text{ stacks} = 9 \times 10^8 \text{ in.}^3.
\]

(5) Calculate the height. To determine the height of the bills, use the equation:

\[
\text{Height of money} = \frac{\text{volume of bills}}{\text{area of field}},
\]

\[
\text{Height of money} = \frac{9 \times 10^8 \text{ in.}^3}{6 \times 10^6 \text{ in.}^2} = 1.33 \times 10^2 \text{ in.},
\]

\[
\text{Height of money} \approx 1 \times 10^2 \text{ in.} = 100 \text{ in.}
\]

The height of the money will be about 100 in. high. Converting this value to feet gives

\[
100 \text{ in.} \times \frac{1 \text{ ft}}{12 \text{ in.}} = 8.33 \text{ ft} \approx 8 \text{ ft}.
\]

**Discussion**

The final approximate value is much higher than the early estimate of 3 in., but the other early estimate of 10 ft (120 in.) was roughly correct. How did the approximation measure up to your first guess? What can this exercise tell you in terms of rough "guesstimates" versus carefully calculated approximations?

---

**Check Your Understanding**

Using mental math and your understanding of fundamental units, approximate the area of a regulation basketball court. Describe the process you used to arrive at your final approximation.

**Solution**

An average male is about two meters tall. It would take approximately 15 men laid out end to end to end to cover the length, and about 7 to cover the width. That gives an approximate area of \(420\) m\(^2\).

---

**Glossary**

- **accuracy**: the degree to which a measured value agrees with correct value for that measurement
- **approximation**: an estimated value based on prior experience and reasoning
- **classical physics**: physics that was developed from the Renaissance to the end of the 19th century
- **conversion factor**: a ratio expressing how many of one unit are equal to another unit
- **derived units**: units that can be calculated using algebraic combinations of the fundamental units
- **English units**: system of measurement used in the United States; includes units of measurement such as feet, gallons, and pounds
- **fundamental units**: units that can only be expressed relative to the procedure used to measure them
- **kilogram**: the SI unit for mass, abbreviated (kg)
- **law**: a description, using concise language or a mathematical formula, a generalized pattern in nature that is supported by scientific evidence and repeated experiments
- **meter**: the SI unit for length, abbreviated (m)
- **method of adding percents**: the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation
Section Summary

1.1 Physics: An Introduction

- Science seeks to discover and describe the underlying order and simplicity in nature.
- Physics is the most basic of the sciences, concerning itself with energy, matter, space and time, and their interactions.
- Scientific laws and theories express the general truths of nature and the body of knowledge they encompass. These laws of nature are rules that all natural processes appear to follow.

1.2 Physical Quantities and Units

- Physical quantities are a characteristic or property of an object that can be measured or calculated from other measurements.
- Units are standards for expressing and comparing the measurement of physical quantities. All units can be expressed as combinations of four fundamental units.
- The four fundamental units we will use in this text are the meter (for length), the kilogram (for mass), the second (for time), and the ampere (for electric current). These units are part of the metric system, which uses powers of 10 to relate quantities over the vast ranges encountered in nature.
- The four fundamental units are abbreviated as follows: meter, m; kilogram, kg; second, s; and ampere, A. The metric system also uses a standard set of prefixes to denote each order of magnitude greater than or lesser than the fundamental unit itself.
- Unit conversions involve changing a value expressed in one type of unit to another type of unit. This is done by using conversion factors, which are ratios relating equal quantities of different units.

1.3 Accuracy, Precision, and Significant Figures

- Accuracy of a measured value refers to how close a measurement is to the correct value. The uncertainty in a measurement is an estimate of the amount by which the measurement result may differ from this value.
- Precision of measured values refers to how close the agreement is between repeated measurements.
- The precision of a measuring tool is related to the size of its measurement increments. The smaller the measurement
increment, the more precise the tool.

- Significant figures express the precision of a measuring tool.
- When multiplying or dividing measured values, the final answer can contain only as many significant figures as the least precise value.
- When adding or subtracting measured values, the final answer cannot contain more decimal places than the least precise value.

### 1.4 Approximation

Scientists often approximate the values of quantities to perform calculations and analyze systems.

#### Conceptual Questions

1. **Physics: An Introduction**
   1. Models are particularly useful in relativity and quantum mechanics, where conditions are outside those normally encountered by humans. What is a model?
   2. How does a model differ from a theory?
   3. If two different theories describe experimental observations equally well, can one be said to be more valid than the other (assuming both use accepted rules of logic)?
   4. What determines the validity of a theory?
   5. Certain criteria must be satisfied if a measurement or observation is to be believed. Will the criteria necessarily be as strict for an expected result as for an unexpected result?
   6. Can the validity of a model be limited, or must it be universally valid? How does this compare to the required validity of a theory or a law?
   7. Classical physics is a good approximation to modern physics under certain circumstances. What are they?
   8. When is it necessary to use relativistic quantum mechanics?
   9. Can classical physics be used to accurately describe a satellite moving at a speed of 7500 m/s? Explain why or why not.

2. **Physical Quantities and Units**
   10. Identify some advantages of metric units.

3. **Accuracy, Precision, and Significant Figures**
   11. What is the relationship between the accuracy and uncertainty of a measurement?
   12. Prescriptions for vision correction are given in units called diopters (D). Determine the meaning of that unit. Obtain information (perhaps by calling an optometrist or performing an internet search) on the minimum uncertainty with which corrections in diopters are determined and the accuracy with which corrective lenses can be produced. Discuss the sources of uncertainties in both the prescription and accuracy in the manufacture of lenses.
Problems & Exercises

1.2 Physical Quantities and Units

1. The speed limit on some interstate highways is roughly 100 km/h. (a) What is this in meters per second? (b) How many miles per hour is this?

2. A car is traveling at a speed of 33 m/s. (a) What is its speed in kilometers per hour? (b) Is it exceeding the 90 km/h speed limit?

3. Show that \( 1.0 \text{ m/s} = 3.6 \text{ km/h} \). Hint: Show the explicit steps involved in converting \( 1.0 \text{ m/s} = 3.6 \text{ km/h} \).

4. American football is played on a 100-yd-long field, excluding the end zones. How long is the field in meters? (Assume that 1 meter equals 3.281 feet.)

5. Soccer fields vary in size. A large soccer field is 115 m long and 85 m wide. What are its dimensions in feet and inches? (Assume that 1 meter equals 3.281 feet.)

6. What is the height in meters of a person who is 6 ft 1.0 in. tall? (Assume that 1 meter equals 39.37 in.)

7. Mount Everest, at 29,028 feet, is the tallest mountain on the Earth. What is its height in kilometers? (Assume that 1 kilometer equals 3.281 feet.)

8. The speed of sound is measured to be 342 m/s on a certain day. What is this in km/h?

9. Tectonic plates are large segments of the Earth’s crust that move slowly. Suppose that one such plate has an average speed of 4.0 cm/year. (a) What distance does it move in 1 s at this speed? (b) What is its speed in kilometers per million years?

10. (a) Refer to Table 1.3 to determine the average distance between the Earth and the Sun. Then calculate the average speed of the Earth in its orbit in kilometers per second. (b) What is this in meters per second?

1.3 Accuracy, Precision, and Significant Figures

Express your answers to problems in this section to the correct number of significant figures and proper units.

11. Suppose that your bathroom scale reads your mass as 65 kg with a 3% uncertainty. What is the uncertainty in your mass (in kilograms)?

12. A good-quality measuring tape can be off by 0.50 cm over a distance of 20 m. What is its percent uncertainty?

13. (a) A car speedometer has a 5.0% uncertainty. What is the range of possible speeds when it reads 90 km/h? (b) What is its speed in kilometers per million years?

14. An infant’s pulse rate is measured to be 130 ± 5 beats/min. What is the percent uncertainty in this measurement?

15. (a) Suppose that a person has an average heart rate of 72.0 beats/min. How many beats does he or she have in 2.0 y? (b) In 2.00 y? (c) In 2.000 y?

16. A can contains 375 mL of soda. How much is left after 308 mL is removed?

17. State how many significant figures are proper in the results of the following calculations: (a) \( (106.7)(98.2)/(46.210)(1.01) \) (b) \( (18.7)^2 \) (c) \( (1.60 \times 10^{-19})(3712) \).

18. (a) How many significant figures are in the numbers 99 and 100? (b) If the uncertainty in each number is 1, what is the percent uncertainty in each? (c) Which is a more meaningful way to express the accuracy of these two numbers, significant figures or percent uncertainties?

19. (a) If your speedometer has an uncertainty of 2.0 km/h at a speed of 90 km/h, what is the percent uncertainty? (b) If it has the same percent uncertainty when it reads 60 km/h, what is the range of speeds you could be going?

20. (a) A person’s blood pressure is measured to be 120 ± 2 mm Hg. What is its percent uncertainty? (b) Assuming the same percent uncertainty, what is the uncertainty in a blood pressure measurement of 80 mm Hg?

21. A person measures his or her heart rate by counting the number of beats in 30 s. If 40 ± 1 beats are counted in 30.0 ± 0.5 s, what is the heart rate and its uncertainty in beats per minute?

22. What is the area of a circle 3.102 cm in diameter?

23. If a marathon runner averages 9.5 m/h, how long does it take him or her to run a 26.22-mi marathon?

24. A marathon runner completes a 42.188-km course in 2 h, 30 min, and 12 s. There is an uncertainty of 25 m in the distance traveled and an uncertainty of 1 s in the elapsed time. (a) Calculate the percent uncertainty in the distance. (b) Calculate the uncertainty in the elapsed time. (c) What is the average speed in meters per second? (d) What is the uncertainty in the average speed?

25. The sides of a small rectangular box are measured to be 1.80 ± 0.01 cm, 2.05 ± 0.02 cm, and 3.1 ± 0.1 cm long. Calculate its volume and uncertainty in cubic centimeters.

26. When non-metric units were used in the United Kingdom, a unit of mass called the pound-mass (lbm) was employed, where 1 lbm = 0.4539 kg. (a) If there is an uncertainty of 0.0001 kg in the pound-mass unit, what is its percent uncertainty? (b) Based on that percent uncertainty, what mass in pound-mass has an uncertainty of 1 kg when converted to kilograms?

27. The length and width of a rectangular room are measured to be 3.955 ± 0.005 m and 3.050 ± 0.005 m. Calculate the area of the room and its uncertainty in square meters.

28. A car engine moves a piston with a circular cross section of 7.500 ± 0.002 cm diameter a distance of 3.250 ± 0.001 cm to compress the gas in the cylinder. (a) By what amount is the gas decreased in volume in cubic centimeters? (b) Find the uncertainty in this volume.

1.4 Approximation

29. How many heartbeats are there in a lifetime?
30. A generation is about one-third of a lifetime. Approximately how many generations have passed since the year 0 AD?

31. How many times longer than the mean life of an extremely unstable atomic nucleus is the lifetime of a human? (Hint: The lifetime of an unstable atomic nucleus is on the order of $10^{-22}$ s.)

32. Calculate the approximate number of atoms in a bacterium. Assume that the average mass of an atom in the bacterium is ten times the mass of a hydrogen atom. (Hint: The mass of a hydrogen atom is on the order of $10^{-27}$ kg and the mass of a bacterium is on the order of $10^{-15}$ kg.)

33. Approximately how many atoms thick is a cell membrane, assuming all atoms there average about twice the size of a hydrogen atom?

34. (a) What fraction of Earth's diameter is the greatest ocean depth? (b) The greatest mountain height?

35. (a) Calculate the number of cells in a hummingbird assuming the mass of an average cell is ten times the mass of a bacterium. (b) Making the same assumption, how many cells are there in a human?

36. Assuming one nerve impulse must end before another can begin, what is the maximum firing rate of a nerve in impulses per second?
2 KINEMATICS

Figure 2.1 The motion of an American kestrel through the air can be described by the bird's displacement, speed, velocity, and acceleration. When it flies in a straight line without any change in direction, its motion is said to be one dimensional. (credit: Vince Maidens, Wikimedia Commons)

Chapter Outline

2.1. Displacement
- Define position, displacement, distance, and distance traveled.
- Explain the relationship between position and displacement.
- Distinguish between displacement and distance traveled.
- Calculate displacement and distance given initial position, final position, and the path between the two.

2.2. Vectors, Scalars, and Coordinate Systems
- Define and distinguish between scalar and vector quantities.
- Assign a coordinate system for a scenario involving one-dimensional motion.

2.3. Time, Velocity, and Speed
- Explain the relationships between instantaneous velocity, average velocity, instantaneous speed, average speed, displacement, and time.
- Calculate velocity and speed given initial position, initial time, final position, and final time.
- Derive a graph of velocity vs. time given a graph of position vs. time.
- Interpret a graph of velocity vs. time.

2.4. Acceleration
- Define and distinguish between instantaneous acceleration, average acceleration, and deceleration.
- Calculate acceleration given initial time, initial velocity, final time, and final velocity.

2.5. Motion Equations for Constant Acceleration in One Dimension
- Calculate displacement of an object that is not accelerating, given initial position and velocity.
- Calculate final velocity of an accelerating object, given initial velocity, acceleration, and time.
- Calculate displacement and final position of an accelerating object, given initial position, initial velocity, time, and acceleration.

2.6. Problem-Solving Basics for One-Dimensional Kinematics
- Apply problem-solving steps and strategies to solve problems of one-dimensional kinematics.
- Apply strategies to determine whether or not the result of a problem is reasonable, and if not, determine the cause.

2.7. Falling Objects
- Describe the effects of gravity on objects in motion.
- Describe the motion of objects that are in free fall.
- Calculate the position and velocity of objects in free fall.

2.8. Graphical Analysis of One-Dimensional Motion
- Describe a straight-line graph in terms of its slope and y-intercept.
- Determine average velocity or instantaneous velocity from a graph of position vs. time.
- Determine average or instantaneous acceleration from a graph of velocity vs. time.
- Derive a graph of velocity vs. time from a graph of position vs. time.
• Derive a graph of acceleration vs. time from a graph of velocity vs. time.

**Introduction to One-Dimensional Kinematics**

Objects are in motion everywhere we look. Everything from a tennis game to a space-probe flyby of the planet Neptune involves motion. When you are resting, your heart moves blood through your veins. And even in inanimate objects, there is continuous motion in the vibrations of atoms and molecules. Questions about motion are interesting in and of themselves: How long will it take for a space probe to get to Mars? Where will a football land if it is thrown at a certain angle? But an understanding of motion is also key to understanding other concepts in physics. An understanding of acceleration, for example, is crucial to the study of force.

Our formal study of physics begins with kinematics which is defined as the study of motion without considering its causes. The word “kinematics” comes from a Greek term meaning motion and is related to other English words such as “cinema” (movies) and “kinesiology” (the study of human motion). In one-dimensional kinematics and Two-Dimensional Kinematics we will study only the motion of a football, for example, without worrying about what forces cause or change its motion. Such considerations come in other chapters. In this chapter, we examine the simplest type of motion—namely, motion along a straight line, or one-dimensional motion. In Two-Dimensional Kinematics, we apply concepts developed here to study motion along curved paths (two- and three-dimensional motion); for example, that of a car rounding a curve.

(No media type is supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42122/1.9/#concept-trailer-1d-kinematics)

**2.1 Displacement**

**Position**

In order to describe the motion of an object, you must first be able to describe its position—where it is at any particular time. More precisely, you need to specify its position relative to a convenient reference frame. Earth is often used as a reference frame, and we often describe the position of an object as it relates to stationary objects in that reference frame. For example, a rocket launch would be described in terms of the position of the rocket with respect to the Earth as a whole, while a professor’s position could be described in terms of where she is in relation to the nearby white board. (See Figure 2.3.) In other cases, we use reference frames that are not stationary but are in motion relative to the Earth. To describe the position of a person in an airplane, for example, we use the airplane, not the Earth, as the reference frame. (See Figure 2.4.)

**Displacement**

If an object moves relative to a reference frame (for example, if a professor moves to the right relative to a white board or a passenger moves toward the rear of an airplane), then the object’s position changes. This change in position is known as displacement. The word “displacement” implies that an object has moved, or has been displaced.

| Displacement
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement is the change in position of an object:</td>
</tr>
<tr>
<td>[ \Delta x = x_f - x_0 ]</td>
</tr>
<tr>
<td>where ( \Delta x ) is displacement, ( x_f ) is the final position, and ( x_0 ) is the initial position.</td>
</tr>
</tbody>
</table>

In this text the upper case Greek letter \( \Delta \) (delta) always means “change in” whatever quantity follows it; thus, \( \Delta x \) means
change in position. Always solve for displacement by subtracting initial position \( x_0 \) from final position \( x_f \).

Note that the SI unit for displacement is the meter (m) (see Physical Quantities and Units), but sometimes kilometers, miles, feet, and other units of length are used. Keep in mind that when units other than the meter are used in a problem, you may need to convert them into meters to complete the calculation.

\[
\Delta x = x_f - x_0 = 2.0 \text{ m}
\]

\[ \Delta x = x_f - x_0 = +2.0 \text{ m} \]

**Figure 2.3** A professor paces left and right while lecturing. Her position relative to Earth is given by \( x \). The \( +2.0 \text{ m} \) displacement of the professor relative to Earth is represented by an arrow pointing to the right.

\[
\Delta x = x_f - x_0 = -4.0 \text{ m}
\]

**Figure 2.4** A passenger moves from his seat to the back of the plane. His location relative to the airplane is given by \( x \). The \( -4.0 \text{-m} \) displacement of the passenger relative to the plane is represented by an arrow toward the rear of the plane. Notice that the arrow representing his displacement is twice as long as the arrow representing the displacement of the professor (he moves twice as far) in Figure 2.3.

Note that displacement has a direction as well as a magnitude. The professor’s displacement is 2.0 m to the right, and the airline passenger’s displacement is 4.0 m toward the rear. In one-dimensional motion, direction can be specified with a plus or minus sign. When you begin a problem, you should select which direction is positive (usually that will be to the right or up, but you are free to select positive as being any direction). The professor’s initial position is \( x_0 = 1.5 \text{ m} \) and her final position is \( x_f = 3.5 \text{ m} \). Thus her displacement is

\[
\Delta x = x_f - x_0 = 3.5 \text{ m} - 1.5 \text{ m} = +2.0 \text{ m}.
\]

In this coordinate system, motion to the right is positive, whereas motion to the left is negative. Similarly, the airplane passenger’s initial position is \( x_0 = 6.0 \text{ m} \) and his final position is \( x_f = 2.0 \text{ m} \), so his displacement is
His displacement is negative because his motion is toward the rear of the plane, or in the negative \( x \) direction in our coordinate system.

**Distance**

Although displacement is described in terms of direction, distance is not. **Distance** is defined to be the magnitude or size of displacement between two positions. Note that the distance between two positions is not the same as the distance traveled between them. **Distance traveled** is the total length of the path traveled between two positions. Distance has no direction and, thus, no sign. For example, the distance the professor walks is 2.0 m. The distance the airplane passenger walks is 4.0 m.

### Misconception Alert: Distance Traveled vs. Magnitude of Displacement

It is important to note that the distance traveled, however, can be greater than the magnitude of the displacement (by magnitude, we mean just the size of the displacement without regard to its direction; that is, just a number with a unit). For example, the professor could pace back and forth many times, perhaps walking a distance of 150 m during a lecture, yet still end up only 2.0 m to the right of her starting point. In this case her displacement would be +2.0 m, the magnitude of her displacement would be 2.0 m, but the distance she traveled would be 150 m. In kinematics we usually deal with the magnitude of displacement, and almost never with distance traveled. One way to think about this is to assume you marked the start of the motion and the end of the motion. The displacement is simply the difference in the position of the two marks and is independent of the path taken in traveling between the two marks. The distance traveled, however, is the total length of the path taken between the two marks.

### Check Your Understanding

A cyclist rides 3 km west and then turns around and rides 2 km east. (a) What is her displacement? (b) What distance does she ride? (c) What is the magnitude of her displacement?

**Solution**

\[
\Delta x = x_f - x_0 = 2.0 \text{ m} - 6.0 \text{ m} = -4.0 \text{ m}. \quad (2.3)
\]

(a) The rider’s displacement is \( \Delta x = x_f - x_0 = -1 \text{ km} \). (The displacement is negative because we take east to be positive and west to be negative.)

(b) The distance traveled is \( 3 \text{ km} + 2 \text{ km} = 5 \text{ km} \).

(c) The magnitude of the displacement is \( 1 \text{ km} \).
2.2 Vectors, Scalars, and Coordinate Systems

Figure 2.6 The motion of this Eclipse Concept jet can be described in terms of the distance it has traveled (a scalar quantity) or its displacement in a specific direction (a vector quantity). In order to specify the direction of motion, its displacement must be described based on a coordinate system. In this case, it may be convenient to choose motion toward the left as positive motion (it is the forward direction for the plane), although in many cases, the \( x \)-coordinate runs from left to right, with motion to the right as positive and motion to the left as negative. (credit: Armchair Aviator, Flickr)

What is the difference between distance and displacement? Whereas displacement is defined by both direction and magnitude, distance is defined only by magnitude. Displacement is an example of a vector quantity. Distance is an example of a scalar quantity. A vector is any quantity with both magnitude and direction. Other examples of vectors include a velocity of 90 km/h east and a force of 500 newtons straight down.

The direction of a vector in one-dimensional motion is given simply by a plus (\(+\)) or minus (\(-\)) sign. Vectors are represented graphically by arrows. An arrow used to represent a vector has a length proportional to the vector’s magnitude (e.g., the larger the magnitude, the longer the length of the vector) and points in the same direction as the vector.

Some physical quantities, like distance, either have no direction or none is specified. A scalar is any quantity that has a magnitude, but no direction. For example, a 20ºC temperature, the 250 kilocalories (250 Calories) of energy in a candy bar, a 90 km/h speed limit, a person’s 1.8 m height, and a distance of 2.0 m are all scalars—quantities with no specified direction. Note, however, that a scalar can be negative, such as a \(-20ºC\) temperature. In this case, the minus sign indicates a point on a scale rather than a direction. Scalars are never represented by arrows.

Coordinate Systems for One-Dimensional Motion

In order to describe the direction of a vector quantity, you must designate a coordinate system within the reference frame. For one-dimensional motion, this is a simple coordinate system consisting of a one-dimensional coordinate line. In general, when describing horizontal motion, motion to the right is usually considered positive, and motion to the left is considered negative. With vertical motion, motion up is usually positive and motion down is negative. In some cases, however, as with the jet in Figure 2.6, it can be more convenient to switch the positive and negative directions. For example, if you are analyzing the motion of falling objects, it can be useful to define downwards as the positive direction. If people in a race are running to the left, it is useful to define left as the positive direction. It does not matter as long as the system is clear and consistent. Once you assign a positive direction and start solving a problem, you cannot change it.

Figure 2.7 It is usually convenient to consider motion upward or to the right as positive (\(+\)) and motion downward or to the left as negative (\(-\)).
Check Your Understanding

A person's speed can stay the same as he or she rounds a corner and changes direction. Given this information, is speed a scalar or a vector quantity? Explain.

Solution

Speed is a scalar quantity. It does not change at all with direction changes; therefore, it has magnitude only. If it were a vector quantity, it would change as direction changes (even if its magnitude remained constant).

2.3 Time, Velocity, and Speed

There is more to motion than distance and displacement. Questions such as, “How long does a foot race take?” and “What was the runner’s speed?” cannot be answered without an understanding of other concepts. In this section we add definitions of time, velocity, and speed to expand our description of motion.

Time

As discussed in Physical Quantities and Units, the most fundamental physical quantities are defined by how they are measured. This is the case with time. Every measurement of time involves measuring a change in some physical quantity. It may be a number on a digital clock, a heartbeat, or the position of the Sun in the sky. In physics, the definition of time is simple—time is change, or the interval over which change occurs. It is impossible to know that time has passed unless something changes. The amount of time or change is calibrated by comparison with a standard. The SI unit for time is the second, abbreviated s. We might, for example, observe that a certain pendulum makes one full swing every 0.75 s. We could then use the pendulum to measure time by counting its swings or, of course, by connecting the pendulum to a clock mechanism that registers time on a dial. This allows us to not only measure the amount of time, but also to determine a sequence of events.

How does time relate to motion? We are usually interested in elapsed time for a particular motion, such as how long it takes an airplane passenger to get from his seat to the back of the plane. To find elapsed time, we note the time at the beginning and end of the motion and subtract the two. For example, a lecture may start at 11:00 A.M. and end at 11:50 A.M., so that the elapsed time would be 50 min. Elapsed time \( \Delta t \) is the difference between the ending time and beginning time,

\[
\Delta t = t_f - t_0,
\]

where \( \Delta t \) is the change in time or elapsed time, \( t_f \) is the time at the end of the motion, and \( t_0 \) is the time at the beginning of the motion. (As usual, the delta symbol, \( \Delta \), means the change in the quantity that follows it.)

Life is simpler if the beginning time \( t_0 \) is taken to be zero, as when we use a stopwatch. If we were using a stopwatch, it would simply read zero at the start of the lecture and 50 min at the end. If \( t_0 = 0 \), then \( \Delta t = t_f \equiv t \).

In this text, for simplicity’s sake,

- motion starts at time equal to zero \( (t_0 = 0) \)
- the symbol \( t \) is used for elapsed time unless otherwise specified \( (\Delta t = t_f \equiv t) \)

Velocity

Your notion of velocity is probably the same as its scientific definition. You know that if you have a large displacement in a small amount of time you have a large velocity, and that velocity has units of distance divided by time, such as miles per hour or kilometers per hour.
Average Velocity

**Average velocity** is displacement (change in position) divided by the time of travel,

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0}.$$  \(\text{(2.5)}\)

where \(\bar{v}\) is the average (indicated by the bar over the \(v\)) velocity, \(\Delta x\) is the change in position (or displacement), and \(x_f\) and \(x_0\) are the final and beginning positions at times \(t_f\) and \(t_0\), respectively. If the starting time \(t_0\) is taken to be zero, then the average velocity is simply

$$\bar{v} = \frac{\Delta x}{t}.$$  \(\text{(2.6)}\)

Notice that this definition indicates that velocity is a vector because displacement is a vector. It has both magnitude and direction. The SI unit for velocity is meters per second or m/s, but many other units, such as km/h, mi/h (also written as mph), and cm/s, are in common use. Suppose, for example, an airplane passenger took 5 seconds to move −4 m (the negative sign indicates that displacement is toward the back of the plane). His average velocity would be

$$\bar{v} = \frac{\Delta x}{t} = \frac{-4 \text{ m}}{5 \text{ s}} = -0.8 \text{ m/s}.$$  \(\text{(2.7)}\)

The minus sign indicates the average velocity is also toward the rear of the plane.

The average velocity of an object does not tell us anything about what happens to it between the starting point and ending point, however. For example, we cannot tell from average velocity whether the airplane passenger stops momentarily or backs up before he goes to the back of the plane. To get more details, we must consider smaller segments of the trip over smaller time intervals.

![Figure 2.9 A more detailed record of an airplane passenger heading toward the back of the plane, showing smaller segments of his trip.](image)

The smaller the time intervals considered in a motion, the more detailed the information. When we carry this process to its logical conclusion, we are left with an infinitesimally small interval. Over such an interval, the average velocity becomes the **instantaneous velocity** or the **velocity at a specific instant**. A car’s speedometer, for example, shows the magnitude (but not the direction) of the instantaneous velocity of the car. (Police give tickets based on instantaneous velocity, but when calculating how long it will take to get from one place to another on a road trip, you need to use average velocity.) **Instantaneous velocity** \(v\) is the average velocity at a specific instant in time (or over an infinitesimally small time interval).

Mathematically, finding instantaneous velocity, \(v\), at a precise instant \(t\) can involve taking a limit, a calculus operation beyond the scope of this text. However, under many circumstances, we can find precise values for instantaneous velocity without calculus.

**Speed**

In everyday language, most people use the terms “speed” and “velocity” interchangeably. In physics, however, they do not have the same meaning and they are distinct concepts. One major difference is that speed has no direction. Thus **speed is a scalar**. Just as we need to distinguish between instantaneous velocity and average velocity, we also need to distinguish between instantaneous speed and average speed.

**Instantaneous speed** is the magnitude of instantaneous velocity. For example, suppose the airplane passenger at one instant had an instantaneous velocity of −3.0 m/s (the minus meaning toward the rear of the plane). At that same time his instantaneous speed was 3.0 m/s. Or suppose that at one time during a shopping trip your instantaneous velocity is 40 km/h due north. Your instantaneous speed at that instant would be 40 km/h—the same magnitude but without a direction. Average speed, however, is very different from average velocity. **Average speed** is the distance traveled divided by elapsed time.

We have noted that distance traveled can be greater than the magnitude of displacement. So average speed can be greater than average velocity, which is displacement divided by time. For example, if you drive to a store and return home in half an hour, and
your car’s odometer shows the total distance traveled was 6 km, then your average speed was 12 km/h. Your average velocity, however, was zero, because your displacement for the round trip is zero. (Displacement is change in position and, thus, is zero for a round trip.) Thus average speed is not simply the magnitude of average velocity.

Figure 2.10 During a 30-minute round trip to the store, the total distance traveled is 6 km. The average speed is 12 km/h. The displacement for the round trip is zero, since there was no net change in position. Thus the average velocity is zero.

Another way of visualizing the motion of an object is to use a graph. A plot of position or of velocity as a function of time can be very useful. For example, for this trip to the store, the position, velocity, and speed-vs.-time graphs are displayed in Figure 2.11. (Note that these graphs depict a very simplified model of the trip. We are assuming that speed is constant during the trip, which is unrealistic given that we’ll probably stop at the store. But for simplicity’s sake, we will model it with no stops or changes in speed. We are also assuming that the route between the store and the house is a perfectly straight line.)
Figure 2.11 Position vs. time, velocity vs. time, and speed vs. time on a trip. Note that the velocity for the return trip is negative.

Making Connections: Take-Home Investigation—Getting a Sense of Speed

If you have spent much time driving, you probably have a good sense of speeds between about 10 and 70 miles per hour. But what are these in meters per second? What do we mean when we say that something is moving at 10 m/s? To get a better sense of what these values really mean, do some observations and calculations on your own:

- calculate typical car speeds in meters per second
- estimate jogging and walking speed by timing yourself; convert the measurements into both m/s and mi/h
- determine the speed of an ant, snail, or falling leaf
A commuter train travels from Baltimore to Washington, DC, and back in 1 hour and 45 minutes. The distance between the two stations is approximately 40 miles. What is (a) the average velocity of the train, and (b) the average speed of the train in m/s?

Solution
(a) The average velocity of the train is zero because \( x_f = x_0 \); the train ends up at the same place it starts.

(b) The average speed of the train is calculated below. Note that the train travels 40 miles one way and 40 miles back, for a total distance of 80 miles.

\[
\text{average speed} = \frac{\text{distance}}{\text{time}} = \frac{80 \text{ miles}}{105 \text{ minutes}}
\]

\[
= \frac{80 \text{ miles}}{105 \text{ minutes}} \times \frac{5280 \text{ feet}}{1 \text{ mile}} \times \frac{1 \text{ meter}}{3.28 \text{ feet}} \times \frac{1 \text{ minute}}{60 \text{ seconds}} = 20 \text{ m/s}
\]

2.4 Acceleration

Figure 2.12 A plane decelerates, or slows down, as it comes in for landing in St. Maarten. Its acceleration is opposite in direction to its velocity. (credit: Steve Corey, Flickr)

In everyday conversation, to accelerate means to speed up. The accelerator in a car can in fact cause it to speed up. The greater the acceleration, the greater the change in velocity over a given time. The formal definition of acceleration is consistent with these notions, but more inclusive.

### Average Acceleration

Average Acceleration is the rate at which velocity changes,

\[
\ddot{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}
\]

where \( \ddot{a} \) is average acceleration, \( v \) is velocity, and \( t \) is time. (The bar over the \( a \) means average acceleration.)

Because acceleration is velocity in m/s divided by time in s, the SI units for acceleration are \( \text{m/s}^2 \), meters per second squared or meters per second per second, which literally means by how many meters per second the velocity changes every second.

Recall that velocity is a vector—it has both magnitude and direction. This means that a change in velocity can be a change in magnitude (or speed), but it can also be a change in direction. For example, if a car turns a corner at constant speed, it is accelerating because its direction is changing. The quicker you turn, the greater the acceleration. So there is an acceleration when velocity changes either in magnitude (an increase or decrease in speed) or in direction, or both.

### Acceleration as a Vector

Acceleration is a vector in the same direction as the change in velocity, \( \Delta v \). Since velocity is a vector, it can change either in magnitude or in direction. Acceleration is therefore a change in either speed or direction, or both.

Keep in mind that although acceleration is in the direction of the change in velocity, it is not always in the direction of motion. When an object slows down, its acceleration is opposite to the direction of its motion. This is known as deceleration.
Figure 2.13 A subway train in Sao Paulo, Brazil, decelerates as it comes into a station. It is accelerating in a direction opposite to its direction of motion. (credit: Yusuke Kawasaki, Flickr)

### Misconception Alert: Deceleration vs. Negative Acceleration

Deceleration always refers to acceleration in the direction opposite to the direction of the velocity. Deceleration always reduces speed. Negative acceleration, however, is acceleration in the negative direction in the chosen coordinate system. Negative acceleration may or may not be deceleration, and deceleration may or may not be considered negative acceleration. For example, consider Figure 2.14.
Figure 2.14 (a) This car is speeding up as it moves toward the right. It therefore has positive acceleration in our coordinate system. (b) This car is slowing down as it moves toward the right. Therefore, it has negative acceleration in our coordinate system, because its acceleration is toward the left. The car is also decelerating: the direction of its acceleration is opposite to its direction of motion. (c) This car is moving toward the left, but slowing down over time. Therefore, its acceleration is positive in our coordinate system because it is toward the right. However, the car is decelerating because its acceleration is opposite to its motion. (d) This car is speeding up as it moves toward the left. It has negative acceleration because it is accelerating toward the left. However, because its acceleration is in the same direction as its motion, it is speeding up (not decelerating).

Example 2.1 Calculating Acceleration: A Racehorse Leaves the Gate

A racehorse coming out of the gate accelerates from rest to a velocity of 15.0 m/s due west in 1.80 s. What is its average acceleration?
Strategy

First we draw a sketch and assign a coordinate system to the problem. This is a simple problem, but it always helps to visualize it. Notice that we assign east as positive and west as negative. Thus, in this case, we have negative velocity.

We can solve this problem by identifying $\Delta v$ and $\Delta t$ from the given information and then calculating the average acceleration directly from the equation:

$$a = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}.$$

Solution

1. Identify the knowns. $v_0 = 0$, $v_f = -15.0 \text{ m/s}$ (the negative sign indicates direction toward the west), $\Delta t = 1.80 \text{ s}$.

2. Find the change in velocity. Since the horse is going from zero to $-15.0 \text{ m/s}$, its change in velocity equals its final velocity:

$$\Delta v = v_f = -15.0 \text{ m/s}.$$

3. Plug in the known values ($\Delta v$ and $\Delta t$) and solve for the unknown $\bar{a}$.

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-15.0 \text{ m/s}}{1.80 \text{ s}} = -8.33 \text{ m/s}^2.$$

Discussion

The negative sign for acceleration indicates that acceleration is toward the west. An acceleration of $8.33 \text{ m/s}^2$ due west means that the horse increases its velocity by $8.33 \text{ m/s}$ due west each second, that is, $8.33 \text{ meters per second per second}$, which we write as $8.33 \text{ m/s}^2$. This is truly an average acceleration, because the ride is not smooth. We shall see later that an acceleration of this magnitude would require the rider to hang on with a force nearly equal to his weight.

Instantaneous Acceleration

Instantaneous acceleration $a$, or the acceleration at a specific instant in time, is obtained by the same process as discussed for instantaneous velocity in Time, Velocity, and Speed—that is, by considering an infinitesimally small interval of time. How do we find instantaneous acceleration using only algebra? The answer is that we choose an average acceleration that is representative of the motion. Figure 2.17 shows graphs of instantaneous acceleration versus time for two very different motions. In Figure 2.17(a), the acceleration varies slightly and the average over the entire interval is nearly the same as the instantaneous acceleration at any time. In this case, we should treat this motion as if it had a constant acceleration equal to the
average (in this case about 1.8 m/s²). In Figure 2.17(b), the acceleration varies drastically over time. In such situations it is best to consider smaller time intervals and choose an average acceleration for each. For example, we could consider motion over the time intervals from 0 to 1.0 s and from 1.0 to 3.0 s as separate motions with accelerations of +3.0 m/s² and −2.0 m/s², respectively.

![Figure 2.17](image1.png)

**Figure 2.17** Graphs of instantaneous acceleration versus time for two different one-dimensional motions. (a) Here acceleration varies only slightly and is always in the same direction, since it is positive. The average over the interval is nearly the same as the acceleration at any given time. (b) Here the acceleration varies greatly, perhaps representing a package on a post office conveyor belt that is accelerated forward and backward as it bumps along. It is necessary to consider small time intervals (such as from 0 to 1.0 s) with constant or nearly constant acceleration in such a situation.

The next several examples consider the motion of the subway train shown in Figure 2.18. In (a) the shuttle moves to the right, and in (b) it moves to the left. The examples are designed to further illustrate aspects of motion and to illustrate some of the reasoning that goes into solving problems.

![Figure 2.18](image2.png)

**Figure 2.18** One-dimensional motion of a subway train considered in Example 2.2, Example 2.3, Example 2.4, Example 2.5, Example 2.6, and Example 2.7. Here we have chosen the x-axis so that + means to the right and − means to the left for displacements, velocities, and accelerations. (a) The subway train moves to the right from x₀ to xᵢ. Its displacement Δx is +2.0 km. (b) The train moves to the left from x₀' to xᵢ'. Its displacement Δx' is −1.5 km. (Note that the prime symbol (') is used simply to distinguish between displacement in the two different situations. The distances of travel and the size of the cars are on different scales to fit everything into the diagram.)
Example 2.2 Calculating Displacement: A Subway Train

What are the magnitude and sign of displacements for the motions of the subway train shown in parts (a) and (b) of Figure 2.18?

**Strategy**

A drawing with a coordinate system is already provided, so we don’t need to make a sketch, but we should analyze it to make sure we understand what it is showing. Pay particular attention to the coordinate system. To find displacement, we use the equation \( \Delta x = x_f - x_0 \). This is straightforward since the initial and final positions are given.

**Solution**

1. Identify the knowns. In the figure we see that \( x_f = 6.70 \text{ km} \) and \( x_0 = 4.70 \text{ km} \) for part (a), and \( x'_f = 3.75 \text{ km} \) and \( x'_0 = 5.25 \text{ km} \) for part (b).

2. Solve for displacement in part (a).

\[
\Delta x = x_f - x_0 = 6.70 \text{ km} - 4.70 \text{ km} = +2.00 \text{ km}
\]

(2.12)

3. Solve for displacement in part (b).

\[
\Delta x' = x'_f - x'_0 = 3.75 \text{ km} - 5.25 \text{ km} = -1.50 \text{ km}
\]

(2.13)

**Discussion**

The direction of the motion in (a) is to the right and therefore its displacement has a positive sign, whereas motion in (b) is to the left and thus has a negative sign.

Example 2.3 Comparing Distance Traveled with Displacement: A Subway Train

What are the distances traveled for the motions shown in parts (a) and (b) of the subway train in Figure 2.18?

**Strategy**

To answer this question, think about the definitions of distance and distance traveled, and how they are related to displacement. Distance between two positions is defined to be the magnitude of displacement, which was found in Example 2.2. Distance traveled is the total length of the path traveled between the two positions. (See Displacement.) In the case of the subway train shown in Figure 2.18, the distance traveled is the same as the distance between the initial and final positions of the train.

**Solution**

1. The displacement for part (a) was +2.00 km. Therefore, the distance between the initial and final positions was 2.00 km, and the distance traveled was 2.00 km.

2. The displacement for part (b) was −1.5 km. Therefore, the distance between the initial and final positions was 1.50 km, and the distance traveled was 1.50 km.

**Discussion**

Distance is a scalar. It has magnitude but no sign to indicate direction.

Example 2.4 Calculating Acceleration: A Subway Train Speeding Up

Suppose the train in Figure 2.18(a) accelerates from rest to 30.0 km/h in the first 20.0 s of its motion. What is its average acceleration during that time interval?

**Strategy**

It is worth it at this point to make a simple sketch:

![Figure 2.19](image-url)
This problem involves three steps. First we must determine the change in velocity, then we must determine the change in time, and finally we use these values to calculate the acceleration.

**Solution**

1. Identify the knowns. \( v_0 = 0 \) (the trains starts at rest), \( v_f = 30.0 \text{ km/h} \), and \( \Delta t = 20.0 \text{ s} \).
2. Calculate \( \Delta v \). Since the train starts from rest, its change in velocity is \( \Delta v = +30.0 \text{ km/h} \), where the plus sign means velocity to the right.
3. Plug in known values and solve for the unknown, \( a \).
   \[
   a = \frac{\Delta v}{\Delta t} = \frac{+30.0 \text{ km/h}}{20.0 \text{ s}}
   \]
4. Since the units are mixed (we have both hours and seconds for time), we need to convert everything into SI units of meters and seconds. (See Physical Quantities and Units for more guidance.)
   \[
   a = \left( +30 \text{ km/h} \right) \left( \frac{10^3 \text{ m}}{1 \text{ km}} \right) \left( \frac{1 \text{ h}}{3600 \text{ s}} \right) = 0.417 \text{ m/s}^2
   \]

**Discussion**

The plus sign means that acceleration is to the right. This is reasonable because the train starts from rest and ends up with a velocity to the right (also positive). So acceleration is in the same direction as the change in velocity, as is always the case.

---

**Example 2.5 Calculate Acceleration: A Subway Train Slowing Down**

Now suppose that at the end of its trip, the train in Figure 2.18(a) slows to a stop from a speed of 30.0 km/h in 8.00 s. What is its average acceleration while stopping?

**Strategy**

\[
\begin{align*}
\nu_0 &= 30.0 \text{ km/h} \\
\nu_f &= 0 \text{ km/h} \\
\Delta t &= 8.00 \text{ s} \\
a &= ?
\end{align*}
\]

Figure 2.20

In this case, the train is decelerating and its acceleration is negative because it is toward the left. As in the previous example, we must find the change in velocity and the change in time and then solve for acceleration.

**Solution**

1. Identify the knowns. \( \nu_0 = 30.0 \text{ km/h} \), \( \nu_f = 0 \text{ km/h} \) (the train is stopped, so its velocity is 0), and \( \Delta t = 8.00 \text{ s} \).
2. Solve for the change in velocity, \( \Delta \nu \).
   \[
   \Delta \nu = \nu_f - \nu_0 = 0 - 30.0 \text{ km/h} = -30.0 \text{ km/h}
   \]
3. Plug in the knowns, \( \Delta \nu \) and \( \Delta t \), and solve for \( \ddot{a} \).
   \[
   \ddot{a} = \frac{\Delta \nu}{\Delta t} = \frac{-30.0 \text{ km/h}}{8.00 \text{ s}}
   \]
4. Convert the units to meters and seconds.
   \[
   \ddot{a} = \frac{\Delta \nu}{\Delta t} \left( \frac{10^3 \text{ m}}{1 \text{ km}} \right) \left( \frac{1 \text{ h}}{3600 \text{ s}} \right) = -1.04 \text{ m/s}^2
   \]

**Discussion**

The minus sign indicates that acceleration is to the left. This sign is reasonable because the train initially has a positive velocity in this problem, and a negative acceleration would oppose the motion. Again, acceleration is in the same direction as the change in velocity, which is negative here. This acceleration can be called a deceleration because it has a direction opposite to the velocity.
The graphs of position, velocity, and acceleration vs. time for the trains in Example 2.4 and Example 2.5 are displayed in Figure 2.21. (We have taken the velocity to remain constant from 20 to 40 s, after which the train decelerates.)

**Figure 2.21** (a) Position of the train over time. Notice that the train’s position changes slowly at the beginning of the journey, then more and more quickly as it picks up speed. Its position then changes more slowly as it slows down at the end of the journey. In the middle of the journey, while the velocity remains constant, the position changes at a constant rate. (b) Velocity of the train over time. The train’s velocity increases as it accelerates at the beginning of the journey. It remains the same in the middle of the journey (where there is no acceleration). It decreases as the train decelerates at the end of the journey. (c) The acceleration of the train over time. The train has positive acceleration as it speeds up at the beginning of the journey. It has no acceleration as it travels at constant velocity in the middle of the journey. Its acceleration is negative as it slows down at the end of the journey.

**Example 2.6 Calculating Average Velocity: The Subway Train**

What is the average velocity of the train in part b of Example 2.2, and shown again below, if it takes 5.00 min to make its trip?
Average velocity is displacement divided by time. It will be negative here, since the train moves to the left and has a negative displacement.

Solution
1. Identify the knowns. \( x'_f = 3.75 \text{ km} \), \( x'_0 = 5.25 \text{ km} \), \( \Delta t = 5.00 \text{ min} \).
2. Determine displacement, \( \Delta x' \). We found \( \Delta x' \) to be \(-1.5 \text{ km}\) in Example 2.2.
3. Solve for average velocity.

\[
\bar{v} = \frac{\Delta x'}{\Delta t} = \frac{-1.50 \text{ km}}{5.00 \text{ min}}
\]  

(2.19)

4. Convert units.

\[
\bar{v} = \frac{\Delta x'}{\Delta t} = \left( \frac{-1.50 \text{ km}}{5.00 \text{ min}} \right) \left( \frac{60 \text{ min}}{1 \text{ h}} \right) = -18.0 \text{ km/h}
\]  

(2.20)

Discussion
The negative velocity indicates motion to the left.

Example 2.7 Calculating Deceleration: The Subway Train

Finally, suppose the train in Figure 2.22 slows to a stop from a velocity of \( 20.0 \text{ km/h} \) in \( 10.0 \text{ s} \). What is its average acceleration?

Strategy
Once again, let’s draw a sketch:

As before, we must find the change in velocity and the change in time to calculate average acceleration.

Solution
1. Identify the knowns. \( v_0 = -20 \text{ km/h} \), \( v_f = 0 \text{ km/h} \), \( \Delta t = 10.0 \text{ s} \).
2. Calculate \( \Delta v \). The change in velocity here is actually positive, since

\[
\Delta v = v_f - v_0 = 0 - (-20 \text{ km/h}) = +20 \text{ km/h}. \tag{2.21}
\]

3. Solve for \( \ddot{a} \).

\[
\ddot{a} = \frac{\Delta v}{\Delta t} = \frac{+20.0 \text{ km/h}}{10.0 \text{ s}} \tag{2.22}
\]
\[
\ddot{a} = \left(\frac{+20.0 \text{ km/h}}{10.0 \text{ s}}\right) \left(10^3 \text{ m} \frac{1 \text{ km}}{1 \text{ h}} \frac{1 \text{ h}}{3600 \text{ s}}\right) = +0.556 \text{ m/s}^2
\]  

\section*{Discussion}

The plus sign means that acceleration is to the right. This is reasonable because the train initially has a negative velocity (to the left) in this problem and a positive acceleration opposes the motion (and so it is to the right). Again, acceleration is in the same direction as the change in velocity, which is positive here. As in Example 2.5, this acceleration can be called a deceleration since it is in the direction opposite to the velocity.

\section*{Sign and Direction}

Perhaps the most important thing to note about these examples is the signs of the answers. In our chosen coordinate system, plus means the quantity is to the right and minus means it is to the left. This is easy to imagine for displacement and velocity. But it is a little less obvious for acceleration. Most people interpret negative acceleration as the slowing of an object. This was not the case in Example 2.7, where a positive acceleration slowed a negative velocity. The crucial distinction was that the acceleration was in the opposite direction from the velocity. In fact, a negative acceleration will increase a negative velocity. For example, the train moving to the left in Figure 2.22 is sped up by an acceleration to the left. In that case, both \(v\) and \(a\) are negative. The plus and minus signs give the directions of the accelerations. If acceleration has the same sign as the velocity, the object is speeding up. If acceleration has the opposite sign as the velocity, the object is slowing down.

\section*{Check Your Understanding}

An airplane lands on a runway traveling east. Describe its acceleration.

\textbf{Solution}

If we take east to be positive, then the airplane has negative acceleration, as it is accelerating toward the west. It is also decelerating: its acceleration is opposite in direction to its velocity.

\section*{Moving Man Simulation}

Learn about position, velocity, and acceleration graphs. Move the little man back and forth with the mouse and plot his motion. Set the position, velocity, or acceleration and let the simulation move the man for you.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42100/1.17/#MovingMan)

\section*{2.5 Motion Equations for Constant Acceleration in One Dimension}

![Kinematic equations can help us describe and predict the motion of moving objects such as these kayaks racing in Newbury, England. (credit: Barry Skeates, Flickr)](Figure 2.25)

We might know that the greater the acceleration of, say, a car moving away from a stop sign, the greater the displacement in a given time. But we have not developed a specific equation that relates acceleration and displacement. In this section, we develop some convenient equations for kinematic relationships, starting from the definitions of displacement, velocity, and acceleration already covered.

\textbf{Notation:} \(t, x, v, a\)

First, let us make some simplifications in notation. Taking the initial time to be zero, as if time is measured with a stopwatch, is a great simplification. Since elapsed time is \(\Delta t = t_f - t_0\), taking \(t_0 = 0\) means that \(\Delta t = t_f\), the final time on the stopwatch.
When initial time is taken to be zero, we use the subscript 0 to denote initial values of position and velocity. That is, \( x_0 \) is the initial position and \( v_0 \) is the initial velocity. We put no subscripts on the final values. That is, \( t \) is the final time, \( x \) is the final position, and \( v \) is the final velocity. This gives a simpler expression for elapsed time—now, \( \Delta t = t \). It also simplifies the expression for displacement, which is now \( \Delta x = x - x_0 \). Also, it simplifies the expression for change in velocity, which is now \( \Delta v = v - v_0 \). To summarize, using the simplified notation, with the initial time taken to be zero,

\[
\begin{align*}
\Delta t &= t \\
\Delta x &= x - x_0 \\
\Delta v &= v - v_0
\end{align*}
\]

where the subscript 0 denotes an initial value and the absence of a subscript denotes a final value in whatever motion is under consideration.

We now make the important assumption that acceleration is constant. This assumption allows us to avoid using calculus to find instantaneous acceleration. Since acceleration is constant, the average and instantaneous accelerations are equal. That is,

\[ \ddot{a} = a = \text{constant}, \]

so we use the symbol \( a \) for acceleration at all times. Assuming acceleration to be constant does not seriously limit the situations we can study nor degrade the accuracy of our treatment. For one thing, acceleration is constant in a great number of situations. Furthermore, in many other situations we can accurately describe motion by assuming a constant acceleration equal to the average acceleration for that motion. Finally, in motions where acceleration changes drastically, such as a car accelerating to top speed and then braking to a stop, the motion can be considered in separate parts, each of which has its own constant acceleration.

### Solving for Displacement (\( \Delta x \)) and Final Position (\( x \)) from Average Velocity when Acceleration (\( a \)) is Constant

To get our first two new equations, we start with the definition of average velocity:

\[ \bar{v} = \frac{\Delta x}{\Delta t}. \]  \hspace{1cm} (2.26)

Substituting the simplified notation for \( \Delta x \) and \( \Delta t \) yields

\[ \bar{v} = \frac{x - x_0}{t}. \]  \hspace{1cm} (2.27)

Solving for \( x \) yields

\[ x = x_0 + \bar{v}t, \]  \hspace{1cm} (2.28)

where the average velocity is

\[ \bar{v} = \frac{v_0 + v}{2} \] (constant \( a \)).  \hspace{1cm} (2.29)

The equation \( \bar{v} = \frac{v_0 + v}{2} \) reflects the fact that, when acceleration is constant, \( v \) is just the simple average of the initial and final velocities. For example, if you steadily increase your velocity (that is, with constant acceleration) from 30 to 60 km/h, then your average velocity during this steady increase is 45 km/h. Using the equation \( \bar{v} = \frac{v_0 + v}{2} \) to check this, we see that

\[ \bar{v} = \frac{30 \text{ km/h} + 60 \text{ km/h}}{2} = 45 \text{ km/h}, \]

which seems logical.

### Example 2.8 Calculating Displacement: How Far does the Jogger Run?

A jogger runs down a straight stretch of road with an average velocity of 4.00 m/s for 2.00 min. What is his final position, taking his initial position to be zero?

**Strategy**

Draw a sketch.
The final position \( x \) is given by the equation

\[ x = x_0 + \bar{v} t. \]  

(2.31)

To find \( x \), we identify the values of \( x_0 \), \( \bar{v} \), and \( t \) from the statement of the problem and substitute them into the equation.

**Solution**

1. Identify the knowns. \( \bar{v} = 4.00 \text{ m/s} \), \( \Delta t = 2.00 \text{ min} \), and \( x_0 = 0 \text{ m} \).

2. Enter the known values into the equation.

\[ x = x_0 + \bar{v} t = 0 + (4.00 \text{ m/s})(120 \text{ s}) = 480 \text{ m} \]  

(2.32)

**Discussion**

Velocity and final displacement are both positive, which means they are in the same direction.

The equation \( x = x_0 + \bar{v} t \) gives insight into the relationship between displacement, average velocity, and time. It shows, for example, that displacement is a linear function of average velocity. (By linear function, we mean that displacement depends on \( \bar{v} \) rather than on \( \bar{v}^2 \). When graphed, linear functions look like straight lines with a constant slope.) On a car trip, for example, we will get twice as far in a given time if we average 90 km/h than if we average 45 km/h.

**Solving for Final Velocity**

We can derive another useful equation by manipulating the definition of acceleration.

\[ a = \frac{\Delta v}{\Delta t} \]  

(2.33)

Substituting the simplified notation for \( \Delta v \) and \( \Delta t \) gives us
\[ a = \frac{v - v_0}{t} \text{ (constant } a) \].

Solving for \( v \) yields

\[ v = v_0 + at \text{ (constant } a) \].

**Example 2.9 Calculating Final Velocity: An Airplane Slowing Down after Landing**

An airplane lands with an initial velocity of 70.0 m/s and then decelerates at 1.50 m/s\(^2\) for 40.0 s. What is its final velocity?

**Strategy**

Draw a sketch. We draw the acceleration vector in the direction opposite the velocity vector because the plane is decelerating.

**Solution**

1. Identify the knowns. \( v_0 = 70.0 \text{ m/s}, a = -1.50 \text{ m/s}^2, t = 40.0 \text{ s} \).
2. Identify the unknown. In this case, it is final velocity, \( v_f \).
3. Determine which equation to use. We can calculate the final velocity using the equation \( v = v_0 + at \).
4. Plug in the known values and solve.

\[ v = v_0 + at = 70.0 \text{ m/s} + (-1.50 \text{ m/s}^2)(40.0 \text{ s}) = 10.0 \text{ m/s} \] (2.36)

**Discussion**

The final velocity is much less than the initial velocity, as desired when slowing down, but still positive. With jet engines, reverse thrust could be maintained long enough to stop the plane and start moving it backward. That would be indicated by a negative final velocity, which is not the case here.

In addition to being useful in problem solving, the equation \( v = v_0 + at \) gives us insight into the relationships among velocity, acceleration, and time. From it we can see, for example, that

- final velocity depends on how large the acceleration is and how long it lasts
- if the acceleration is zero, then the final velocity equals the initial velocity \( (v = v_0) \), as expected (i.e., velocity is constant)
- if \( a \) is negative, then the final velocity is less than the initial velocity

(All of these observations fit our intuition, and it is always useful to examine basic equations in light of our intuition and experiences to check that they do indeed describe nature accurately.)
Making Connections: Real-World Connection

Figure 2.30 The Space Shuttle Endeavor blasts off from the Kennedy Space Center in February 2010. (credit: Matthew Simantov, Flickr)

An intercontinental ballistic missile (ICBM) has a larger average acceleration than the Space Shuttle and achieves a greater velocity in the first minute or two of flight (actual ICBM burn times are classified—short-burn-time missiles are more difficult for an enemy to destroy). But the Space Shuttle obtains a greater final velocity, so that it can orbit the earth rather than come directly back down as an ICBM does. The Space Shuttle does this by accelerating for a longer time.

Solving for Final Position When Velocity is Not Constant ($a \neq 0$)

We can combine the equations above to find a third equation that allows us to calculate the final position of an object experiencing constant acceleration. We start with

$$v = v_0 + at. \quad (2.37)$$

Adding $v_0$ to each side of this equation and dividing by 2 gives

$$\frac{v_0 + v}{2} = v_0 + \frac{1}{2}at. \quad (2.38)$$

Since $\frac{v_0 + v}{2} = \ddot{v}$ for constant acceleration, then

$$\ddot{v} = v_0 + \frac{1}{2}at. \quad (2.39)$$

Now we substitute this expression for $\ddot{v}$ into the equation for displacement, $x = x_0 + \ddot{v}t$, yielding

$$x = x_0 + v_0t + \frac{1}{2}at^2 \ (\text{constant } a). \quad (2.40)$$

Example 2.10 Calculating Displacement of an Accelerating Object: Dragsters

Dragsters can achieve average accelerations of 26.0 m/s$^2$. Suppose such a dragster accelerates from rest at this rate for 5.56 s. How far does it travel in this time?
Figure 2.31 U.S. Army Top Fuel pilot Tony “The Sarge” Schumacher begins a race with a controlled burnout. (credit: Lt. Col. William Thurmond. Photo Courtesy of U.S. Army.)

Strategy
Draw a sketch.

![Sketch of motion](image)

We are asked to find displacement, which is \( x \) if we take \( x_0 \) to be zero. (Think about it like the starting line of a race. It can be anywhere, but we call it 0 and measure all other positions relative to it.) We can use the equation \( x = x_0 + v_0 t + \frac{1}{2} a t^2 \) once we identify \( v_0 \), \( a \), and \( t \) from the statement of the problem.

Solution

1. Identify the knowns. Starting from rest means that \( v_0 = 0 \), \( a \) is given as \( 26.0 \text{ m/s}^2 \) and \( t \) is given as 5.56 s.

2. Plug the known values into the equation to solve for the unknown \( x \):

\[
x = x_0 + v_0 t + \frac{1}{2} a t^2.
\]

Since the initial position and velocity are both zero, this simplifies to

\[
x = \frac{1}{2} a t^2.
\]

Substituting the identified values of \( a \) and \( t \) gives

\[
x = \frac{1}{2} (26.0 \text{ m/s}^2)(5.56 \text{ s})^2,
\]

yielding

\[
x = 402 \text{ m}.
\]

Discussion

If we convert 402 m to miles, we find that the distance covered is very close to one quarter of a mile, the standard distance for drag racing. So the answer is reasonable. This is an impressive displacement in only 5.56 s, but top-notch dragsters can do a quarter mile in even less time than this.

What else can we learn by examining the equation \( x = x_0 + v_0 t + \frac{1}{2} a t^2 \)? We see that:

- displacement depends on the square of the elapsed time when acceleration is not zero. In Example 2.10, the dragster covers only one fourth of the total distance in the first half of the elapsed time.
if acceleration is zero, then the initial velocity equals average velocity \(v_0 = \bar{v}\) and \(x = x_0 + v_0 t + \frac{1}{2} a t^2\) becomes \(x = x_0 + v_0 t\)

**Solving for Final Velocity when Velocity Is Not Constant \((a \neq 0)\)**

A fourth useful equation can be obtained from another algebraic manipulation of previous equations.

If we solve \(v = v_0 + a t\) for \(t\), we get

\[
t = \frac{v - v_0}{a}.
\]

Substituting this and \(\bar{v} = \frac{v_0 + v}{2}\) into \(x = x_0 + \bar{v} t\), we get

\[
v^2 = v_0^2 + 2a(x - x_0) \quad \text{(constant}\ a).\]

**Example 2.11 Calculating Final Velocity: Dragsters**

Calculate the final velocity of the dragster in Example 2.10 without using information about time.

**Strategy**

Draw a sketch.

**Figure 2.33**

The equation \(v^2 = v_0^2 + 2a(x - x_0)\) is ideally suited to this task because it relates velocities, acceleration, and displacement, and no time information is required.

**Solution**

1. Identify the known values. We know that \(v_0 = 0\), since the dragster starts from rest. Then we note that \(x - x_0 = 402\) m (this was the answer in Example 2.10). Finally, the average acceleration was given to be \(a = 26.0\) m/s\(^2\).
2. Plug the knowns into the equation \(v^2 = v_0^2 + 2a(x - x_0)\) and solve for \(v\).

\[
v^2 = 0 + 2(26.0\ \text{m/s}^2)(402\ \text{m}).
\]

Thus

\[
v^2 = 2.09 \times 10^4\ \text{m}^2/\text{s}^2.
\]

To get \(v\), we take the square root:

\[
v = \sqrt{2.09 \times 10^4\ \text{m}^2/\text{s}^2} = 145\ \text{m/s}.
\]

**Discussion**

145 m/s is about 522 km/h or about 324 mi/h, but even this breakneck speed is short of the record for the quarter mile. Also, note that a square root has two values; we took the positive value to indicate a velocity in the same direction as the acceleration.

An examination of the equation \(v^2 = v_0^2 + 2a(x - x_0)\) can produce further insights into the general relationships among physical quantities:

- The final velocity depends on how large the acceleration is and the distance over which it acts.
- For a fixed deceleration, a car that is going twice as fast doesn’t simply stop in twice the distance—it takes much further to...
Putting Equations Together

In the following examples, we further explore one-dimensional motion, but in situations requiring slightly more algebraic manipulation. The examples also give insight into problem-solving techniques. The box below provides easy reference to the equations needed.

Summary of Kinematic Equations (constant \(a\))

\[
\begin{align*}
\dot{x} &= x_0 + \dot{v}t \\
\dot{v} &= \frac{v_0 + v}{2} \\
v &= v_0 + at \\
x &= x_0 + v_0t + \frac{1}{2}at^2 \\
v^2 &= v_0^2 + 2a(x-x_0)
\end{align*}
\]

Example 2.12 Calculating Displacement: How Far Does a Car Go When Coming to a Halt?

On dry concrete, a car can decelerate at a rate of 7.00 m/s\(^2\), whereas on wet concrete it can decelerate at only 5.00 m/s\(^2\). Find the distances necessary to stop a car moving at 30.0 m/s (about 110 km/h) (a) on dry concrete and (b) on wet concrete. (c) Repeat both calculations, finding the displacement from the point where the driver sees a traffic light turn red, taking into account his reaction time of 0.500 s to get his foot on the brake.

Strategy

Draw a sketch.

![Figure 2.34](image)

\(\Delta x = ?\)

\(v_0 = 30.0 \text{ m/s}\)

\(v_f = 0 \text{ m/s}\)

\(a_{\text{dry}} = -7.00 \text{ m/s}^2\)

\(a_{\text{wet}} = -5.00 \text{ m/s}^2\)

In order to determine which equations are best to use, we need to list all of the known values and identify exactly what we need to solve for. We shall do this explicitly in the next several examples, using tables to set them off.

Solution for (a)

1. Identify the knowns and what we want to solve for. We know that \(v_0 = 30.0 \text{ m/s}\); \(v = 0\); \(a = -7.00 \text{ m/s}^2\) (\(a\) is negative because it is in a direction opposite to velocity). We take \(x_0\) to be 0. We are looking for displacement \(\Delta x\), or \(x - x_0\).

2. Identify the equation that will help us solve the problem. The best equation to use is

\[v^2 = v_0^2 + 2a(x-x_0)\]  (2.55)

This equation is best because it includes only one unknown, \(x\). We know the values of all the other variables in this equation. (There are other equations that would allow us to solve for \(x\), but they require us to know the stopping time, \(t\), which we do not know. We could use them but it would entail additional calculations.)

3. Rearrange the equation to solve for \(x\).
4. Enter known values.

$$x - x_0 = \frac{v^2 - v_0^2}{2a}$$ \quad (2.56)

Thus,

$$x - 0 = \frac{0^2 - (30.0 \text{ m/s})^2}{2(-7.00 \text{ m/s}^2)}$$ \quad (2.57)

Hence,

$$x = 64.3 \text{ m on dry concrete.}$$ \quad (2.58)

**Solution for (b)**

This part can be solved in exactly the same manner as Part A. The only difference is that the deceleration is $$-5.00 \text{ m/s}^2$$. The result is

$$x_{\text{wet}} = 90.0 \text{ m on wet concrete.}$$ \quad (2.59)

**Solution for (c)**

Once the driver reacts, the stopping distance is the same as it is in Parts A and B for dry and wet concrete. So to answer this question, we need to calculate how far the car travels during the reaction time, and then add that to the stopping time. It is reasonable to assume that the velocity remains constant during the driver's reaction time.

1. Identify the knowns and what we want to solve for. We know that $$\bar{v} = 30.0 \text{ m/s}$$; $$t_{\text{reaction}} = 0.500 \text{ s}$$; $$a_{\text{reaction}} = 0$$.

   We take $$x_0 - \text{reaction}$$ to be 0. We are looking for $$x_{\text{reaction}}$$.

2. Identify the best equation to use.

   $$x = x_0 + \bar{v} \cdot t$$ works well because the only unknown value is $$x$$, which is what we want to solve for.

3. Plug in the knowns to solve the equation.

   $$x = 0 + (30.0 \text{ m/s})(0.500 \text{ s}) = 15.0 \text{ m.}$$ \quad (2.60)

   This means the car travels 15.0 m while the driver reacts, making the total displacements in the two cases of dry and wet concrete 15.0 m greater than if he reacted instantly.

4. Add the displacement during the reaction time to the displacement when braking.

   $$x_{\text{braking}} + x_{\text{reaction}} = x_{\text{total}}$$ \quad (2.61)

   a. 64.3 m + 15.0 m = 79.3 m when dry

   b. 90.0 m + 15.0 m = 105 m when wet

**Figure 2.35** The distance necessary to stop a car varies greatly, depending on road conditions and driver reaction time. Shown here are the braking distances for dry and wet pavement, as calculated in this example, for a car initially traveling at 30.0 m/s. Also shown are the total distances traveled from the point where the driver first sees a light turn red, assuming a 0.500 s reaction time.

**Discussion**

The displacements found in this example seem reasonable for stopping a fast-moving car. It should take longer to stop a car on wet rather than dry pavement. It is interesting that reaction time adds significantly to the displacements. But more important is the general approach to solving problems. We identify the knowns and the quantities to be determined and then find an appropriate equation. There is often more than one way to solve a problem. The various parts of this example can in
Example 2.13 Calculating Time: A Car Merges into Traffic

Suppose a car merges into freeway traffic on a 200-m-long ramp. If its initial velocity is 10.0 m/s and it accelerates at 2.00 m/s\(^2\), how long does it take to travel the 200 m up the ramp? (Such information might be useful to a traffic engineer.)

**Strategy**

Draw a sketch.

**Solution**

1. Identify the knowns and what we want to solve for. We know that \( v_0 = 10 \text{ m/s} \); \( a = 2.00 \text{ m/s}^2 \); and \( x = 200 \text{ m} \).

2. We need to solve for \( t \). Choose the best equation. \( x = x_0 + v_0 t + \frac{1}{2}at^2 \) works best because the only unknown in the equation is the variable \( t \) for which we need to solve.

3. We will need to rearrange the equation to solve for \( t \). In this case, it will be easier to plug in the knowns first.

\[
200 \text{ m} = 0 \text{ m} + (10.0 \text{ m/s})t + \frac{1}{2}(2.00 \text{ m/s}^2)t^2
\]

(2.62)

4. Simplify the equation. The units of meters (m) cancel because they are in each term. We can get the units of seconds (s) to cancel by taking \( t = t \text{ s} \), where \( t \) is the magnitude of time and s is the unit. Doing so leaves

\[
200 = 10t + t^2.
\]

(2.63)

5. Use the quadratic formula to solve for \( t \).

(a) Rearrange the equation to get 0 on one side of the equation.

\[
t^2 + 10t - 200 = 0
\]

(2.64)

This is a quadratic equation of the form

\[
a t^2 + b t + c = 0,
\]

(2.65)

where the constants are \( a = 1.00 \), \( b = 10.0 \), and \( c = -200 \).

(b) Its solutions are given by the quadratic formula:

\[
t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

(2.66)

This yields two solutions for \( t \), which are

\[
t = 10.0 \text{ and } -20.0.
\]

(2.67)

In this case, then, the time is \( t = t \) in seconds, or

\[
t = 10.0 \text{ s and } -20.0 \text{ s}.
\]

(2.68)
A negative value for time is unreasonable, since it would mean that the event happened 20 s before the motion began. We can discard that solution. Thus,

\[ t = 10.0 \text{ s}. \]  

**(Discussion)**

Whenever an equation contains an unknown squared, there will be two solutions. In some problems both solutions are meaningful, but in others, such as the above, only one solution is reasonable. The 10.0 s answer seems reasonable for a typical freeway on-ramp.

With the basics of kinematics established, we can go on to many other interesting examples and applications. In the process of developing kinematics, we have also glimpsed a general approach to problem solving that produces both correct answers and insights into physical relationships. **Problem-Solving Basics** discusses problem-solving basics and outlines an approach that will help you succeed in this invaluable task.

**Making Connections: Take-Home Experiment—Breaking News**

We have been using SI units of meters per second squared to describe some examples of acceleration or deceleration of cars, runners, and trains. To achieve a better feel for these numbers, one can measure the braking deceleration of a car doing a slow (and safe) stop. Recall that, for average acceleration, \( \ddot{a} = \frac{\Delta v}{\Delta t} \). While traveling in a car, slowly apply the brakes as you come up to a stop sign. Have a passenger note the initial speed in miles per hour and the time taken (in seconds) to stop. From this, calculate the deceleration in miles per hour per second. Convert this to meters per second squared and compare with other decelerations mentioned in this chapter. Calculate the distance traveled in braking.

**Check Your Understanding**

A rocket accelerates at a rate of \( 20 \text{ m/s}^2 \) during launch. How long does it take the rocket to reach a velocity of \( 400 \text{ m/s} \)?

**Solution**

To answer this, choose an equation that allows you to solve for time \( t \), given only \( a \), \( v_0 \), and \( v \).

\[ v = v_0 + at \]  

Rearrange to solve for \( t \).

\[ t = \frac{v - v_0}{a} = \frac{400 \text{ m/s} - 0 \text{ m/s}}{20 \text{ m/s}^2} = 20 \text{ s} \]  

**2.6 Problem-Solving Basics for One-Dimensional Kinematics**

Problem-solving skills are obviously essential to success in a quantitative course in physics. More importantly, the ability to apply broad physical principles, usually represented by equations, to specific situations is a very powerful form of knowledge. It is much more powerful than memorizing a list of facts. Analytical skills and problem-solving abilities can be applied to new situations, whereas a list of facts cannot be made long enough to contain every possible circumstance. Such analytical skills are useful both for solving problems in this text and for applying physics in everyday and professional life.
Problem-Solving Steps

While there is no simple step-by-step method that works for every problem, the following general procedures facilitate problem solving and make it more meaningful. A certain amount of creativity and insight is required as well.

Step 1
Examine the situation to determine which physical principles are involved. It often helps to draw a simple sketch at the outset. You will also need to decide which direction is positive and note that on your sketch. Once you have identified the physical principles, it is much easier to find and apply the equations representing those principles. Although finding the correct equation is essential, keep in mind that equations represent physical principles, laws of nature, and relationships among physical quantities. Without a conceptual understanding of a problem, a numerical solution is meaningless.

Step 2
Make a list of what is given or can be inferred from the problem as stated (identify the knowns). Many problems are stated very succinctly and require some inspection to determine what is known. A sketch can also be very useful at this point. Formally identifying the knowns is of particular importance in applying physics to real-world situations. Remember, “stopped” means velocity is zero, and we often can take initial time and position as zero.

Step 3
Identify exactly what needs to be determined in the problem (identify the unknowns). In complex problems, especially, it is not always obvious what needs to be found or in what sequence. Making a list can help.

Step 4
Find an equation or set of equations that can help you solve the problem. Your list of knowns and unknowns can help here. It is easiest if you can find equations that contain only one unknown—that is, all of the other variables are known, so you can easily solve for the unknown. If the equation contains more than one unknown, then an additional equation is needed to solve the problem. In some problems, several unknowns must be determined to get at the one needed most. In such problems it is especially important to keep physical principles in mind to avoid going astray in a sea of equations. You may have to use two (or more) different equations to get the final answer.

Step 5
Substitute the knowns along with their units into the appropriate equation, and obtain numerical solutions complete with units. This step produces the numerical answer; it also provides a check on units that can help you find errors. If the units of the answer are incorrect, then an error has been made. However, be warned that correct units do not guarantee that the numerical part of the answer is also correct.

Step 6
Check the answer to see if it is reasonable: Does it make sense? This final step is extremely important—the goal of physics is to accurately describe nature. To see if the answer is reasonable, check both its magnitude and its sign, in addition to its units. Your judgment will improve as you solve more and more physics problems, and it will become possible for you to make finer and finer judgments regarding whether nature is adequately described by the answer to a problem. This step brings the problem back to its conceptual meaning. If you can judge whether the answer is reasonable, you have a deeper understanding of physics than just being able to mechanically solve a problem.

When solving problems, we often perform these steps in different order, and we also tend to do several steps simultaneously. There is no rigid procedure that will work every time. Creativity and insight grow with experience, and the basics of problem solving become almost automatic. One way to get practice is to work out the text’s examples for yourself as you read. Another is to work as many end-of-section problems as possible, starting with the easiest to build confidence and progressing to the more difficult. Once you become involved in physics, you will see it all around you, and you can begin to apply it to situations you encounter outside the classroom, just as is done in many of the applications in this text.

Unreasonable Results

Physics must describe nature accurately. Some problems have results that are unreasonable because one premise is unreasonable or because certain premises are inconsistent with one another. The physical principle applied correctly then produces an unreasonable result. For example, if a person starting a foot race accelerates at $0.40 \text{m/s}^2$ for 100 s, his final speed will be 40 m/s (about 150 km/h)—clearly unreasonable because the time of 100 s is an unreasonable premise. The physics is correct in a sense, but there is more to describing nature than just manipulating equations correctly. Checking the result of a problem to see if it is reasonable does more than help uncover errors in problem solving—it also builds intuition in judging whether nature is being accurately described.

Use the following strategies to determine whether an answer is reasonable and, if it is not, to determine what is the cause.

Step 1
Solve the problem using strategies as outlined in the format followed in the worked examples in the text. In the example given in the preceding paragraph, you would identify the given as the acceleration and time and use the equation below to find the unknown final velocity.

$$v = v_0 + at = 0 + (0.40 \text{m/s}^2)(100 \text{ s}) = 40 \text{ m/s}. \quad (2.72)$$
Step 2

*Check to see if the answer is reasonable.* Is it too large or too small, or does it have the wrong sign, improper units, ...? In this case, you may need to convert meters per second into a more familiar unit, such as miles per hour.

\[
\left( \frac{40 \text{ m}}{\text{s}} \right) \left( \frac{3.28 \text{ ft}}{\text{m}} \right) \left( \frac{1 \text{ mi}}{5280 \text{ ft}} \right) \left( \frac{60 \text{ s}}{60 \text{ min}} \right) \left( \frac{60 \text{ min}}{1 \text{ h}} \right) = 89 \text{ mph}
\]  

(2.73)

This velocity is about four times greater than a person can run—so it is too large.

Step 3

*If the answer is unreasonable, look for what specifically could cause the identified difficulty.* In the example of the runner, there are only two assumptions that are suspect. The acceleration could be too great or the time too long. First look at the acceleration and think about what the number means. If someone accelerates at 0.40 \( \text{m/s}^2 \), their velocity is increasing by 0.4 m/s each second. Does this seem reasonable? If so, the time must be too long. It is not possible for someone to accelerate at a constant rate of 0.40 \( \text{m/s}^2 \) for 100 s (almost two minutes).

2.7 Falling Objects

Falling objects form an interesting class of motion problems. For example, we can estimate the depth of a vertical mine shaft by dropping a rock into it and listening for the rock to hit the bottom. By applying the kinematics developed so far to falling objects, we can examine some interesting situations and learn much about gravity in the process.

Gravity

The most remarkable and unexpected fact about falling objects is that, if air resistance and friction are negligible, then in a given location all objects fall toward the center of Earth with the *same constant acceleration, independent of their mass*. This experimentally determined fact is unexpected, because we are so accustomed to the effects of air resistance and friction that we expect light objects to fall slower than heavy ones.

![Image of a hammer and a feather falling](image)

*Figure 2.38 A hammer and a feather will fall with the same constant acceleration if air resistance is considered negligible. This is a general characteristic of gravity not unique to Earth, as astronaut David R. Scott demonstrated on the Moon in 1971, where the acceleration due to gravity is only 1.67 \( \text{m/s}^2 \).*

In the real world, air resistance can cause a lighter object to fall slower than a heavier object of the same size. A tennis ball will reach the ground after a hard baseball dropped at the same time. (It might be difficult to observe the difference if the height is not large.) Air resistance opposes the motion of an object through the air, while friction between objects—such as between clothes and a laundry chute or between a stone and a pool into which it is dropped—also opposes motion between them. For the ideal situations of these first few chapters, an object *falling without air resistance or friction* is defined to be in *free-fall*.

The force of gravity causes objects to fall toward the center of Earth. The acceleration due to gravity is *constant*, which means we can apply the kinematics equations to any falling object where air resistance and friction are negligible. This opens a broad class of interesting situations to us. The acceleration due to gravity is so important that its magnitude is given its own symbol, \( g \). It is constant at any given location on Earth and has the average value

\[
g = 9.80 \text{ m/s}^2
\]

(2.74)

Although \( g \) varies from 9.78 \( \text{m/s}^2 \) to 9.83 \( \text{m/s}^2 \), depending on latitude, altitude, underlying geological formations, and local topography, the average value of 9.80 \( \text{m/s}^2 \) will be used in this text unless otherwise specified. The direction of the acceleration due to gravity is *downward (towards the center of Earth)*. In fact, its direction defines what we call vertical. Note that whether the acceleration \( a \) in the kinematic equations has the value \( +g \) or \( -g \) depends on how we define our coordinate system. If we define the upward direction as positive, then \( a = -g = -9.80 \text{ m/s}^2 \), and if we define the downward direction as negative, then \( a = g = 9.80 \text{ m/s}^2 \).
One-Dimensional Motion Involving Gravity

The best way to see the basic features of motion involving gravity is to start with the simplest situations and then progress toward more complex ones. So we start by considering straight up and down motion with no air resistance or friction. These assumptions mean that the velocity (if there is any) is vertical. If the object is dropped, we know the initial velocity is zero. Once the object has left contact with whatever held or threw it, the object is in free-fall. Under these circumstances, the motion is one-dimensional and has constant acceleration of magnitude $g$. We will also represent vertical displacement with the symbol $y$ and use $x$ for horizontal displacement.

**Kinematic Equations for Objects in Free-Fall where Acceleration = $-g$**

\[
\begin{align*}
    v &= v_0 - gt \\ 
    y &= y_0 + v_0t - \frac{1}{2}gt^2 \\
    v^2 &= v_0^2 - 2g(y - y_0)
\end{align*}
\]

**Example 2.14 Calculating Position and Velocity of a Falling Object: A Rock Thrown Upward**

A person standing on the edge of a high cliff throws a rock straight up with an initial velocity of 13.0 m/s. The rock misses the edge of the cliff as it falls back to earth. Calculate the position and velocity of the rock 1.00 s, 2.00 s, and 3.00 s after it is thrown, neglecting the effects of air resistance.

**Strategy**

Draw a sketch.

![Diagram](image)

$v_0 = 13.0 \text{ m/s}$

$a = -9.8 \text{ m/s}^2$

We are asked to determine the position $y$ at various times. It is reasonable to take the initial position $y_0$ to be zero. This problem involves one-dimensional motion in the vertical direction. We use plus and minus signs to indicate direction, with up being positive and down negative. Since up is positive, and the rock is thrown upward, the initial velocity must be positive too. The acceleration due to gravity is downward, so $a$ is negative. It is crucial that the initial velocity and the acceleration due to gravity have opposite signs. Opposite signs indicate that the acceleration due to gravity opposes the initial motion and will slow and eventually reverse it.

Since we are asked for values of position and velocity at three times, we will refer to these as $y_1$ and $v_1$; $y_2$ and $v_2$; and $y_3$ and $v_3$.

**Solution for Position**

$y_1$

1. Identify the knowns. We know that $y_0 = 0$; $v_0 = 13.0 \text{ m/s}$; $a = -g = -9.8 \text{ m/s}^2$; and $t = 1.00 \text{ s}$.

2. Identify the best equation to use. We will use $y = y_0 + v_0t + \frac{1}{2}at^2$ because it includes only one unknown, $y$ (or $y_1$, here), which is the value we want to find.

3. Plug in the known values and solve for $y_1$.

\[
y = 0 + (13.0 \text{ m/s})(1.00 \text{ s}) + \frac{1}{2}(-9.8 \text{ m/s}^2)(1.00 \text{ s})^2 = 8.10 \text{ m}
\]

**Discussion**

The rock is 8.10 m above its starting point at $t = 1.00 \text{ s}$, since $y_1 > y_0$. It could be moving up or down; the only way to tell is to calculate $v_1$ and find out if it is positive or negative.

**Solution for Velocity**
1. Identify the knowns. We know that \( y_0 = 0 \); \( v_0 = 13.0 \text{ m/s} \); \( a = -g = -9.80 \text{ m/s}^2 \); and \( t = 1.00 \text{ s} \). We also know from the solution above that \( y_1 = 8.10 \text{ m} \).

2. Identify the best equation to use. The most straightforward is \( v = v_0 - gt \) (from \( v = v_0 + at \), where \( a = \) gravitational acceleration = \(-g\)).

3. Plug in the knowns and solve.

\[
v_1 = v_0 - gt = 13.0 \text{ m/s} - (9.80 \text{ m/s}^2)(1.00 \text{ s}) = 3.20 \text{ m/s}
\] (2.79)

**Discussion**

The positive value for \( v_1 \) means that the rock is still heading upward at \( t = 1.00 \text{ s} \). However, it has slowed from its original 13.0 m/s, as expected.

**Solution for Remaining Times**

The procedures for calculating the position and velocity at \( t = 2.00 \text{ s} \) and \( 3.00 \text{ s} \) are the same as those above. The results are summarized in Table 2.1 and illustrated in Figure 2.40.

<table>
<thead>
<tr>
<th>Time, ( t )</th>
<th>Position, ( y )</th>
<th>Velocity, ( v )</th>
<th>Acceleration, ( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00 s</td>
<td>8.10 m</td>
<td>3.20 m/s</td>
<td>(-9.80 \text{ m/s}^2)</td>
</tr>
<tr>
<td>2.00 s</td>
<td>6.40 m</td>
<td>(-6.60 \text{ m/s})</td>
<td>(-9.80 \text{ m/s}^2)</td>
</tr>
<tr>
<td>3.00 s</td>
<td>(-5.10 \text{ m})</td>
<td>(-16.4 \text{ m/s})</td>
<td>(-9.80 \text{ m/s}^2)</td>
</tr>
</tbody>
</table>

Graphing the data helps us understand it more clearly.
Figure 2.40 Vertical position, vertical velocity, and vertical acceleration vs. time for a rock thrown vertically up at the edge of a cliff. Notice that velocity changes linearly with time and that acceleration is constant. Misconception Alert! Notice that the position vs. time graph shows vertical position only. It is easy to get the impression that the graph shows some horizontal motion—the shape of the graph looks like the path of a projectile. But this is not the case; the horizontal axis is time, not space. The actual path of the rock in space is straight up, and straight down.

Discussion
The interpretation of these results is important. At 1.00 s the rock is above its starting point and heading upward, since \( y_1 \) and \( v_1 \) are both positive. At 2.00 s, the rock is still above its starting point, but the negative velocity means it is moving downward. At 3.00 s, both \( y_3 \) and \( v_3 \) are negative, meaning the rock is below its starting point and continuing to move downward. Notice that when the rock is at its highest point (at 1.5 s), its velocity is zero, but its acceleration is still \(-9.80 \text{ m/s}^2\). Its acceleration is \(-9.80 \text{ m/s}^2\) for the whole trip—while it is moving up and while it is moving down. Note that the values for \( y \) are the positions (or displacements) of the rock, not the total distances traveled. Finally, note that free-fall applies to upward motion as well as downward. Both have the same acceleration—the acceleration due to gravity, which remains constant the entire time. Astronauts training in the famous Vomit Comet, for example, experience free-fall while arcing up as well as down, as we will discuss in more detail later.

Making Connections: Take-Home Experiment—Reaction Time
A simple experiment can be done to determine your reaction time. Have a friend hold a ruler between your thumb and index finger, separated by about 1 cm. Note the mark on the ruler that is right between your fingers. Have your friend drop the ruler...
unexpectedly, and try to catch it between your two fingers. Note the new reading on the ruler. Assuming acceleration is that due to gravity, calculate your reaction time. How far would you travel in a car (moving at 30 m/s) if the time it took your foot to go from the gas pedal to the brake was twice this reaction time?

Example 2.15 Calculating Velocity of a Falling Object: A Rock Thrown Down

What happens if the person on the cliff throws the rock straight down, instead of straight up? To explore this question, calculate the velocity of the rock when it is 5.10 m below the starting point, and has been thrown downward with an initial speed of 13.0 m/s.

Strategy

Draw a sketch.

![Figure 2.41](image)

Since up is positive, the final position of the rock will be negative because it finishes below the starting point at \( y_0 = 0 \). Similarly, the initial velocity is downward and therefore negative, as is the acceleration due to gravity. We expect the final velocity to be negative since the rock will continue to move downward.

Solution

1. Identify the knowns. \( y_0 = 0 \); \( y_1 = -5.10 \) m; \( v_0 = -13.0 \) m/s; \( a = -g = -9.80 \) m/s\(^2\).

2. Choose the kinematic equation that makes it easiest to solve the problem. The equation \( v^2 = v_0^2 + 2a(y - y_0) \) works well because the only unknown in it is \( v \). (We will plug \( y_1 \) in for \( y_0 \).)

3. Enter the known values

\[
v^2 = (-13.0 \text{ m/s})^2 + 2(-9.80 \text{ m/s}^2)(-5.10 \text{ m} - 0 \text{ m}) = 268.96 \text{ m}^2/\text{s}^2,
\]

where we have retained extra significant figures because this is an intermediate result.

Taking the square root, and noting that a square root can be positive or negative, gives

\[
v = \pm 16.4 \text{ m/s}.
\]

The negative root is chosen to indicate that the rock is still heading down. Thus,

\[
v = -16.4 \text{ m/s}.
\]

Discussion

Note that this is exactly the same velocity the rock had at this position when it was thrown straight upward with the same initial speed. (See Example 2.14 and Figure 2.42(a).) This is not a coincidental result. Because we only consider the acceleration due to gravity in this problem, the speed of a falling object depends only on its initial speed and its vertical position relative to the starting point. For example, if the velocity of the rock is calculated at a height of 8.10 m above the starting point (using the method from Example 2.14) when the initial velocity is 13.0 m/s straight up, a result of \( \pm 3.20 \) m/s is obtained. Here both signs are meaningful; the positive value occurs when the rock is at 8.10 m and heading up, and the negative value occurs when the rock is at 8.10 m and heading back down. It has the same speed but the opposite direction.
Figure 2.42 (a) A person throws a rock straight up, as explored in Example 2.14. The arrows are velocity vectors at 0, 1.00, 2.00, and 3.00 s. (b) A person throws a rock straight down from a cliff with the same initial speed as before, as in Example 2.15. Note that at the same distance below the point of release, the rock has the same velocity in both cases.

Another way to look at it is this: In Example 2.14, the rock is thrown up with an initial velocity of 13.0 m/s. It rises and then falls back down. When its position is $y = 0$ on its way back down, its velocity is $-13.0$ m/s. That is, it has the same speed on its way down as on its way up. We would then expect its velocity at a position of $y = -5.10$ m to be the same whether we have thrown it upwards at $+13.0$ m/s or thrown it downwards at $-13.0$ m/s. The velocity of the rock on its way down from $y = 0$ is the same whether we have thrown it up or down to start with, as long as the speed with which it was initially thrown is the same.

Example 2.16 Find $g$ from Data on a Falling Object

The acceleration due to gravity on Earth differs slightly from place to place, depending on topography (e.g., whether you are on a hill or in a valley) and subsurface geology (whether there is dense rock like iron ore as opposed to light rock like salt beneath you.) The precise acceleration due to gravity can be calculated from data taken in an introductory physics laboratory course. An object, usually a metal ball for which air resistance is negligible, is dropped and the time it takes to fall a known distance is measured. See, for example, Figure 2.43. Very precise results can be produced with this method if sufficient care is taken in measuring the distance fallen and the elapsed time.
Figure 2.43 Positions and velocities of a metal ball released from rest when air resistance is negligible. Velocity is seen to increase linearly with time while displacement increases with time squared. Acceleration is a constant and is equal to gravitational acceleration.

Suppose the ball falls 1.0000 m in 0.45173 s. Assuming the ball is not affected by air resistance, what is the precise acceleration due to gravity at this location?

**Strategy**

Draw a sketch.
We need to solve for acceleration $a$. Note that in this case, displacement is downward and therefore negative, as is acceleration.

**Solution**

1. Identify the knowns. $y_0 = 0$; $y = -1.0000$ m; $t = 0.45173$; $v_0 = 0$.

2. Choose the equation that allows you to solve for $a$ using the known values.

\[ y = y_0 + v_0t + \frac{1}{2}at^2 \]  

(2.83)

3. Substitute 0 for $v_0$ and rearrange the equation to solve for $a$. Substituting 0 for $v_0$ yields

\[ y = y_0 + \frac{1}{2}at^2. \]  

(2.84)

Solving for $a$ gives

\[ a = \frac{2(y - y_0)}{t^2}. \]  

(2.85)

4. Substitute known values yields

\[ a = \frac{2(-1.0000 \text{ m} - 0)}{(0.45173 \text{ s})^2} = -9.8010 \text{ m/s}^2, \]  

(2.86)

so, because $a = -g$ with the directions we have chosen,

\[ g = 9.8010 \text{ m/s}^2. \]  

(2.87)

**Discussion**

The negative value for $a$ indicates that the gravitational acceleration is downward, as expected. We expect the value to be somewhere around the average value of 9.80 m/s², so 9.8010 m/s² makes sense. Since the data going into the calculation are relatively precise, this value for $g$ is more precise than the average value of 9.80 m/s²; it represents the local value for the acceleration due to gravity.

**Check Your Understanding**

A chunk of ice breaks off a glacier and falls 30.0 meters before it hits the water. Assuming it falls freely (there is no air resistance), how long does it take to hit the water?

**Solution**

We know that initial position $y_0 = 0$, final position $y = -30.0$ m, and $a = -g = -9.80 \text{ m/s}^2$. We can then use the equation $y = y_0 + v_0t + \frac{1}{2}at^2$ to solve for $t$. Inserting $a = -g$, we obtain

\[ y = 0 + 0 - \frac{1}{2}gt^2 \]  

(2.88)

\[ t^2 = \frac{2y}{-g} \]

\[ t = \pm \sqrt{\frac{2y}{-g}} = \pm \sqrt{\frac{2(-30.0 \text{ m})}{-9.80 \text{ m/s}^2}} = \pm \sqrt{6.12 \text{ s}^2} = 2.47 \text{ s} \approx 2.5 \text{ s} \]

where we take the positive value as the physically relevant answer. Thus, it takes about 2.5 seconds for the piece of ice to
hit the water.

**Equation Grapher**

Learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (e.g. \( y = bx \)) to see how they add to generate the polynomial curve.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42102/1.20/#graphing_polynomials)

**Figure 2.45**

### 2.8 Graphical Analysis of One-Dimensional Motion

A graph, like a picture, is worth a thousand words. Graphs not only contain numerical information; they also reveal relationships between physical quantities. This section uses graphs of position, velocity, and acceleration versus time to illustrate one-dimensional kinematics.

**Slopes and General Relationships**

First note that graphs in this text have perpendicular axes, one horizontal and the other vertical. When two physical quantities are plotted against one another in such a graph, the horizontal axis is usually considered to be an **independent variable** and the vertical axis a **dependent variable**. If we call the horizontal axis the \( x \)-axis and the vertical axis the \( y \)-axis, as in **Figure 2.46**, a straight-line graph has the general form

\[
 y = mx + b. \tag{2.89}
\]

Here \( m \) is the **slope**, defined to be the rise divided by the run (as seen in the figure) of the straight line. The letter \( b \) is used for the **\( y \)-intercept**, which is the point at which the line crosses the vertical axis.

**Figure 2.46** A straight-line graph. The equation for a straight line is \( y = mx + b \).

**Graph of Position vs. Time (\( a = 0 \), so \( v \) is constant)**

Time is usually an independent variable that other quantities, such as position, depend upon. A graph of position versus time would, thus, have \( x \) on the vertical axis and \( t \) on the horizontal axis. **Figure 2.47** is just such a straight-line graph. It shows a graph of position versus time for a jet-powered car on a very flat dry lake bed in Nevada.
Using the relationship between dependent and independent variables, we see that the slope in the graph above is average velocity \( \bar{v} \) and the intercept is position at time zero—that is, \( x_0 \). Substituting these symbols into \( y = mx + b \) gives

\[
x = \bar{v}t + x_0
\]

or

\[
x = x_0 + \bar{v}t.
\]

Thus a graph of position versus time gives a general relationship among displacement (change in position), velocity, and time, as well as giving detailed numerical information about a specific situation.

**The Slope of \( x \) vs. \( t \)**

The slope of the graph of position \( x \) vs. time \( t \) is velocity \( v \).

\[
slope = \frac{\Delta x}{\Delta t} = v
\]

Notice that this equation is the same as that derived algebraically from other motion equations in Motion Equations for Constant Acceleration in One Dimension.

From the figure we can see that the car has a position of 525 m at 0.50 s and 2000 m at 6.40 s. Its position at other times can be read from the graph; furthermore, information about its velocity and acceleration can also be obtained from the graph.

**Example 2.17 Determining Average Velocity from a Graph of Position versus Time: Jet Car**

Find the average velocity of the car whose position is graphed in Figure 2.47.

**Strategy**

The slope of a graph of \( x \) vs. \( t \) is average velocity, since slope equals rise over run. In this case, rise = change in position and run = change in time, so that

\[
slope = \frac{\Delta x}{\Delta t} = \bar{v}.
\]

Since the slope is constant here, any two points on the graph can be used to find the slope. (Generally speaking, it is most accurate to use two widely separated points on the straight line. This is because any error in reading data from the graph is proportionally smaller if the interval is larger.)

**Solution**

1. Choose two points on the line. In this case, we choose the points labeled on the graph: (6.4 s, 2000 m) and (0.50 s, 525 m). (Note, however, that you could choose any two points.)
2. Substitute the \( x \) and \( t \) values of the chosen points into the equation. Remember in calculating change (\( \Delta \)) we always use final value minus initial value.

\[
\bar{v} = \frac{\Delta x}{\Delta t} = \frac{2000 \text{ m} - 525 \text{ m}}{6.4 \text{ s} - 0.50 \text{ s}}
\]
yielding \[ \ddot{v} = 250 \, \text{m/s}. \] (2.95)

**Discussion**

This is an impressively large land speed (900 km/h, or about 560 mi/h): much greater than the typical highway speed limit of 60 mi/h (27 m/s or 96 km/h), but considerably shy of the record of 343 m/s (1234 km/h or 766 mi/h) set in 1997.

**Graphs of Motion when \( a \) is constant but \( a \neq 0 \)**

The graphs in Figure 2.48 below represent the motion of the jet-powered car as it accelerates toward its top speed, but only during the time when its acceleration is constant. Time starts at zero for this motion (as if measured with a stopwatch), and the position and velocity are initially 200 m and 15 m/s, respectively.
Figure 2.48 Graphs of motion of a jet-powered car during the time span when its acceleration is constant. (a) The slope of an $x$ vs. $t$ graph is velocity. This is shown at two points, and the instantaneous velocities obtained are plotted in the next graph. Instantaneous velocity at any point is the slope of the tangent at that point. (b) The slope of the $v$ vs. $t$ graph is constant for this part of the motion, indicating constant acceleration. (c) Acceleration has the constant value of $5.0 \text{ m/s}^2$ over the time interval plotted.
The graph of position versus time in Figure 2.48(a) is a curve rather than a straight line. The slope of the curve becomes steeper as time progresses, showing that the velocity is increasing over time. The slope at any point on a position-versus-time graph is the instantaneous velocity at that point. It is found by drawing a straight line tangent to the curve at the point of interest and taking the slope of this straight line. Tangent lines are shown for two points in Figure 2.48(a). If this is done at every point on the curve and the values are plotted against time, then the graph of velocity versus time shown in Figure 2.48(b) is obtained. Furthermore, the slope of the graph of velocity versus time is acceleration, which is shown in Figure 2.48(c).

Example 2.18 Determining Instantaneous Velocity from the Slope at a Point: Jet Car

Calculate the velocity of the jet car at a time of 25 s by finding the slope of the \( x \) vs. \( t \) graph in the graph below.

![Graph showing x vs. t data](image)

The slope of an \( x \) vs. \( t \) graph is velocity. This is shown at two points. Instantaneous velocity at any point is the slope of the tangent at that point.

**Strategy**

The slope of a curve at a point is equal to the slope of a straight line tangent to the curve at that point. This principle is illustrated in Figure 2.50, where Q is the point at \( t = 25 \text{ s} \).

**Solution**

1. Find the tangent line to the curve at \( t = 25 \text{ s} \).
2. Determine the endpoints of the tangent. These correspond to a position of 1300 m at time 19 s and a position of 3120 m at time 32 s.
3. Plug these endpoints into the equation to solve for the slope, \( v \).

\[
\text{slope} = v_Q = \frac{\Delta x_Q}{\Delta t_Q} = \frac{(3120 \text{ m} - 1300 \text{ m})}{(32 \text{ s} - 19 \text{ s})}
\]

Thus,

\[
v_Q = \frac{1820 \text{ m}}{13 \text{ s}} = 140 \text{ m/s}.
\]
This is the value given in this figure’s table for $v$ at $t = 25$ s. The value of 140 m/s for $v_Q$ is plotted in Figure 2.50. The entire graph of $v$ vs. $t$ can be obtained in this fashion.

Carrying this one step further, we note that the slope of a velocity versus time graph is acceleration. Slope is rise divided by run; on a $v$ vs. $t$ graph, rise = change in velocity $\Delta v$ and run = change in time $\Delta t$.

The Slope of $v$ vs. $t$

The slope of a graph of velocity $v$ vs. time $t$ is acceleration $a$.

$$\text{slope} = \frac{\Delta v}{\Delta t} = a$$  \hspace{1cm} (2.98)

Since the velocity versus time graph in Figure 2.48(b) is a straight line, its slope is the same everywhere, implying that acceleration is constant. Acceleration versus time is graphed in Figure 2.48(c).

Additional general information can be obtained from Figure 2.50 and the expression for a straight line, $y = mx + b$.

In this case, the vertical axis $y$ is $V$, the intercept $b$ is $v_0$, the slope $m$ is $a$, and the horizontal axis $x$ is $t$. Substituting these symbols yields

$$v = v_0 + at.$$  \hspace{1cm} (2.99)

A general relationship for velocity, acceleration, and time has again been obtained from a graph. Notice that this equation was also derived algebraically from other motion equations in Motion Equations for Constant Acceleration in One Dimension.

It is not accidental that the same equations are obtained by graphical analysis as by algebraic techniques. In fact, an important way to discover physical relationships is to measure various physical quantities and then make graphs of one quantity against another to see if they are correlated in any way. Correlations imply physical relationships and might be shown by smooth graphs such as those above. From such graphs, mathematical relationships can sometimes be postulated. Further experiments are then performed to determine the validity of the hypothesized relationships.

Graphs of Motion Where Acceleration is Not Constant

Now consider the motion of the jet car as it goes from 165 m/s to its top velocity of 250 m/s, graphed in Figure 2.51. Time again starts at zero, and the initial velocity is 165 m/s. (This was the final velocity of the car in the motion graphed in Figure 2.48.) Acceleration gradually decreases from 5.0 m/s$^2$ to zero when the car hits 250 m/s. The velocity increases until 55 s and then becomes constant, since acceleration decreases to zero at 55 s and remains zero afterward.
Figure 2.51 Graphs of motion of a jet-powered car as it reaches its top velocity. This motion begins where the motion in Figure 2.48 ends. (a) The velocity gradually approaches its top value. The slope of this graph is acceleration. It is plotted in the final graph. (b) Acceleration gradually declines to zero when velocity becomes constant. Notice in each of the three graphs that the acceleration drops down to zero and the velocity levels out. This results in a position-time graph that is almost linear. A close-up of the position time graph would show a slight curvature, as indicated in the velocity graph.

Example 2.19 Calculating Acceleration from a Graph of Velocity versus Time

Calculate the acceleration of the jet car at a time of 25 s by finding the slope of the $v\ vs.\ t$ graph in Figure 2.51(a).

**Strategy**
The slope of the curve at $t = 25\ s$ is equal to the slope of the line tangent at that point, as illustrated in Figure 2.51(a).

**Solution**
Determine endpoints of the tangent line from the figure, and then plug them into the equation to solve for slope, $a$.

$$ \text{slope} = \frac{\Delta v}{\Delta t} = \frac{260 \text{ m/s} - 210 \text{ m/s}}{51 \text{ s} - 1.0 \text{ s}} $$

$$ a = \frac{50 \text{ m/s}}{50 \text{ s}} = 1.0 \text{ m/s}^2. $$

**Discussion**
Note that this value for $a$ is consistent with the value plotted in Figure 2.51(b) at $t = 25\ s$.

A graph of position versus time can be used to generate a graph of velocity versus time, and a graph of velocity versus time can be used to generate a graph of acceleration versus time. We do this by finding the slope of the graphs at every point. If the graph is linear (i.e., a line with a constant slope), it is easy to find the slope at any point and you have the slope for every point.

Graphical analysis of motion can be used to describe both specific and general characteristics of kinematics. Graphs can also be used for other topics in physics. An important aspect of exploring physical relationships is to graph them and look for underlying relationships.
Check Your Understanding

A graph of velocity vs. time of a ship coming into a harbor is shown below. (a) Describe the motion of the ship based on the graph. (b) What would a graph of the ship’s acceleration look like?

![Graph of velocity vs. time](image)

**Figure 2.52**

**Solution**

(a) The ship moves at constant velocity and then begins to decelerate at a constant rate. At some point, its deceleration rate decreases. It maintains this lower deceleration rate until it stops moving.

(b) A graph of acceleration vs. time would show zero acceleration in the first leg, large and constant negative acceleration in the second leg, and constant negative acceleration.

![Graph of acceleration vs. time](image)

**Figure 2.53**

**Glossary**

- **acceleration**: the rate of change in velocity; the change in velocity over time
- **acceleration due to gravity**: acceleration of an object as a result of gravity
- **average acceleration**: the change in velocity divided by the time over which it changes
- **average speed**: distance traveled divided by time during which motion occurs
- **average velocity**: displacement divided by time over which displacement occurs
- **deceleration**: acceleration in the direction opposite to velocity; acceleration that results in a decrease in velocity
- **dependent variable**: the variable that is being measured; usually plotted along the $y$-axis
- **displacement**: the change in position of an object
- **distance**: the magnitude of displacement between two positions
- **distance traveled**: the total length of the path traveled between two positions
- **elapsed time**: the difference between the ending time and beginning time
- **free-fall**: the state of movement that results from gravitational force only
- **independent variable**: the variable that the dependent variable is measured with respect to; usually plotted along the $x$-axis
- **instantaneous acceleration**: acceleration at a specific point in time
- **instantaneous speed**: magnitude of the instantaneous velocity
- **instantaneous velocity**: velocity at a specific instant, or the average velocity over an infinitesimal time interval
- **kinematics**: the study of motion without considering its causes
- **model**: simplified description that contains only those elements necessary to describe the physics of a physical situation
**position**: the location of an object at a particular time

**scalar**: a quantity that is described by magnitude, but not direction

**slope**: the difference in \( y \)-value (the rise) divided by the difference in \( x \)-value (the run) of two points on a straight line

**time**: change, or the interval over which change occurs

**vector**: a quantity that is described by both magnitude and direction

**y-intercept**: the \( y \)-value when \( x = 0 \), or when the graph crosses the \( y \)-axis

### Section Summary

#### 2.1 Displacement
- Kinematics is the study of motion without considering its causes. In this chapter, it is limited to motion along a straight line, called one-dimensional motion.
- Displacement is the change in position of an object.
- In symbols, displacement \( \Delta x \) is defined to be
  \[
  \Delta x = x_f - x_0,
  \]
  where \( x_0 \) is the initial position and \( x_f \) is the final position. In this text, the Greek letter \( \Delta \) (delta) always means “change in” whatever quantity follows it. The SI unit for displacement is the meter (m). Displacement has a direction as well as a magnitude.
- When you start a problem, assign which direction will be positive.
- Distance is the magnitude of displacement between two positions.
- Distance traveled is the total length of the path traveled between two positions.

#### 2.2 Vectors, Scalars, and Coordinate Systems
- A vector is any quantity that has magnitude and direction.
- A scalar is any quantity that has magnitude but no direction.
- Displacement and velocity are vectors, whereas distance and speed are scalars.
- In one-dimensional motion, direction is specified by a plus or minus sign to signify left or right, up or down, and the like.

#### 2.3 Time, Velocity, and Speed
- Time is measured in terms of change, and its SI unit is the second (s). Elapsed time for an event is
  \[
  \Delta t = t_f - t_0,
  \]
  where \( t_f \) is the final time and \( t_0 \) is the initial time. The initial time is often taken to be zero, as if measured with a stopwatch; the elapsed time is then just \( t \).
- Average velocity \( \bar{v} \) is defined as displacement divided by the travel time. In symbols, average velocity is
  \[
  \bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0}.
  \]
- The SI unit for velocity is m/s.
- Velocity is a vector and thus has a direction.
- Instantaneous velocity \( v \) is the velocity at a specific instant or the average velocity for an infinitesimal interval.
- Instantaneous speed is the magnitude of the instantaneous velocity.
- Instantaneous speed is a scalar quantity, as it has no direction specified.
- Average speed is the total distance traveled divided by the elapsed time. (Average speed is not the magnitude of the average velocity.) Speed is a scalar quantity; it has no direction associated with it.

#### 2.4 Acceleration
- Acceleration is the rate at which velocity changes. In symbols, average acceleration \( \bar{a} \) is
  \[
  \bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}.
  \]
- The SI unit for acceleration is \( \text{m/s}^2 \).
- Acceleration is a vector, and thus has a both a magnitude and direction.
- Acceleration can be caused by either a change in the magnitude or the direction of the velocity.
- Instantaneous acceleration \( a \) is the acceleration at a specific instant in time.
- Deceleration is an acceleration with a direction opposite to that of the velocity.
2.5 Motion Equations for Constant Acceleration in One Dimension

- To simplify calculations we take acceleration to be constant, so that $\dot{a} = a$ at all times.
- We also take initial time to be zero.
- Initial position and velocity are given a subscript 0; final values have no subscript. Thus,
  \[\begin{align*}
  \Delta t &= t \\
  \Delta x &= x - x_0 \\
  \Delta v &= v - v_0
  \end{align*}\]
- The following kinematic equations for motion with constant $a$ are useful:
  \[\begin{align*}
  x &= x_0 + \frac{v + \dot{v}}{2}t \\
  \dot{v} &= \frac{v_0 + v}{2} \\
  v &= v_0 + at \\
  x &= x_0 + v_0t + \frac{1}{2}at^2 \\
  v^2 &= v_0^2 + 2a(x - x_0)
  \end{align*}\]
- In vertical motion, $y$ is substituted for $x$.

2.6 Problem-Solving Basics for One-Dimensional Kinematics

- The six basic problem solving steps for physics are:
  - Step 1. Examine the situation to determine which physical principles are involved.
  - Step 2. Make a list of what is given or can be inferred from the problem as stated (identify the knowns).
  - Step 3. Identify exactly what needs to be determined in the problem (identify the unknowns).
  - Step 4. Find an equation or set of equations that can help you solve the problem.
  - Step 5. Substitute the knowns along with their units into the appropriate equation, and obtain numerical solutions complete with units.
  - Step 6. Check the answer to see if it is reasonable: Does it make sense?

2.7 Falling Objects

- An object in free-fall experiences constant acceleration if air resistance is negligible.
- On Earth, all free-falling objects have an acceleration due to gravity $g$, which averages
  \[g = 9.80 \text{ m/s}^2.\]
- Whether the acceleration $a$ should be taken as $+g$ or $-g$ is determined by your choice of coordinate system. If you choose the upward direction as positive, $a = -g = -9.80 \text{ m/s}^2$ is negative. In the opposite case, $a = +g = 9.80 \text{ m/s}^2$ is positive. Since acceleration is constant, the kinematic equations above can be applied with the appropriate $+g$ or $-g$ substituted for $a$.
- For objects in free-fall, up is normally taken as positive for displacement, velocity, and acceleration.

2.8 Graphical Analysis of One-Dimensional Motion

- Graphs of motion can be used to analyze motion.
- Graphical solutions yield identical solutions to mathematical methods for deriving motion equations.
- The slope of a graph of displacement $x$ vs. time $t$ is velocity $v$.
- The slope of a graph of velocity $v$ vs. time $t$ graph is acceleration $a$.
- Average velocity, instantaneous velocity, and acceleration can all be obtained by analyzing graphs.

Conceptual Questions

2.1 Displacement

1. Give an example in which there are clear distinctions among distance traveled, displacement, and magnitude of displacement. Specifically identify each quantity in your example.

2. Under what circumstances does distance traveled equal magnitude of displacement? What is the only case in which magnitude of displacement and displacement are exactly the same?
3. Bacteria move back and forth by using their flagella (structures that look like little tails). Speeds of up to 50 μm/s \( (50 \times 10^{-6} \text{ m/s}) \) have been observed. The total distance traveled by a bacterium is large for its size, while its displacement is small. Why is this?

2.2 Vectors, Scalars, and Coordinate Systems

4. A student writes, “A bird that is diving for prey has a speed of \(-10 \text{ m/s}\).” What is wrong with the student’s statement? What has the student actually described? Explain.

5. What is the speed of the bird in Exercise 2.4?

6. Acceleration is the change in velocity over time. Given this information, is acceleration a vector or a scalar quantity? Explain.

7. A weather forecast states that the temperature is predicted to be \(-5^\circ\text{C}\) the following day. Is this temperature a vector or a scalar quantity? Explain.

2.3 Time, Velocity, and Speed

8. Give an example (but not one from the text) of a device used to measure time and identify what change in that device indicates a change in time.

9. There is a distinction between average speed and the magnitude of average velocity. Give an example that illustrates the difference between these two quantities.

10. Does a car’s odometer measure position or displacement? Does its speedometer measure speed or velocity?

11. If you divide the total distance traveled on a car trip (as determined by the odometer) by the time for the trip, are you calculating the average speed or the magnitude of the average velocity? Under what circumstances are these two quantities the same?

12. How are instantaneous velocity and instantaneous speed related to one another? How do they differ?

2.4 Acceleration

13. Is it possible for speed to be constant while acceleration is not zero? Give an example of such a situation.

14. Is it possible for velocity to be constant while acceleration is not zero? Explain.

15. Give an example in which velocity is zero yet acceleration is not.

16. If a subway train is moving to the left (has a negative velocity) and then comes to a stop, what is the direction of its acceleration? Is the acceleration positive or negative?

17. Plus and minus signs are used in one-dimensional motion to indicate direction. What is the sign of an acceleration that reduces the magnitude of a negative velocity? Of a positive velocity?

2.6 Problem-Solving Basics for One-Dimensional Kinematics

18. What information do you need in order to choose which equation or equations to use to solve a problem? Explain.

19. What is the last thing you should do when solving a problem? Explain.

2.7 Falling Objects

20. What is the acceleration of a rock thrown straight upward on the way up? At the top of its flight? On the way down?

21. An object that is thrown straight up falls back to Earth. This is one-dimensional motion. (a) When is its velocity zero? (b) Does its velocity change direction? (c) Does the acceleration due to gravity have the same sign on the way up as on the way down?

22. Suppose you throw a rock nearly straight up at a coconut in a palm tree, and the rock misses on the way up but hits the coconut on the way down. Neglecting air resistance, how does the speed of the rock when it hits the coconut on the way down compare with what it would have been if it had hit the coconut on the way up? Is it more likely to dislodge the coconut on the way up or down? Explain.

23. If an object is thrown straight up and air resistance is negligible, then its speed when it returns to the starting point is the same as when it was released. If air resistance were not negligible, how would its speed upon return compare with its initial speed? How would the maximum height to which it rises be affected?

24. The severity of a fall depends on your speed when you strike the ground. All factors but the acceleration due to gravity being the same, how many times higher could a safe fall on the Moon be than on Earth (gravitational acceleration on the Moon is about 1/6 that of the Earth)?

25. How many times higher could an astronaut jump on the Moon than on Earth if his takeoff speed is the same in both locations (gravitational acceleration on the Moon is about 1/6 of \( g \) on Earth)?
2.8 Graphical Analysis of One-Dimensional Motion

26. (a) Explain how you can use the graph of position versus time in Figure 2.54 to describe the change in velocity over time. Identify (b) the time \( t_a, t_b, t_c, t_d, \) or \( t_e \) at which the instantaneous velocity is greatest, (c) the time at which it is zero, and (d) the time at which it is negative.

![Figure 2.54](image)

27. (a) Sketch a graph of velocity versus time corresponding to the graph of position versus time given in Figure 2.55. (b) Identify the time or times \( t_a, t_b, t_c, \) etc.) at which the instantaneous velocity is greatest. (c) At which times is it zero? (d) At which times is it negative?

![Figure 2.55](image)
28. (a) Explain how you can determine the acceleration over time from a velocity versus time graph such as the one in Figure 2.56. (b) Based on the graph, how does acceleration change over time?

![Figure 2.56](image)

29. (a) Sketch a graph of acceleration versus time corresponding to the graph of velocity versus time given in Figure 2.57. (b) Identify the time or times \( t_a, t_b, t_c \), etc.) at which the acceleration is greatest. (c) At which times is it zero? (d) At which times is it negative?

![Figure 2.57](image)

30. Consider the velocity vs. time graph of a person in an elevator shown in Figure 2.58. Suppose the elevator is initially at rest. It then accelerates for 3 seconds, maintains that velocity for 15 seconds, then decelerates for 5 seconds until it stops. The acceleration for the entire trip is not constant so we cannot use the equations of motion from Motion Equations for Constant Acceleration in One Dimension for the complete trip. (We could, however, use them in the three individual sections where acceleration is a constant.) Sketch graphs of (a) position vs. time and (b) acceleration vs. time for this trip.

![Figure 2.58](image)
31. A cylinder is given a push and then rolls up an inclined plane. If the origin is the starting point, sketch the position, velocity, and acceleration of the cylinder vs. time as it goes up and then down the plane.
2.1 Displacement

![Diagram of displacement](Figure 2.59)

1. Find the following for path A in Figure 2.59: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

2. Find the following for path B in Figure 2.59: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

3. Find the following for path C in Figure 2.59: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

4. Find the following for path D in Figure 2.59: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

2.3 Time, Velocity, and Speed

5. (a) Calculate Earth’s average speed relative to the Sun. (b) What is its average velocity over a period of one year?

6. A helicopter blade spins at exactly 100 revolutions per minute. Its tip is 5.00 m from the center of rotation. (a) Calculate the average speed of the blade tip in the helicopter’s frame of reference. (b) What is its average velocity over one revolution?

7. The North American and European continents are moving apart at a rate of about 3 cm/yr. At this rate how long will it take them to drift 500 km farther apart than they are at present?

8. Land west of the San Andreas fault in southern California is moving at an average velocity of about 6 cm/yr northwest relative to land east of the fault. Los Angeles is west of the fault and may thus someday be at the same latitude as San Francisco, which is east of the fault. How far in the future will this occur if the displacement to be made is 590 km northwest, assuming the motion remains constant?

9. On May 26, 1934, a streamlined, stainless steel diesel train called the Zephyr set the world’s nonstop long-distance speed record for trains. Its run from Denver to Chicago took 13 hours, 4 minutes, 58 seconds, and was witnessed by more than a million people along the route. The total distance traveled was 1633.8 km. What was its average speed in km/h and m/s?

10. Tidal friction is slowing the rotation of the Earth. As a result, the orbit of the Moon is increasing in radius at a rate of approximately 4 cm/year. Assuming this to be a constant rate, how many years will pass before the radius of the Moon’s orbit increases by $3.84 \times 10^6$ m (1%)?

11. A student drove to the university from her home and noted that the odometer reading of her car increased by 12.0 km. The trip took 18.0 min. (a) What was her average speed? (b) If the straight-line distance from her home to the university is 10.3 km in a direction 25.0° south of east, what was her average velocity? (c) If she returned home by the same path 7 h 30 min after she left, what were her average speed and velocity for the entire trip?

12. The speed of propagation of the action potential (an electrical signal) in a nerve cell depends (inversely) on the diameter of the axon (nerve fiber). If the nerve cell connecting the spinal cord to your feet is 1.1 m long, and the nerve impulse speed is 18 m/s, how long does it take for the nerve signal to travel this distance?

13. Conversations with astronauts on the lunar surface were characterized by a kind of echo in which the earthbound person’s voice was so loud in the astronaut’s space helmet that it was picked up by the astronaut’s microphone and transmitted back to Earth. It is reasonable to assume that the echo time equals the time necessary for the radio wave to travel from the Earth to the Moon and back (that is, neglecting any time delays in the electronic equipment). Calculate the distance from Earth to the Moon given that the echo time was 2.56 s and that radio waves travel at the speed of light $(3.00 \times 10^8$ m/s).

14. A football quarterback runs 15.0 m straight down the playing field in 2.50 s. He is then hit and pushed 3.00 m straight backward in 1.75 s. He breaks the tackle and runs straight forward another 21.0 m in 5.20 s. Calculate his average velocity (a) for each of the three intervals and (b) for the entire motion.

15. The planetary model of the atom pictures electrons orbiting the atomic nucleus much as planets orbit the Sun. In this model you can view hydrogen, the simplest atom, as having a single electron in a circular orbit $1.06 \times 10^{-10}$ m in diameter. (a) If the average speed of the electron in this orbit is known to be $2.20 \times 10^6$ m/s, calculate the number of revolutions per second it makes about the nucleus. (b) What is the electron’s average velocity per revolution?

2.4 Acceleration

16. A cheetah can accelerate from rest to a speed of 30.0 m/s in 7.00 s. What is its acceleration?

17. Professional Application

Dr. John Paul Stapp was U.S. Air Force officer who studied the effects of extreme deceleration on the human body. On December 10, 1954, Stapp rode a rocket sled, accelerating from rest to a top speed of 282 m/s (1015 km/h) in 5.00 s, and was brought jarringly back to rest in only 1.40 s! Calculate his (a) acceleration and (b) deceleration. Express each in multiples of $g$ (9.80 m/s$^2$) by taking its ratio to the acceleration of gravity.

18. A commuter backs her car out of her garage with an acceleration of $1.40 \text{ m/s}^2$. (a) How long does it take her to reach a speed of 2.00 m/s? (b) If she then brakes to a stop in 0.800 s, what is her deceleration?
19. Assume that an intercontinental ballistic missile goes from rest to a suborbital speed of 6.50 km/s in 60.0 s (the actual speed and time are classified). What is its average acceleration in m/s² and in multiples of \( g \) (9.80 m/s²)?

2.5 Motion Equations for Constant Acceleration in One Dimension

20. An Olympic-class sprinter starts a race with an acceleration of 4.50 m/s². (a) What is her speed 2.40 s later? (b) Sketch a graph of her position vs. time for this period.

21. A well-thrown ball is caught in a well-padded mitt. If the deceleration of the ball is 2.10×10⁴ m/s², and 1.85 ms (1 ms = 10⁻³ s) elapses from the time the ball first touches the mitt until it stops, what was the initial velocity of the ball?

22. A bullet in a gun is accelerated from the firing chamber to the end of the barrel at an average rate of 6.20×10⁵ m/s² for 8.10×10⁻⁴ s. What is its muzzle velocity (that is, its final velocity)?

23. (a) A light-rail commuter train accelerates at a rate of 1.35 m/s². How long does it take to reach its top speed of 80.0 km/h, starting from rest? (b) The same train ordinarily decelerates at a rate of 1.65 m/s². How long does it take to come to a stop from its top speed? (c) In emergencies the train can decelerate more rapidly, coming to rest from 80.0 km/h in 8.30 s. What is its emergency deceleration in m/s²?

24. While entering a freeway, a car accelerates from rest at a rate of 2.40 m/s² for 12.0 s. (a) Draw a sketch of the situation. (b) List the knowns in this problem. (c) How far does the car travel in those 12.0 s? To solve this part, first identify the unknown, and then discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, check your units, and discuss whether the answer is reasonable. (d) What is the car’s final velocity? Solve for this unknown in the same manner as in part (c), showing all steps explicitly.

25. At the end of a race, a runner decelerates from a velocity of 9.00 m/s at a rate of 2.00 m/s². (a) How far does she travel in the next 5.00 s? (b) What is her final velocity? (c) Evaluate the result. Does it make sense?

26. Professional Application:

Blood is accelerated from rest to 30.0 cm/s in a distance of 1.80 cm by the left ventricle of the heart. (a) Make a sketch of the situation. (b) List the knowns in this problem. (c) How long does the acceleration take? To solve this part, first identify the unknown, and then discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, checking your units. (d) Is the answer reasonable when compared with the time for a heartbeat?

27. In a slap shot, a hockey player accelerates the puck from a velocity of 8.00 m/s to 40.0 m/s in the same direction. If this shot takes 3.33×10⁻² s, calculate the distance over which the puck accelerates.

28. A powerful motorcycle can accelerate from rest to 26.8 m/s (100 km/h) in only 3.90 s. (a) What is its average acceleration? (b) How far does it travel in that time?

29. Freight trains can produce only relatively small accelerations and decelerations. (a) What is the final velocity of a freight train that accelerates at a rate of 0.0500 m/s² for 8.00 min, starting with an initial velocity of 4.00 m/s? (b) If the train can slow down at a rate of 0.550 m/s², how long will it take to come to a stop from this velocity? (c) How far will it travel in each case?

30. A fireworks shell is accelerated from rest to a velocity of 65.0 m/s over a distance of 0.250 m. (a) How long did the acceleration last? (b) Calculate the acceleration.

31. A swan on a lake gets airborne by flapping its wings and running on top of the water. (a) If the swan must reach a velocity of 6.00 m/s to take off and it accelerates from rest at an average rate of 0.350 m/s², how far will it travel before becoming airborne? (b) How long does this take?

32. Professional Application:

A woodpecker’s brain is specially protected from large accelerations by tendon-like attachments inside the skull. While pecking on a tree, the woodpecker’s head comes to a stop from an initial velocity of 0.600 m/s in a distance of only 2.00 mm. (a) Find the acceleration in m/s² and in multiples of \( g \) (9.80 m/s²). (b) Calculate the stopping time. (c) The tendons cradling the brain stretch, making its stopping distance 4.50 mm (greater than the head and, hence, less deceleration of the brain). What is the brain’s deceleration, expressed in multiples of \( g \)?

33. An unwary football player collides with a padded goalpost while running at a velocity of 7.50 m/s and comes to a full stop after compressing the padding and his body 0.350 m. (a) What is his deceleration? (b) How long does the collision last?

34. In World War II, there were several reported cases of airmen who jumped from their flaming airplanes with no parachute to escape certain death. Some fell about 20,000 feet (6000 m), and some of them survived, with few life-threatening injuries. For these lucky pilots, the tree branches and snow drifts on the ground allowed their deceleration to be relatively small. If we assume that a pilot’s speed upon impact was 123 mph (54 m/s), then what was his deceleration? Assume that the trees and snow stopped him over a distance of 3.0 m.

35. Consider a grey squirrel falling out of a tree to the ground. (a) If we ignore air resistance in this case (only for the sake of this problem), determine a squirrel’s velocity just before hitting the ground, assuming it fell from a height of 3.0 m. (b) If the squirrel stops in a distance of 2.0 cm through bending its limbs, compare its deceleration with that of the airmen in the previous problem.

36. An express train passes through a station. It enters with an initial velocity of 22.0 m/s and decelerates at a rate of 0.150 m/s² as it goes through. The station is 210 m long. (a) How long did the nose of the train stay in the station? (b) How fast is it going when the nose leaves the station? (c) If the train is 130 m long, when does the end of the train leave the station? (d) What is the velocity of the end of the train as it leaves?
37. Dragsters can actually reach a top speed of 145 m/s in only 4.45 s—considerably less time than given in Example 2.10 and Example 2.11. (a) Calculate the average acceleration for such a dragster. (b) Find the final velocity of this dragster starting from rest and accelerating at the rate found in (a) for 402 m (a quarter mile) without using any information on time. (c) Why is the final velocity greater than that used to find the average acceleration? Hint: Consider whether the assumption of constant acceleration is valid for a dragster. If not, discuss whether the acceleration would be greater at the beginning or end of the run and what effect that would have on the final velocity.

38. A bicycle racer sprints at the end of a race to clinch a victory. The racer has an initial velocity of 11.5 m/s and accelerates at the rate of 0.500 m/s² for 7.00 s. (a) What is his final velocity? (b) The racer continues at this velocity to the finish line. If he was 300 m from the finish line when he started to accelerate, how much time did he save? (c) One other racer was 5.00 m ahead when the winner started to accelerate, but he was unable to accelerate, and traveled at 11.8 m/s until the finish line. How far ahead of him (in meters and in seconds) did the winner finish?

39. In 1967, New Zealander Burt Munro set the world record for an Indian motorcycle, on the Bonneville Salt Flats in Utah, with a maximum speed of 183.58 m/h. The one-way course was 5.00 mi long. Acceleration rates are often described by the time it takes to reach 60.0 m/h from rest. If this time was 4.00 s, and Burt accelerated at this rate until he reached his maximum speed, how long did it take Burt to complete the course?

40. (a) A world record was set for the men's 100-m dash in the 2008 Olympic Games in Beijing by Usain Bolt of Jamaica. Bolt “coasted” across the finish line with a time of 9.69 s. If we assume that Bolt accelerated for 3.00 s to reach his maximum speed, and maintained that speed for the rest of the race, calculate his maximum speed and his acceleration. (b) During the same Olympics, Bolt also set the world record in the 200-m dash with a time of 19.30 s. Using the same assumptions as for the 100-m dash, what was his maximum speed for this race?

2.7 Falling Objects

Assume air resistance is negligible unless otherwise stated.

41. Calculate the displacement and velocity at times of (a) 0.500, (b) 1.00, (c) 1.50, and (d) 2.00 s for a ball thrown straight up with an initial velocity of 15.0 m/s. Take the point of release to be \( y_0 = 0 \).

42. Calculate the displacement and velocity at times of (a) 0.500, (b) 1.00, (c) 1.50, (d) 2.00, and (e) 2.50 s for a rock thrown straight down with an initial velocity of 14.0 m/s from the Verrazano Narrows Bridge in New York City. The roadway of this bridge is 70.0 m above the water.

43. A basketball referee tosses the ball straight up for the starting tip-off. At what velocity must a basketball player leave the ground to rise 1.25 m above the floor in an attempt to get the ball?

44. A rescue helicopter is hovering over a person whose boat has sunk. One of the rescuers throws a life preserver straight down to the victim with an initial velocity of 1.40 m/s and observes that it takes 1.8 s to reach the water. (a) List the knowns in this problem. (b) How high above the water was the preserver released? Note that the downdraft of the helicopter reduces the effects of air resistance on the falling life preserver, so that an acceleration equal to that of gravity is reasonable.

45. A dolphin in an aquatic show jumps straight up out of the water at a velocity of 13.0 m/s. (a) List the knowns in this problem. (b) How high does his body rise above the water? To solve this part, first note that the final velocity is now a known and identify its value. Then identify the unknown, and discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, checking units, and discuss whether the answer is reasonable. (c) How long is the dolphin in the air? Neglect any effects due to his size or orientation.

46. A swimmer bounces straight up from a diving board and falls feet first into a pool. She starts with a velocity of 4.00 m/s, and her takeoff point is 1.80 m above the pool. (a) How long are her feet in the air? (b) What is her highest point above the board? (c) What is her velocity when her feet hit the water?

47. (a) Calculate the height of a cliff if it takes 2.35 s for a rock to hit the ground when it is thrown straight up from the cliff with an initial velocity of 8.00 m/s. (b) How long would it take to reach the ground if it is thrown straight down with the same speed?

48. A very strong, but inept, shot putter puts the shot straight up vertically with an initial velocity of 11.0 m/s. How long does he have to get out of the way if the shot was released at a height of 2.20 m, and he is 1.80 m tall?

49. You throw a ball straight up with an initial velocity of 15.0 m/s. It passes a tree branch on the way up at a height of 7.00 m. How much additional time will pass before the ball passes the tree branch on the way back down?

50. A kangaroo can jump over an object 2.50 m high. (a) Calculate its vertical speed when it leaves the ground. (b) How long is it in the air?

51. Standing at the base of one of the cliffs of Mt. Arapiles in Victoria, Australia, a hiker hears a rock break loose from a height of 105 m. He can’t see the rock right away but then does, 1.50 s later. (a) How far above the hiker is the rock when he can see it? (b) How much time does he have to move before the rock hits his head?

52. An object is dropped from a height of 75.0 m above ground level. (a) Determine the distance traveled during the first second. (b) Determine the final velocity at which the object hits the ground. (c) Determine the distance traveled during the last second of motion before hitting the ground.

53. There is a 250-m-high cliff at Half Dome in Yosemite National Park in California. Suppose a boulder breaks loose from the top of this cliff. (a) How fast will it be going when it strikes the ground? (b) Assuming a reaction time of 0.300 s, how long will a tourist at the bottom have to get out of the way after hearing the sound of the rock breaking loose (neglecting the height of the tourist, which would become negligible anyway if hit)? The speed of sound is 335 m/s on this day.
54. A ball is thrown straight up. It passes a 2.00-m-high window 7.50 m off the ground on its path up and takes 0.312 s to go past the window. What was the ball's initial velocity? Hint: First consider only the distance along the window, and solve for the ball's velocity at the bottom of the window. Next, consider only the distance from the ground to the bottom of the window, and solve for the initial velocity using the velocity at the bottom of the window as the final velocity.

55. Suppose you drop a rock into a dark well and, using precision equipment, you measure the time for the sound of a splash to return. (a) Neglecting the time required for sound to travel up the well, calculate the distance to the water if the sound returns in 2.0000 s. (b) Now calculate the distance taking into account the time for sound to travel up the well. The speed of sound is 332.00 m/s in this well.

56. A steel ball is dropped onto a hard floor from a height of 1.50 m and rebounds to a height of 1.45 m. (a) Calculate its velocity just before it strikes the floor. (b) Calculate its velocity just after it leaves the floor on its way back up. (c) Calculate its acceleration during contact with the floor if that contact lasts 0.0800 ms \((8.00\times10^{-5} \text{ s})\). (d) How much did the ball compress during its collision with the floor, assuming the floor is absolutely rigid?

57. A coin is dropped from a hot-air balloon that is 300 m above the ground and rising at 10.0 m/s upward. For the coin, find (a) the maximum height reached, (b) its position and velocity 4.00 s after being released, and (c) the time before it hits the ground.

58. A soft tennis ball is dropped onto a hard floor from a height of 1.50 m and rebounds to a height of 1.10 m. (a) Calculate its velocity just before it strikes the floor. (b) Calculate its velocity just after it leaves the floor on its way back up. (c) Calculate its acceleration during contact with the floor if that contact lasts 3.50 ms \((3.50\times10^{-3} \text{ s})\). (d) How much did the ball compress during its collision with the floor, assuming the floor is absolutely rigid?

2.8 Graphical Analysis of One-Dimensional Motion

Note: There is always uncertainty in numbers taken from graphs. If your answers differ from expected values, examine them to see if they are within data extraction uncertainties estimated by you.

59. (a) By taking the slope of the curve in Figure 2.60, verify that the velocity of the jet car is 115 m/s at \(t = 20 \text{ s}\). (b) By taking the slope of the curve at any point in Figure 2.61, verify that the jet car's acceleration is \(5.0 \text{ m/s}^2\).

60. Using approximate values, calculate the slope of the curve in Figure 2.62 to verify that the velocity at \(t = 30.0 \text{ s}\) is approximately 0.24 m/s. Assume all values are known to 2 significant figures.
62. By taking the slope of the curve in Figure 2.63, verify that the acceleration is $3.2 \text{ m/s}^2$ at $t = 10 \text{ s}$.

![Velocity vs. Time](image1)

**Figure 2.63**

63. Construct the position graph for the subway shuttle train as shown in Figure 2.18(a). Your graph should show the position of the train, in kilometers, from $t = 0$ to $20 \text{ s}$. You will need to use the information on acceleration and velocity given in the examples for this figure.

![Position vs. Time](image2)

**Figure 2.64**

64. (a) Take the slope of the curve in Figure 2.64 to find the jogger’s velocity at $t = 2.5 \text{ s}$. (b) Repeat at $7.5 \text{ s}$. These values must be consistent with the graph in Figure 2.65.

![Velocity vs. Time](image3)

**Figure 2.65**

![Acceleration vs. Time](image4)

**Figure 2.66**
65. A graph of \( v(t) \) is shown for a world-class track sprinter in a 100-m race. (See Figure 2.67). (a) What is his average velocity for the first 4 s? (b) What is his instantaneous velocity at \( t = 5 \) s? (c) What is his average acceleration between 0 and 4 s? (d) What is his time for the race?

![Runner Velocity vs. Time](image)

**Figure 2.67**

66. Figure 2.68 shows the position graph for a particle for 6 s. (a) Draw the corresponding Velocity vs. Time graph. (b) What is the acceleration between 0 s and 2 s? (c) What happens to the acceleration at exactly 2 s?

![Position vs. Time](image)

**Figure 2.68**
TWO-DIMENSIONAL KINEMATICS

Chapter Outline

3.1. Kinematics in Two Dimensions: An Introduction
- Observe that motion in two dimensions consists of horizontal and vertical components.
- Understand the independence of horizontal and vertical vectors in two-dimensional motion.

3.2. Vector Addition and Subtraction: Graphical Methods
- Understand the rules of vector addition, subtraction, and multiplication.
- Apply graphical methods of vector addition and subtraction to determine the displacement of moving objects.

3.3. Vector Addition and Subtraction: Analytical Methods
- Understand the rules of vector addition and subtraction using analytical methods.
- Apply analytical methods to determine vertical and horizontal component vectors.
- Apply analytical methods to determine the magnitude and direction of a resultant vector.

3.4. Projectile Motion
- Identify and explain the properties of a projectile, such as acceleration due to gravity, range, maximum height, and trajectory.
- Determine the location and velocity of a projectile at different points in its trajectory.
- Apply the principle of independence of motion to solve projectile motion problems.

3.5. Addition of Velocities
- Apply principles of vector addition to determine relative velocity.
- Explain the significance of the observer in the measurement of velocity.

Introduction to Two-Dimensional Kinematics

The arc of a basketball, the orbit of a satellite, a bicycle rounding a curve, a swimmer diving into a pool, blood gushing out of a wound, and a puppy chasing its tail are but a few examples of motions along curved paths. In fact, most motions in nature follow curved paths rather than straight lines. Motion along a curved path on a flat surface or a plane (such as that of a ball on a pool table or a skater on an ice rink) is two-dimensional, and thus described by two-dimensional kinematics. Motion not confined to a plane, such as a car following a winding mountain road, is described by three-dimensional kinematics. Both two- and three-dimensional kinematics are simple extensions of the one-dimensional kinematics developed for straight-line motion in the previous chapter. This simple extension will allow us to apply physics to many more situations, and it will also yield unexpected insights about nature.

(This media type is not supported in this reader. Click to open media in browser.)
### 3.1 Kinematics in Two Dimensions: An Introduction

**Figure 3.2** Walkers and drivers in a city like New York are rarely able to travel in straight lines to reach their destinations. Instead, they must follow roads and sidewalks, making two-dimensional, zigzagged paths. (credit: Margaret W. Carruthers)

**Two-Dimensional Motion: Walking in a City**

Suppose you want to walk from one point to another in a city with uniform square blocks, as pictured in **Figure 3.3**.

**Figure 3.3** A pedestrian walks a two-dimensional path between two points in a city. In this scene, all blocks are square and are the same size.

The straight-line path that a helicopter might fly is blocked to you as a pedestrian, and so you are forced to take a two-dimensional path, such as the one shown. You walk 14 blocks in all, 9 east followed by 5 north. What is the straight-line distance?

An old adage states that the shortest distance between two points is a straight line. The two legs of the trip and the straight-line path form a right triangle, and so the Pythagorean theorem, \( a^2 + b^2 = c^2 \), can be used to find the straight-line distance.

\[
c = \sqrt{a^2 + b^2}
\]

**Figure 3.4** The Pythagorean theorem relates the length of the legs of a right triangle, labeled \( a \) and \( b \), with the hypotenuse, labeled \( c \). The relationship is given by: \( a^2 + b^2 = c^2 \). This can be rewritten, solving for \( c : c = \sqrt{a^2 + b^2} \).

The hypotenuse of the triangle is the straight-line path, and so in this case its length in units of city blocks is \( \sqrt{(9 \text{ blocks})^2 + (5 \text{ blocks})^2} = 10.3 \text{ blocks} \), considerably shorter than the 14 blocks you walked. (Note that we are using three significant figures in the answer. Although it appears that “9” and “5” have only one significant digit, they are discrete numbers. In this case “9 blocks” is the same as “9.0 or 9.00 blocks.” We have decided to use three significant figures in the answer in order to show the result more precisely.)
The fact that the straight-line distance (10.3 blocks) in Figure 3.5 is less than the total distance walked (14 blocks) is one example of a general characteristic of vectors. (Recall that vectors are quantities that have both magnitude and direction.)

As for one-dimensional kinematics, we use arrows to represent vectors. The length of the arrow is proportional to the vector’s magnitude. The arrow’s length is indicated by hash marks in Figure 3.3 and Figure 3.5. The arrow points in the same direction as the vector. For two-dimensional motion, the path of an object can be represented with three vectors: one vector shows the straight-line path between the initial and final points of the motion, one vector shows the horizontal component of the motion, and one vector shows the vertical component of the motion. The horizontal and vertical components of the motion add together to give the straight-line path. For example, observe the three vectors in Figure 3.5. The first represents a 9-block displacement east. The second represents a 5-block displacement north. These vectors are added to give the third vector, with a 10.3-block total displacement. The third vector is the straight-line path between the two points. Note that in this example, the vectors that we are adding are perpendicular to each other and thus form a right triangle. This means that we can use the Pythagorean theorem to calculate the magnitude of the total displacement. (Note that we cannot use the Pythagorean theorem to add vectors that are not perpendicular. We will develop techniques for adding vectors having any direction, not just those perpendicular to one another, in Vector Addition and Subtraction: Graphical Methods and Vector Addition and Subtraction: Analytical Methods.)

The Independence of Perpendicular Motions

The person taking the path shown in Figure 3.5 walks east and then north (two perpendicular directions). How far he or she walks east is only affected by his or her motion eastward. Similarly, how far he or she walks north is only affected by his or her motion northward.

The horizontal and vertical components of two-dimensional motion are independent of each other. Any motion in the horizontal direction does not affect motion in the vertical direction, and vice versa. This is true in a simple scenario like that of walking in one direction first, followed by another. It is also true of more complicated motion involving movement in two directions at once. For example, let’s compare the motions of two baseballs. One baseball is dropped from rest. At the same instant, another is thrown horizontally from the same height and follows a curved path. A stroboscope has captured the positions of the balls at fixed time intervals as they fall.

Figure 3.6 This shows the motions of two identical balls—one falls from rest, the other has an initial horizontal velocity. Each subsequent position is an equal time interval. Arrows represent horizontal and vertical velocities at each position. The ball on the right has an initial horizontal velocity, while the ball on the left has no horizontal velocity. Despite the difference in horizontal velocities, the vertical velocities and positions are identical for both balls. This shows that the vertical and horizontal motions are independent.

It is remarkable that for each flash of the strobe, the vertical positions of the two balls are the same. This similarity implies that
the vertical motion is independent of whether or not the ball is moving horizontally. (Assuming no air resistance, the vertical motion of a falling object is influenced by gravity only, and not by any horizontal forces.) Careful examination of the ball thrown horizontally shows that it travels the same horizontal distance between flashes. This is due to the fact that there are no additional forces on the ball in the horizontal direction after it is thrown. This result means that the horizontal velocity is constant, and affected neither by vertical motion nor by gravity (which is vertical). Note that this case is true only for ideal conditions. In the real world, air resistance will affect the speed of the balls in both directions.

The two-dimensional curved path of the horizontally thrown ball is composed of two independent one-dimensional motions (horizontal and vertical). The key to analyzing such motion, called projectile motion, is to resolve (break) it into motions along perpendicular directions. Resolving two-dimensional motion into perpendicular components is possible because the components are independent. We shall see how to resolve vectors in **Vector Addition and Subtraction: Graphical Methods** and **Vector Addition and Subtraction: Analytical Methods**. We will find such techniques to be useful in many areas of physics.

### 3.2 Vector Addition and Subtraction: Graphical Methods

![Figure 3.8](https://legacy.cnx.org/content/m42104/1.13/#fs-id1167062339457)

**Figure 3.8** Displacement can be determined graphically using a scale map, such as this one of the Hawaiian Islands. A journey from Hawai‘i to Moloka‘i has a number of legs, or journey segments. These segments can be added graphically with a ruler to determine the total two-dimensional displacement of the journey. (credit: US Geological Survey)

#### Vectors in Two Dimensions

A **vector** is a quantity that has magnitude and direction. Displacement, velocity, acceleration, and force, for example, are all vectors. In one-dimensional, or straight-line, motion, the direction of a vector can be given simply by a plus or minus sign. In two dimensions (2-d), however, we specify the direction of a vector relative to some reference frame (i.e., coordinate system), using an arrow having length proportional to the vector’s magnitude and pointing in the direction of the vector.

**Figure 3.9** shows such a graphical representation of a vector, using as an example the total displacement for the person walking in a city considered in **Kinematics in Two Dimensions: An Introduction**. We shall use the notation that a boldface symbol, such as $\mathbf{D}$, stands for a vector. Its magnitude is represented by the symbol in italics, $D$, and its direction by $\theta$.

#### Vectors in This Text

In this text, we will represent a vector with a boldface variable. For example, we will represent the quantity force with the vector $\mathbf{F}$, which has both magnitude and direction. The magnitude of the vector will be represented by a variable in italics, such as $F$, and the direction of the variable will be given by an angle $\theta$. 

This OpenStax book is available for free at https://legacy.cnx.org/content/col11951/1.1
Figure 3.9 A person walks 9 blocks east and 5 blocks north. The displacement is 10.3 blocks at an angle $29.1^\circ$ north of east.

Figure 3.10 To describe the resultant vector for the person walking in a city considered in Figure 3.9 graphically, draw an arrow to represent the total displacement vector $D$. Using a protractor, draw a line at an angle $\theta$ relative to the east-west axis. The length $D$ of the arrow is proportional to the vector’s magnitude and is measured along the line with a ruler. In this example, the magnitude $D$ of the vector is 10.3 units, and the direction $\theta$ is $29.1^\circ$ north of east.

Vector Addition: Head-to-Tail Method

The **head-to-tail method** is a graphical way to add vectors, described in Figure 3.11 below and in the steps following. The **tail** of the vector is the starting point of the vector, and the **head** (or tip) of a vector is the final, pointed end of the arrow.

Figure 3.11 Head-to-Tail Method: The head-to-tail method of graphically adding vectors is illustrated for the two displacements of the person walking in a city considered in Figure 3.9. (a) Draw a vector representing the displacement to the east. (b) Draw a vector representing the displacement to the north. The tail of this vector should originate from the head of the first, east-pointing vector. (c) Draw a line from the tail of the east-pointing vector to the head of the north-pointing vector to form the sum or resultant vector $D$. The length of the arrow $D$ is proportional to the vector’s magnitude and is measured to be 10.3 units. Its direction, described as the angle with respect to the east (or horizontal axis) $\theta$ is measured with a protractor to be $29.1^\circ$. 
**Step 1.** Draw an arrow to represent the first vector (9 blocks to the east) using a ruler and protractor.

![Figure 3.12](image)

**Step 2.** Now draw an arrow to represent the second vector (5 blocks to the north). Place the tail of the second vector at the head of the first vector.

![Figure 3.13](image)

**Step 3.** If there are more than two vectors, continue this process for each vector to be added. Note that in our example, we have only two vectors, so we have finished placing arrows tip to tail.

**Step 4.** Draw an arrow from the tail of the first vector to the head of the last vector. This is the resultant, or the sum, of the other vectors.

![Figure 3.14](image)

**Step 5.** To get the magnitude of the resultant, measure its length with a ruler. (Note that in most calculations, we will use the Pythagorean theorem to determine this length.)

**Step 6.** To get the direction of the resultant, measure the angle it makes with the reference frame using a protractor. (Note that in most calculations, we will use trigonometric relationships to determine this angle.)
The graphical addition of vectors is limited in accuracy only by the precision with which the drawings can be made and the precision of the measuring tools. It is valid for any number of vectors.

**Example 3.1 Adding Vectors Graphically Using the Head-to-Tail Method: A Woman Takes a Walk**

Use the graphical technique for adding vectors to find the total displacement of a person who walks the following three paths (displacements) on a flat field. First, she walks 25.0 m in a direction 49.0° north of east. Then, she walks 23.0 m heading 15.0° north of east. Finally, she turns and walks 32.0 m in a direction 68.0° south of east.

**Strategy**

Represent each displacement vector graphically with an arrow, labeling the first \( \mathbf{A} \), the second \( \mathbf{B} \), and the third \( \mathbf{C} \), making the lengths proportional to the distance and the directions as specified relative to an east-west line. The head-to-tail method outlined above will give a way to determine the magnitude and direction of the resultant displacement, denoted \( \mathbf{R} \).

**Solution**

(1) Draw the three displacement vectors.

(2) Place the vectors head to tail retaining both their initial magnitude and direction.

(3) Draw the resultant vector, \( \mathbf{R} \).

(4) Use a ruler to measure the magnitude of \( \mathbf{R} \), and a protractor to measure the direction of \( \mathbf{R} \). While the direction of the vector can be specified in many ways, the easiest way is to measure the angle between the vector and the nearest
horizontal or vertical axis. Since the resultant vector is south of the eastward pointing axis, we flip the protractor upside down and measure the angle between the eastward axis and the vector.

![Figure 3.18](image)

In this case, the total displacement $R$ is seen to have a magnitude of 50.0 m and to lie in a direction $7.0^\circ$ south of east. By using its magnitude and direction, this vector can be expressed as $R = 50.0 \text{ m}$ and $\theta = 7.0^\circ$ south of east.

**Discussion**

The head-to-tail graphical method of vector addition works for any number of vectors. It is also important to note that the resultant is independent of the order in which the vectors are added. Therefore, we could add the vectors in any order as illustrated in Figure 3.19 and we will still get the same solution.

![Figure 3.19](image)

Here, we see that when the same vectors are added in a different order, the result is the same. This characteristic is true in every case and is an important characteristic of vectors. Vector addition is **commutative**. Vectors can be added in any order.

$$A + B = B + A.$$  \hspace{1cm} (3.1)

(This is true for the addition of ordinary numbers as well—you get the same result whether you add $2 + 3$ or $3 + 2$, for example).

**Vector Subtraction**

Vector subtraction is a straightforward extension of vector addition. To define subtraction (say we want to subtract $B$ from $A$, written $A - B$, we must first define what we mean by subtraction. The **negative** of a vector $B$ is defined to be $-B$; that is, graphically the negative of any vector has the same magnitude but the opposite direction, as shown in Figure 3.20. In other words, $B$ has the same length as $-B$, but points in the opposite direction. Essentially, we just flip the vector so it points in the opposite direction.
The negative of a vector is just another vector of the same magnitude but pointing in the opposite direction. So \( \mathbf{B} \) is the negative of \( -\mathbf{B} \); it has the same length but opposite direction.

The subtraction of vector \( \mathbf{B} \) from vector \( \mathbf{A} \) is then simply defined to be the addition of \( -\mathbf{B} \) to \( \mathbf{A} \). Note that vector subtraction is the addition of a negative vector. The order of subtraction does not affect the results.

\[
\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).
\]

This is analogous to the subtraction of scalars (where, for example, \( 5 - 2 = 5 + (-2) \)). Again, the result is independent of the order in which the subtraction is made. When vectors are subtracted graphically, the techniques outlined above are used, as the following example illustrates.

**Example 3.2 Subtracting Vectors Graphically: A Woman Sailing a Boat**

A woman sailing a boat at night is following directions to a dock. The instructions read to first sail 27.5 m in a direction 66.0º north of east from her current location, and then travel 30.0 m in a direction 112º north of east (or 22.0º west of north). If the woman makes a mistake and travels in the opposite direction for the second leg of the trip, where will she end up? Compare this location with the location of the dock.

**Strategy**

We can represent the first leg of the trip with a vector \( \mathbf{A} \), and the second leg of the trip with a vector \( \mathbf{B} \). The dock is located at a location \( \mathbf{A} + \mathbf{B} \). If the woman mistakenly travels in the opposite direction for the second leg of the journey, she will travel a distance \( \mathbf{B} \) (30.0 m) in the direction 180º - 112º = 68º south of east. We represent this as \( -\mathbf{B} \), as shown below. The vector \( -\mathbf{B} \) has the same magnitude as \( \mathbf{B} \) but is in the opposite direction. Thus, she will end up at a location \( \mathbf{A} + (-\mathbf{B}) \), or \( \mathbf{A} - \mathbf{B} \).
We will perform vector addition to compare the location of the dock, \( \mathbf{A} + \mathbf{B} \), with the location at which the woman mistakenly arrives, \( \mathbf{A} + (\mathbf{-B}) \).

**Solution**

1. To determine the location at which the woman arrives by accident, draw vectors \( \mathbf{A} \) and \( \mathbf{-B} \).
2. Place the vectors head to tail.
3. Draw the resultant vector \( \mathbf{R} \).
4. Use a ruler and protractor to measure the magnitude and direction of \( \mathbf{R} \).

In this case, \( R = 23.0 \text{ m} \) and \( \theta = 7.5^\circ \) south of east.

5. To determine the location of the dock, we repeat this method to add vectors \( \mathbf{A} \) and \( \mathbf{B} \). We obtain the resultant vector \( \mathbf{R}' \):

In this case \( R = 52.9 \text{ m} \) and \( \theta = 90.1^\circ \) north of east.

We can see that the woman will end up a significant distance from the dock if she travels in the opposite direction for the second leg of the trip.

**Discussion**
Because subtraction of a vector is the same as addition of a vector with the opposite direction, the graphical method of subtracting vectors works the same as for addition.

**Multiplication of Vectors and Scalars**

If we decided to walk three times as far on the first leg of the trip considered in the preceding example, then we would walk $3 \times 27.5 \text{ m}$, or $82.5 \text{ m}$, in a direction $66.0^\circ$ north of east. This is an example of multiplying a vector by a positive scalar. Notice that the magnitude changes, but the direction stays the same.

If the scalar is negative, then multiplying a vector by it changes the vector's magnitude and gives the new vector the opposite direction. For example, if you multiply by $-2$, the magnitude doubles but the direction changes. We can summarize these rules in the following way: When vector $\mathbf{A}$ is multiplied by a scalar $c$,

- the magnitude of the vector becomes the absolute value of $c \cdot A$,
- if $c$ is positive, the direction of the vector does not change,
- if $c$ is negative, the direction is reversed.

In our case, $c = 3$ and $A = 27.5 \text{ m}$. Vectors are multiplied by scalars in many situations. Note that division is the inverse of multiplication. For example, dividing by 2 is the same as multiplying by the value $1/2$. The rules for multiplication of vectors by scalars are the same for division; simply treat the divisor as a scalar between 0 and 1.

**Resolving a Vector into Components**

In the examples above, we have been adding vectors to determine the resultant vector. In many cases, however, we will need to do the opposite. We will need to take a single vector and find what other vectors added together produce it. In most cases, this involves determining the perpendicular components of a single vector, for example the $x$- and $y$-components, or the north-south and east-west components.

For example, we may know that the total displacement of a person walking in a city is 10.3 blocks in a direction $29.0^\circ$ north of east and want to find out how many blocks east and north had to be walked. This method is called finding the components (or parts) of the displacement in the east and north directions, and it is the inverse of the process followed to find the total displacement. It is one example of finding the components of a vector. There are many applications in physics where this is a useful thing to do. We will see this soon in Projectile Motion, and much more when we cover forces in Dynamics: Newton's Laws of Motion. Most of these involve finding components along perpendicular axes (such as north and east), so that right triangles are involved. The analytical techniques presented in Vector Addition and Subtraction: Analytical Methods are ideal for finding vector components.

**3.3 Vector Addition and Subtraction: Analytical Methods**

Analytical methods of vector addition and subtraction employ geometry and simple trigonometry rather than the ruler and protractor of graphical methods. Part of the graphical technique is retained, because vectors are still represented by arrows for easy visualization. However, analytical methods are more concise, accurate, and precise than graphical methods, which are limited by the accuracy with which a drawing can be made. Analytical methods are limited only by the accuracy and precision with which physical quantities are known.

**Resolving a Vector into Perpendicular Components**

Analytical techniques and right triangles go hand-in-hand in physics because (among other things) motions along perpendicular directions are independent. We very often need to separate a vector into perpendicular components. For example, given a vector like $\mathbf{A}$ in Figure 3.26, we may wish to find which two perpendicular vectors, $\mathbf{A}_x$ and $\mathbf{A}_y$, add to produce it.
The vector $\mathbf{A}$, with its tail at the origin of an $x$, $y$-coordinate system, is shown together with its $x$- and $y$-components, $\mathbf{A}_x$ and $\mathbf{A}_y$.

These vectors form a right triangle. The analytical relationships among these vectors are summarized below.

$\mathbf{A}_x$ and $\mathbf{A}_y$ are defined to be the components of $\mathbf{A}$ along the $x$- and $y$-axes. The three vectors $\mathbf{A}$, $\mathbf{A}_x$, and $\mathbf{A}_y$ form a right triangle:

$$\mathbf{A}_x + \mathbf{A}_y = \mathbf{A}. \tag{3.3}$$

Note that this relationship between vector components and the resultant vector holds only for vector quantities (which include both magnitude and direction). The relationship does not apply for the magnitudes alone. For example, if $\mathbf{A}_x = 3 \text{ m east}$, $\mathbf{A}_y = 4 \text{ m north}$, and $\mathbf{A} = 5 \text{ m north-east}$, then it is true that the vectors $\mathbf{A}_x + \mathbf{A}_y = \mathbf{A}$. However, it is not true that the sum of the magnitudes of the vectors is also equal. That is,

$$3 \text{ m} + 4 \text{ m} \neq 5 \text{ m} \tag{3.4}$$

Thus,

$$\mathbf{A}_x + \mathbf{A}_y \neq \mathbf{A} \tag{3.5}$$

If the vector $\mathbf{A}$ is known, then its magnitude $A$ (its length) and its angle $\theta$ (its direction) are known. To find $\mathbf{A}_x$ and $\mathbf{A}_y$, its $x$- and $y$-components, we use the following relationships for a right triangle.

$$\mathbf{A}_x = A \cos \theta \tag{3.6}$$

and

$$\mathbf{A}_y = A \sin \theta. \tag{3.7}$$

Suppose, for example, that $\mathbf{A}$ is the vector representing the total displacement of the person walking in a city considered in *Kinematics in Two Dimensions: An Introduction* and *Vector Addition and Subtraction: Graphical Methods*. 

---

**Figure 3.26** The magnitudes of the vector components $\mathbf{A}_x$ and $\mathbf{A}_y$ can be related to the resultant vector $\mathbf{A}$ and the angle $\theta$ with trigonometric identities. Here we see that $A_x = A \cos \theta$ and $A_y = A \sin \theta$. 

**Figure 3.27**
We can use the relationships $A_x = A \cos \theta$ and $A_y = A \sin \theta$ to determine the magnitude of the horizontal and vertical component vectors in this example. Then $A = 10.3$ blocks and $\theta = 29.1^\circ$, so that

$$A_x = A \cos \theta = (10.3 \text{ blocks})(\cos 29.1^\circ) = 9.0 \text{ blocks}$$

$$A_y = A \sin \theta = (10.3 \text{ blocks})(\sin 29.1^\circ) = 5.0 \text{ blocks}.$$  

Calculating a Resultant Vector

If the perpendicular components $A_x$ and $A_y$ of a vector $A$ are known, then $A$ can also be found analytically. To find the magnitude $A$ and direction $\theta$ of a vector from its perpendicular components $A_x$ and $A_y$, relative to the $x$-axis, we use the following relationships:

$$A = \sqrt{A_x^2 + A_y^2}$$

$$\theta = \tan^{-1}(A_y / A_x).$$

Note that the equation $A = \sqrt{A_x^2 + A_y^2}$ is just the Pythagorean theorem relating the legs of a right triangle to the length of the hypotenuse. For example, if $A_x$ and $A_y$ are 9 and 5 blocks, respectively, then $A = \sqrt{9^2 + 5^2} = 10.3$ blocks, again consistent with the example of the person walking in a city. Finally, the direction is $\theta = \tan^{-1}(5/9) = 29.1^\circ$, as before.

**Determining Vectors and Vector Components with Analytical Methods**

Equations $A_x = A \cos \theta$ and $A_y = A \sin \theta$ are used to find the perpendicular components of a vector—that is, to go from $A$ and $\theta$ to $A_x$ and $A_y$. Equations $A = \sqrt{A_x^2 + A_y^2}$ and $\theta = \tan^{-1}(A_y / A_x)$ are used to find a vector from its perpendicular components—that is, to go from $A_x$ and $A_y$ to $A$ and $\theta$. Both processes are crucial to analytical methods.
of vector addition and subtraction.

Adding Vectors Using Analytical Methods

To see how to add vectors using perpendicular components, consider Figure 3.30, in which the vectors \( \mathbf{A} \) and \( \mathbf{B} \) are added to produce the resultant \( \mathbf{R} \).

![Figure 3.30](image)

**Figure 3.30** Vectors \( \mathbf{A} \) and \( \mathbf{B} \) are two legs of a walk, and \( \mathbf{R} \) is the resultant or total displacement. You can use analytical methods to determine the magnitude and direction of \( \mathbf{R} \).

If \( \mathbf{A} \) and \( \mathbf{B} \) represent two legs of a walk (two displacements), then \( \mathbf{R} \) is the total displacement. The person taking the walk ends up at the tip of \( \mathbf{R} \). There are many ways to arrive at the same point. In particular, the person could have walked first in the \( x \)-direction and then in the \( y \)-direction. Those paths are the \( x \)- and \( y \)-components of the resultant, \( \mathbf{R}_x \) and \( \mathbf{R}_y \). If we know \( \mathbf{R}_x \) and \( \mathbf{R}_y \), we can find \( R \) and \( \theta \) using the equations \( A = \sqrt{A_x^2 + A_y^2} \) and \( \theta = \tan^{-1}(A_y/A_x) \). When you use the analytical method of vector addition, you can determine the components or the magnitude and direction of a vector.

**Step 1.** Identify the \( x \)- and \( y \)-axes that will be used in the problem. Then, find the components of each vector to be added along the chosen perpendicular axes. Use the equations \( A_x = A \cos \theta \) and \( A_y = A \sin \theta \) to find the components. In Figure 3.31, these components are \( A_x \), \( A_y \), \( B_x \), and \( B_y \). The angles that vectors \( \mathbf{A} \) and \( \mathbf{B} \) make with the \( x \)-axis are \( \theta_A \) and \( \theta_B \), respectively.

![Figure 3.31](image)

**Figure 3.31** To add vectors \( \mathbf{A} \) and \( \mathbf{B} \), first determine the horizontal and vertical components of each vector. These are the dotted vectors \( \mathbf{A}_x \), \( \mathbf{A}_y \), \( \mathbf{B}_x \) and \( \mathbf{B}_y \) shown in the image.

**Step 2.** Find the components of the resultant along each axis by adding the components of the individual vectors along that axis. That is, as shown in Figure 3.32,

\[
R_x = A_x + B_x
\]

(3.12)

and

\[
R_y = A_y + B_y.
\]

(3.13)
Components along the same axis, say the $x$-axis, are vectors along the same line and, thus, can be added to one another like ordinary numbers. The same is true for components along the $y$-axis. (For example, a 9-block eastward walk could be taken in two legs, the first 3 blocks east and the second 6 blocks east, for a total of 9, because they are along the same direction.) So resolving vectors into components along common axes makes it easier to add them. Now that the components of $\mathbf{R}$ are known, its magnitude and direction can be found.

**Step 3.** To get the magnitude $R$ of the resultant, use the Pythagorean theorem:

$$R = \sqrt{R_x^2 + R_y^2}.$$  \hspace{1cm} (3.14)

**Step 4.** To get the direction of the resultant relative to the $x$-axis:

$$\theta = \tan^{-1}(R_y / R_x).$$  \hspace{1cm} (3.15)

The following example illustrates this technique for adding vectors using perpendicular components.
Example 3.3 Adding Vectors Using Analytical Methods

Add the vector \( \mathbf{A} \) to the vector \( \mathbf{B} \) shown in Figure 3.33, using perpendicular components along the \( x \)- and \( y \)-axes. The \( x \)- and \( y \)-axes are along the east–west and north–south directions, respectively. Vector \( \mathbf{A} \) represents the first leg of a walk in which a person walks 53.0 m in a direction 20.0° north of east. Vector \( \mathbf{B} \) represents the second leg, a displacement of 34.0 m in a direction 63.0° north of east.

![Figure 3.33](image)

**Figure 3.33** Vector \( \mathbf{A} \) has magnitude 53.0 m and direction 20.0° north of the \( x \)-axis. Vector \( \mathbf{B} \) has magnitude 34.0 m and direction 63.0° north of the \( x \)-axis. You can use analytical methods to determine the magnitude and direction of \( \mathbf{R} \).

**Strategy**

The components of \( \mathbf{A} \) and \( \mathbf{B} \) along the \( x \)- and \( y \)-axes represent walking due east and due north to get to the same ending point. Once found, they are combined to produce the resultant.

**Solution**

Following the method outlined above, we first find the components of \( \mathbf{A} \) and \( \mathbf{B} \) along the \( x \)- and \( y \)-axes. Note that \( A = 53.0 \text{ m}, \ \theta_A = 20.0^\circ \), \( B = 34.0 \text{ m}, \ \text{and} \ \theta_B = 63.0^\circ \). We find the \( x \)-components by using \( A_x = A \cos \theta_A \), which gives

\[
A_x = A \cos \theta_A = (53.0 \text{ m})(\cos 20.0^\circ) = 49.8 \text{ m}
\]

and

\[
B_x = B \cos \theta_B = (34.0 \text{ m})(\cos 63.0^\circ) = 15.4 \text{ m}.
\]

Similarly, the \( y \)-components are found using \( A_y = A \sin \theta_A \):

\[
A_y = A \sin \theta_A = (53.0 \text{ m})(\sin 20.0^\circ) = 18.1 \text{ m}
\]

and

\[
B_y = B \sin \theta_B = (34.0 \text{ m})(\sin 63.0^\circ) = 30.3 \text{ m}.
\]

The \( x \)- and \( y \)-components of the resultant are thus

\[
R_x = A_x + B_x = 49.8 \text{ m} + 15.4 \text{ m} = 65.2 \text{ m}
\]

and

\[
R_y = A_y + B_y = 18.1 \text{ m} + 30.3 \text{ m} = 48.4 \text{ m}.
\]

Now we can find the magnitude of the resultant by using the Pythagorean theorem:

\[
R = \sqrt{R_x^2 + R_y^2} = \sqrt{(65.2)^2 + (48.4)^2} \text{ m}
\]

so that

\[
R = 81.2 \text{ m}.
\]
Finally, we find the direction of the resultant:

$$\theta = \tan^{-1}(R_y / R_x) = + \tan^{-1}(48.4 / 65.2).$$  \hspace{1cm} (3.24)

Thus,

$$\theta = \tan^{-1}(0.742) = 36.6^\circ. \hspace{1cm} (3.25)$$

Discussion

This example illustrates the addition of vectors using perpendicular components. Vector subtraction using perpendicular components is very similar—it is just the addition of a negative vector.

Subtraction of vectors is accomplished by the addition of a negative vector. That is, \( \mathbf{A} - \mathbf{B} \equiv \mathbf{A} + (\mathbf{-B}) \). Thus, *the method for the subtraction of vectors using perpendicular components is identical to that for addition*. The components of \( \mathbf{-B} \) are the negatives of the components of \( \mathbf{B} \). The \( x \)- and \( y \)-components of the resultant \( \mathbf{A} - \mathbf{B} = \mathbf{R} \) are thus

$$R_x = A_x + (-B_x)$$  \hspace{1cm} (3.26)

and

$$R_y = A_y + (-B_y)$$  \hspace{1cm} (3.27)

and the rest of the method outlined above is identical to that for addition. (See Figure 3.35.)

Analyzing vectors using perpendicular components is very useful in many areas of physics, because perpendicular quantities are often independent of one another. The next module, *Projectile Motion*, is one of many in which using perpendicular components helps make the picture clear and simplifies the physics.

Figure 3.34 Using analytical methods, we see that the magnitude of \( \mathbf{R} \) is 81.2 m and its direction is 36.6° north of east.

Vector Addition

Learn how to add vectors. Drag vectors onto a graph, change their length and angle, and sum them together. The
3.4 Projectile Motion

Projectile motion is the motion of an object thrown or projected into the air, subject to only the acceleration of gravity. The object is called a projectile, and its path is called its trajectory. The motion of falling objects, as covered in Problem-Solving Basics for One-Dimensional Kinematics, is a simple one-dimensional type of projectile motion in which there is no horizontal movement. In this section, we consider two-dimensional projectile motion, such as that of a football or other object for which air resistance is negligible.

The most important fact to remember here is that motions along perpendicular axes are independent and thus can be analyzed separately. This fact was discussed in Kinematics in Two Dimensions: An Introduction, where vertical and horizontal motions were seen to be independent. The key to analyzing two-dimensional projectile motion is to break it into two motions, one along the horizontal axis and the other along the vertical. (This choice of axes is the most sensible, because acceleration due to gravity is vertical—thus, there will be no acceleration along the horizontal axis when air resistance is negligible.) As is customary, we call the horizontal axis the x-axis and the vertical axis the y-axis. Figure 3.36 illustrates the notation for displacement, where s is defined to be the total displacement and x and y are its components along the horizontal and vertical axes, respectively. The magnitudes of these vectors are s, x, and y. (Note that in the last section we used the notation A to represent a vector with components Ax and Ay. If we continued this format, we would call displacement s with components sx and sy. However, to simplify the notation, we will simply represent the component vectors as x and y.)

Of course, to describe motion we must deal with velocity and acceleration, as well as with displacement. We must find their components along the x- and y-axes, too. We will assume all forces except gravity (such as air resistance and friction, for example) are negligible. The components of acceleration are then very simple: ay = -g = -9.80 m/s². (Note that this definition assumes that the upwards direction is defined as the positive direction. If you arrange the coordinate system instead such that the downwards direction is positive, then acceleration due to gravity takes a positive value.) Because gravity is vertical, ax = 0. Both accelerations are constant, so the kinematic equations can be used.

### Review of Kinematic Equations (constant a)

\[ x = x_0 + v_0 t \]  \hspace{1cm} (3.28)
\[ \dot{v} = \frac{v_0 + v}{2} \]  \hspace{1cm} (3.29)
\[ v = v_0 + at \]  \hspace{1cm} (3.30)
\[ x = x_0 + v_0 t + \frac{1}{2}at^2 \]  \hspace{1cm} (3.31)
\[ v^2 = v_0^2 + 2a(x - x_0) \]  \hspace{1cm} (3.32)

![Figure 3.36](https://legacy.cnx.org/content/col11951/1.1/figures/3.36/3.36.png)

Figure 3.36 The total displacement s of a soccer ball at a point along its path. The vector s has components x and y along the horizontal and vertical axes. Its magnitude is s, and it makes an angle \( \theta \) with the horizontal.

Given these assumptions, the following steps are then used to analyze projectile motion:

**Step 1.** Resolve or break the motion into horizontal and vertical components along the x- and y-axes. These axes are...
perpendicular, so \( A_x = A \cos \theta \) and \( A_y = A \sin \theta \) are used. The magnitude of the components of displacement \( s \) along these axes are \( x \) and \( y \). The magnitudes of the components of the velocity \( \mathbf{v} \) are \( v_x = v \cos \theta \) and \( v_y = v \sin \theta \), where \( v \) is the magnitude of the velocity and \( \theta \) is its direction, as shown in **Figure 3.37**. Initial values are denoted with a subscript 0, as usual.

**Step 2.** Treat the motion as two independent one-dimensional motions, one horizontal and the other vertical. The kinematic equations for horizontal and vertical motion take the following forms:

**Horizontal Motion** ($a_x = 0$)

\[
\begin{align*}
x &= x_0 + v_x t \\
v_x &= v_{0x} = v_x = \text{velocity is a constant.} \\
(\text{assuming positive is up})
\end{align*}
\]

**Vertical Motion**

\[
\begin{align*}
y &= y_0 + \frac{1}{2}(v_{0y} + v_y) t \\
v_y &= v_{0y} - gt \\
y &= y_0 + v_{0y} t - \frac{1}{2}gt^2 \\
v_y^2 &= v_{0y}^2 - 2g(y - y_0).
\end{align*}
\]

**Step 3.** Solve for the unknowns in the two separate motions—one horizontal and one vertical. Note that the only common variable between the motions is time \( t \). The problem solving procedures here are the same as for one-dimensional kinematics and are illustrated in the solved examples below.

**Step 4.** Recombine the two motions to find the total displacement \( s \) and velocity \( \mathbf{v} \). Because the \( x \)- and \( y \)-motions are perpendicular, we determine these vectors by using the techniques outlined in the Vector Addition and Subtraction: Analytical Methods and employing \( A = \sqrt{A_x^2 + A_y^2} \) and \( \theta = \tan^{-1}(A_y / A_x) \) in the following form, where \( \theta \) is the direction of the displacement \( s \) and \( \theta_v \) is the direction of the velocity \( v \):

**Total displacement and velocity**

\[
\begin{align*}
s &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1}(y / x) \\
v &= \sqrt{v_x^2 + v_y^2} \\
\theta_v &= \tan^{-1}(v_y / v_x).
\end{align*}
\]
Figure 3.37 (a) We analyze two-dimensional projectile motion by breaking it into two independent one-dimensional motions along the vertical and horizontal axes. (b) The horizontal motion is simple, because $a_x = 0$ and $v_x$ is thus constant. (c) The velocity in the vertical direction begins to decrease as the object rises; at its highest point, the vertical velocity is zero. As the object falls towards the Earth again, the vertical velocity increases again in magnitude but points in the opposite direction to the initial vertical velocity. (d) The $x$- and $y$-motions are recombined to give the total velocity at any given point on the trajectory.

Example 3.4 A Fireworks Projectile Explodes High and Away

During a fireworks display, a shell is shot into the air with an initial speed of 70.0 m/s at an angle of $75.0^\circ$ above the horizontal, as illustrated in Figure 3.38. The fuse is timed to ignite the shell just as it reaches its highest point above the ground. (a) Calculate the height at which the shell explodes. (b) How much time passed between the launch of the shell and the explosion? (c) What is the horizontal displacement of the shell when it explodes?

**Strategy**

Because air resistance is negligible for the unexploded shell, the analysis method outlined above can be used. The motion can be broken into horizontal and vertical motions in which $a_x = 0$ and $a_y = -g$. We can then define $x_0$ and $y_0$ to be zero and solve for the desired quantities.

**Solution for (a)**

By “height” we mean the altitude or vertical position $y$ above the starting point. The highest point in any trajectory, called the...
apex, is reached when \( v_y = 0 \). Since we know the initial and final velocities as well as the initial position, we use the following equation to find \( y \):

\[
v_y^2 = v_{0y}^2 - 2g(y - y_0).
\]  

(3.45)

![Figure 3.38 The trajectory of a fireworks shell. The fuse is set to explode the shell at the highest point in its trajectory, which is found to be at a height of 233 m and 125 m away horizontally.](image)

Because \( y_0 \) and \( v_y \) are both zero, the equation simplifies to

\[
0 = v_{0y}^2 - 2gy.
\]  

(3.46)

Solving for \( y \) gives

\[
y = \frac{v_{0y}^2}{2g}.
\]  

(3.47)

Now we must find \( v_{0y} \), the component of the initial velocity in the \( y \)-direction. It is given by \( v_{0y} = v_0 \sin \theta \), where \( v_0 \) is the initial velocity of 70.0 m/s, and \( \theta_0 = 75.0^\circ \) is the initial angle. Thus,

\[
v_{0y} = v_0 \sin \theta_0 = (70.0 \text{ m/s}) (\sin 75^\circ) = 67.6 \text{ m/s}.
\]  

(3.48)

and \( y \) is

\[
y = \frac{(67.6 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)},
\]  

(3.49)

so that

\[
y = 233 \text{ m}.
\]  

(3.50)

**Discussion for (a)**

Note that because up is positive, the initial velocity is positive, as is the maximum height, but the acceleration due to gravity is negative. Note also that the maximum height depends only on the vertical component of the initial velocity, so that any projectile with a 67.6 m/s initial vertical component of velocity will reach a maximum height of 233 m (neglecting air resistance). The numbers in this example are reasonable for large fireworks displays, the shells of which do reach such heights before exploding. In practice, air resistance is not completely negligible, and so the initial velocity would have to be somewhat larger than that given to reach the same height.

**Solution for (b)**

As in many physics problems, there is more than one way to solve for the time to the highest point. In this case, the easiest method is to use \( y = y_0 + \frac{1}{2}(v_{0y} + v_y)t \). Because \( y_0 \) is zero, this equation reduces to simply

\[
y = \frac{1}{2}(v_{0y} + v_y)t.
\]  

(3.51)
Note that the final vertical velocity, $v_y$, at the highest point is zero. Thus,

$$ t = \frac{2v_y}{(v_{0y} + v_y)} = \frac{2(233 \text{ m})}{(67.6 \text{ m/s})} = 6.90 \text{ s}. $$

(3.52)

Discussion for (b)

This time is also reasonable for large fireworks. When you are able to see the launch of fireworks, you will notice several seconds pass before the shell explodes. (Another way of finding the time is by using $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$, and solving the quadratic equation for $t$.)

Solution for (c)

Because air resistance is negligible, $a_x = 0$ and the horizontal velocity is constant, as discussed above. The horizontal displacement is horizontal velocity multiplied by time as given by $x = x_0 + v_xt$, where $x_0$ is equal to zero:

$$ x = v_xt, $$

(3.53)

where $v_x$ is the $x$-component of the velocity, which is given by $v_x = v_0 \cos \theta$. Now,

$$ v_x = v_0 \cos \theta_0 = (70.0 \text{ m/s})(\cos 75.0^\circ) = 18.1 \text{ m/s}. $$

(3.54)

The time $t$ for both motions is the same, and so $x$ is

$$ x = (18.1 \text{ m/s})(6.90 \text{ s}) = 125 \text{ m}. $$

(3.55)

Discussion for (c)

The horizontal motion is a constant velocity in the absence of air resistance. The horizontal displacement found here could be useful in keeping the fireworks fragments from falling on spectators. Once the shell explodes, air resistance has a major effect, and many fragments will land directly below.

In solving part (a) of the preceding example, the expression we found for $y$ is valid for any projectile motion where air resistance is negligible. Call the maximum height $y = h$; then,

$$ h = \frac{v_{0y}^2}{2g}. $$

(3.56)

This equation defines the maximum height of a projectile and depends only on the vertical component of the initial velocity.

Defining a Coordinate System

It is important to set up a coordinate system when analyzing projectile motion. One part of defining the coordinate system is to define an origin for the $x$ and $y$ positions. Often, it is convenient to choose the initial position of the object as the origin such that $x_0 = 0$ and $y_0 = 0$. It is also important to define the positive and negative directions in the $x$ and $y$ directions. Typically, we define the positive vertical direction as upwards, and the positive horizontal direction is usually the direction of the object’s motion. When this is the case, the vertical acceleration, $g$, takes a negative value (since it is directed downwards towards the Earth). However, it is occasionally useful to define the coordinates differently. For example, if you are analyzing the motion of a ball thrown downwards from the top of a cliff, it may make sense to define the positive direction downwards since the motion of the ball is solely in the downwards direction. If this is the case, $g$ takes a positive value.

Example 3.5 Calculating Projectile Motion: Hot Rock Projectile

Kilauea in Hawaii is the world’s most continuously active volcano. Very active volcanoes characteristically eject red-hot rocks and lava rather than smoke and ash. Suppose a large rock is ejected from the volcano with a speed of 25.0 m/s and at an angle $35.0^\circ$ above the horizontal, as shown in Figure 3.39. The rock strikes the side of the volcano at an altitude 20.0 m lower than its starting point. (a) Calculate the time it takes the rock to follow this path. (b) What are the magnitude and direction of the rock’s velocity at impact?
Strategy

Again, resolving this two-dimensional motion into two independent one-dimensional motions will allow us to solve for the desired quantities. The time a projectile is in the air is governed by its vertical motion alone. We will solve for $t$ first. While the rock is rising and falling vertically, the horizontal motion continues at a constant velocity. This example asks for the final velocity. Thus, the vertical and horizontal results will be recombined to obtain $v$ and $\theta_v$ at the final time $t$ determined in the first part of the example.

Solution for (a)

While the rock is in the air, it rises and then falls to a final position 20.0 m lower than its starting altitude. We can find the time for this by using

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2.$$  \hfill (3.57)

If we take the initial position $y_0$ to be zero, then the final position is $y = -20.0 \text{ m}$. Now the initial vertical velocity is the vertical component of the initial velocity, found from $v_{0y} = v_0 \sin \theta_0 = (25.0 \text{ m/s}) (\sin 35.0^\circ) = 14.3 \text{ m/s}$. Substituting known values yields

$$-20.0 \text{ m} = (14.3 \text{ m/s})t - \left(\frac{4.90 \text{ m/s}^2}{2}\right)t^2.$$  \hfill (3.58)

Rearranging terms gives a quadratic equation in $t$:

$$\left(\frac{4.90 \text{ m/s}^2}{2}\right)t^2 - (14.3 \text{ m/s})t - (20.0 \text{ m}) = 0.$$  \hfill (3.59)

This expression is a quadratic equation of the form $at^2 + bt + c = 0$, where the constants are $a = 4.90$, $b = -14.3$, and $c = -20.0$. Its solutions are given by the quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$  \hfill (3.60)

This equation yields two solutions: $t = 3.96$ and $t = -1.03$. (It is left as an exercise for the reader to verify these solutions.) The time is $t = 3.96 \text{ s}$ or $-1.03 \text{ s}$. The negative value of time implies an event before the start of motion, and so we discard it. Thus,

$$t = 3.96 \text{ s}.$$  \hfill (3.61)

Discussion for (a)

The time for projectile motion is completely determined by the vertical motion. So any projectile that has an initial vertical velocity of 14.3 m/s and lands 20.0 m below its starting altitude will spend 3.96 s in the air.

Solution for (b)

From the information now in hand, we can find the final horizontal and vertical velocities $v_x$ and $v_y$ and combine them to find the total velocity $v$ and the angle $\theta_v$ it makes with the horizontal. Of course, $v_x$ is constant so we can solve for it at any horizontal location. In this case, we chose the starting point since we know both the initial velocity and initial angle. Therefore:

$$v_x = v_0 \cos \theta_0 = (25.0 \text{ m/s})(\cos 35^\circ) = 20.5 \text{ m/s}.$$  \hfill (3.62)

The final vertical velocity is given by the following equation:

$$v_y = v_{0y} - gt,$$  \hfill (3.63)
where $v_{0y}$ was found in part (a) to be 14.3 m/s. Thus,

$$v_y = 14.3 \text{ m/s} - (9.80 \text{ m/s}^2)(3.96 \text{ s})$$  \hspace{1cm} (3.64)

so that

$$v_y = -24.5 \text{ m/s.}$$  \hspace{1cm} (3.65)

To find the magnitude of the final velocity $v$ we combine its perpendicular components, using the following equation:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(20.5 \text{ m/s})^2 + (-24.5 \text{ m/s})^2},$$  \hspace{1cm} (3.66)

which gives

$$v = 31.9 \text{ m/s.}$$  \hspace{1cm} (3.67)

The direction $\theta_v$ is found from the equation:

$$\theta_v = \tan^{-1}(v_y/v_x)$$  \hspace{1cm} (3.68)

so that

$$\theta_v = \tan^{-1}(-24.5/20.5) = \tan^{-1}(-1.19).$$  \hspace{1cm} (3.69)

Thus,

$$\theta_v = -50.1^\circ.$$  \hspace{1cm} (3.70)

**Discussion for (b)**

The negative angle means that the velocity is $50.1^\circ$ below the horizontal. This result is consistent with the fact that the final vertical velocity is negative and hence downward—as you would expect because the final altitude is 20.0 m lower than the initial altitude. (See Figure 3.39.)

One of the most important things illustrated by projectile motion is that vertical and horizontal motions are independent of each other. Galileo was the first person to fully comprehend this characteristic. He used it to predict the range of a projectile. On level ground, we define range to be the horizontal distance $R$ traveled by a projectile. Galileo and many others were interested in the range of projectiles primarily for military purposes—such as aiming cannons. However, investigating the range of projectiles can shed light on other interesting phenomena, such as the orbits of satellites around the Earth. Let us consider projectile range further.

![Figure 3.40](https://legacy.cnx.org/content/col11951/1.1)

**Figure 3.40** Trajectories of projectiles on level ground. (a) The greater the initial speed $v_0$, the greater the range for a given initial angle. (b) The effect of initial angle $\theta_0$ on the range of a projectile with a given initial speed. Note that the range is the same for $15^\circ$ and $75^\circ$, although the maximum heights of those paths are different.
How does the initial velocity of a projectile affect its range? Obviously, the greater the initial speed \( v_0 \), the greater the range, as shown in Figure 3.40(a). The initial angle \( \theta_0 \) also has a dramatic effect on the range, as illustrated in Figure 3.40(b). For a fixed initial speed, such as might be produced by a cannon, the maximum range is obtained with \( \theta_0 = 45^\circ \). This is true only for conditions neglecting air resistance. If air resistance is considered, the maximum angle is approximately \( 38^\circ \). Interestingly, for every initial angle except \( 45^\circ \), there are two angles that give the same range—the sum of those angles is \( 90^\circ \). The range also depends on the value of the acceleration of gravity \( g \). The lunar astronaut Alan Shepherd was able to drive a golf ball a great distance on the Moon because gravity is weaker there. The range \( R \) of a projectile on level ground for which air resistance is negligible is given by

\[
R = \frac{v_0^2 \sin 2\theta_0}{g},
\]

where \( v_0 \) is the initial speed and \( \theta_0 \) is the initial angle relative to the horizontal. The proof of this equation is left as an end-of-chapter problem (hints are given), but it does fit the major features of projectile range as described.

When we speak of the range of a projectile on level ground, we assume that \( R \) is very small compared with the circumference of the Earth. If, however, the range is large, the Earth curves away below the projectile and acceleration of gravity changes direction along the path. The range is larger than predicted by the range equation given above because the projectile has farther to fall than it would on level ground. (See Figure 3.41.) If the initial speed is great enough, the projectile goes into orbit. This possibility was recognized centuries before it could be accomplished. When an object is in orbit, the Earth curves away from underneath the object at the same rate as it falls. The object thus falls continuously but never hits the surface. These and other aspects of orbital motion, such as the rotation of the Earth, will be covered analytically and in greater depth later in this text.

Once again we see that thinking about one topic, such as the range of a projectile, can lead us to others, such as the Earth orbits. In Addition of Velocities, we will examine the addition of velocities, which is another important aspect of two-dimensional kinematics and will also yield insights beyond the immediate topic.
In each of these situations, an object has a velocity relative to a medium (such as a river) and that medium has a velocity relative to an observer on solid ground. The velocity of the object relative to the observer is the sum of these velocity vectors, as indicated in Figure 3.43 and Figure 3.44. These situations are only two of many in which it is useful to add velocities. In this module, we first re-examine how to add velocities and then consider certain aspects of what relative velocity means.

How do we add velocities? Velocity is a vector (it has both magnitude and direction); the rules of vector addition discussed in Vector Addition and Subtraction: Graphical Methods and Vector Addition and Subtraction: Analytical Methods apply to the addition of velocities, just as they do for any other vectors. In one-dimensional motion, the addition of velocities is simple—they add like ordinary numbers. For example, if a field hockey player is moving at 5 m/s straight toward the goal and drives the ball in the same direction with a velocity of 30 m/s relative to her body, then the velocity of the ball is 35 m/s relative to the stationary, profusely sweating goalkeeper standing in front of the goal.

In two-dimensional motion, either graphical or analytical techniques can be used to add velocities. We will concentrate on analytical techniques. The following equations give the relationships between the magnitude and direction of velocity (v and θ) and its components (v_x and v_y) along the x- and y-axes of an appropriately chosen coordinate system:

\[ v_x = v \cos \theta \]  \hspace{1cm} (3.72)
\[ v_y = v \sin \theta \]  \hspace{1cm} (3.73)
\[ v = \sqrt{v_x^2 + v_y^2} \]  
\[ \theta = \tan^{-1}(v_y/v_x). \]

Figure 3.45 The velocity, \( v \), of an object traveling at an angle \( \theta \) to the horizontal axis is the sum of component vectors \( v_x \) and \( v_y \).

These equations are valid for any vectors and are adapted specifically for velocity. The first two equations are used to find the components of a velocity when its magnitude and direction are known. The last two are used to find the magnitude and direction of velocity when its components are known.

Take-Home Experiment: Relative Velocity of a Boat

Fill a bathtub half-full of water. Take a toy boat or some other object that floats in water. Unplug the drain so water starts to drain. Try pushing the boat from one side of the tub to the other and perpendicular to the flow of water. Which way do you need to push the boat so that it ends up immediately opposite? Compare the directions of the flow of water, heading of the boat, and actual velocity of the boat.

Example 3.6 Adding Velocities: A Boat on a River

Figure 3.46 A boat attempts to travel straight across a river at a speed 0.75 m/s. The current in the river, however, flows at a speed of 1.20 m/s to the right.

Refer to Figure 3.46, which shows a boat trying to go straight across the river. Let us calculate the magnitude and direction of the boat’s velocity relative to an observer on the shore, \( v_{\text{tot}} \). The velocity of the boat, \( v_{\text{boat}} \), is 0.75 m/s in the \( y \)-direction relative to the river and the velocity of the river, \( v_{\text{river}} \), is 1.20 m/s to the right.

Strategy

We start by choosing a coordinate system with its \( x \)-axis parallel to the velocity of the river, as shown in Figure 3.46. Because the boat is directed straight toward the other shore, its velocity relative to the water is parallel to the \( y \)-axis and...
perpendicular to the velocity of the river. Thus, we can add the two velocities by using the equations $v_{\text{tot}} = \sqrt{v_x^2 + v_y^2}$ and $\theta = \tan^{-1}(v_y/v_x)$ directly.

**Solution**

The magnitude of the total velocity is

$$v_{\text{tot}} = \sqrt{v_x^2 + v_y^2}, \quad (3.76)$$

where

$$v_x = v_{\text{river}} = 1.20 \text{ m/s} \quad (3.77)$$

and

$$v_y = v_{\text{boat}} = 0.750 \text{ m/s}. \quad (3.78)$$

Thus,

$$v_{\text{tot}} = \sqrt{(1.20 \text{ m/s})^2 + (0.750 \text{ m/s})^2} \quad (3.79)$$

yielding

$$v_{\text{tot}} = 1.42 \text{ m/s}. \quad (3.80)$$

The direction of the total velocity $\theta$ is given by:

$$\theta = \tan^{-1}(v_y/v_x) = \tan^{-1}(0.750/1.20). \quad (3.81)$$

This equation gives

$$\theta = 32.0^\circ. \quad (3.82)$$

**Discussion**

Both the magnitude $v$ and the direction $\theta$ of the total velocity are consistent with Figure 3.46. Note that because the velocity of the river is large compared with the velocity of the boat, it is swept rapidly downstream. This result is evidenced by the small angle (only $32.0^\circ$) the total velocity has relative to the riverbank.

**Example 3.7 Calculating Velocity: Wind Velocity Causes an Airplane to Drift**

Calculate the wind velocity for the situation shown in Figure 3.47. The plane is known to be moving at 45.0 m/s due north relative to the air mass, while its velocity relative to the ground (its total velocity) is 38.0 m/s in a direction $20.0^\circ$ west of north.
An airplane is known to be heading north at 45.0 m/s, though its velocity relative to the ground is 38.0 m/s at an angle west of north. What is the speed and direction of the wind?

Strategy

In this problem, somewhat different from the previous example, we know the total velocity \( v_{\text{tot}} \) and that it is the sum of two other velocities, \( v_w \) (the wind) and \( v_p \) (the plane relative to the air mass). The quantity \( v_p \) is known, and we are asked to find \( v_w \). None of the velocities are perpendicular, but it is possible to find their components along a common set of perpendicular axes. If we can find the components of \( v_w \), then we can combine them to solve for its magnitude and direction. As shown in Figure 3.47, we choose a coordinate system with its \( x \)-axis due east and its \( y \)-axis due north (parallel to \( v_p \)). (You may wish to look back at the discussion of the addition of vectors using perpendicular components in Vector Addition and Subtraction: Analytical Methods.)

Solution

Because \( v_{\text{tot}} \) is the vector sum of the \( v_w \) and \( v_p \), its \( x \)- and \( y \)-components are the sums of the \( x \)- and \( y \)-components of the wind and plane velocities. Note that the plane only has vertical component of velocity so \( v_{px} = 0 \) and \( v_{py} = v_p \). That is,

\[
v_{\text{tot}} = v_{\text{tot}}x = v_{wx} \tag{3.83}
\]

and

\[
v_{\text{tot}} = v_{\text{tot}}y = v_{wy} + v_p \tag{3.84}
\]

We can use the first of these two equations to find \( v_{wx} \):

\[
v_{wx} = v_{\text{tot}} = v_{\text{tot}} \cos 110^\circ. \tag{3.85}
\]

Because \( v_{\text{tot}} = 38.0 \text{ m/s} \) and \( \cos 110^\circ = -0.342 \) we have

\[
v_{wx} = (38.0 \text{ m/s})(-0.342) = -13 \text{ m/s}. \tag{3.86}
\]

The minus sign indicates motion west which is consistent with the diagram.

Now, to find \( v_{wy} \) we note that

\[
v_{\text{tot}} = v_{\text{tot}}y = v_{wy} + v_p \tag{3.87}
\]

Here \( v_{\text{tot}} = v_{\text{tot}} \sin 110^\circ \); thus,

\[
v_{wy} = (38.0 \text{ m/s})(0.940) - 45.0 \text{ m/s} = -9.29 \text{ m/s}. \tag{3.88}
\]
This minus sign indicates motion south which is consistent with the diagram.

Now that the perpendicular components of the wind velocity $v_{wx}$ and $v_{wy}$ are known, we can find the magnitude and direction of $v_w$. First, the magnitude is

$$v_w = \sqrt{v_{wx}^2 + v_{wy}^2}$$

so that

$$v_w = 16.0 \text{ m/s}$$

The direction is:

$$\theta = \tan^{-1}(v_{wy} / v_{wx}) = \tan^{-1}(-9.29 / -13.0)$$

giving

$$\theta = 35.6^\circ$$

**Discussion**

The wind’s speed and direction are consistent with the significant effect the wind has on the total velocity of the plane, as seen in Figure 3.47. Because the plane is fighting a strong combination of crosswind and head-wind, it ends up with a total velocity significantly less than its velocity relative to the air mass as well as heading in a different direction.

Note that in both of the last two examples, we were able to make the mathematics easier by choosing a coordinate system with one axis parallel to one of the velocities. We will repeatedly find that choosing an appropriate coordinate system makes problem solving easier. For example, in projectile motion we always use a coordinate system with one axis parallel to gravity.

**Relative Velocities and Classical Relativity**

When adding velocities, we have been careful to specify that the velocity is relative to some reference frame. These velocities are called relative velocities. For example, the velocity of an airplane relative to an air mass is different from its velocity relative to the ground. Both are quite different from the velocity of an airplane relative to its passengers (which should be close to zero). Relative velocities are one aspect of relativity, which is defined to be the study of how different observers moving relative to each other measure the same phenomenon.

Nearly everyone has heard of relativity and immediately associates it with Albert Einstein (1879–1955), the greatest physicist of the 20th century. Einstein revolutionized our view of nature with his modern theory of relativity, which we shall study in later chapters. The relative velocities in this section are actually aspects of classical relativity, first discussed correctly by Galileo and Isaac Newton. Classical relativity is limited to situations where speeds are less than about 1% of the speed of light—that is, less than 3,000 km/s. Most things we encounter in daily life move slower than this speed.

Let us consider an example of what two different observers see in a situation analyzed long ago by Galileo. Suppose a sailor at the top of a mast on a moving ship drops his binoculars. Where will it hit the deck? Will it hit at the base of the mast, or will it hit behind the mast because the ship is moving forward? The answer is that if air resistance is negligible, the binoculars will hit at the base of the mast at a point directly below its point of release. Now let us consider what two different observers see when the binoculars drop. One observer is on the ship and the other on shore. The binoculars have no horizontal velocity relative to the observer on the ship, and so he sees them fall straight down the mast. (See Figure 3.48.) To the observer on shore, the binoculars and the ship have the same horizontal velocity, so both move the same distance forward while the binoculars are falling. This observer sees the curved path shown in Figure 3.48. Although the paths look different to the different observers, each sees the same result—the binoculars hit at the base of the mast and not behind it. To get the correct description, it is crucial to correctly specify the velocities relative to the observer.
Figure 3.48 Classical relativity. The same motion as viewed by two different observers. An observer on the moving ship sees the binoculars dropped from the top of its mast fall straight down. An observer on shore sees the binoculars take the curved path, moving forward with the ship. Both observers see the binoculars strike the deck at the base of the mast. The initial horizontal velocity is different relative to the two observers. (The ship is shown moving rather fast to emphasize the effect.)

Example 3.8 Calculating Relative Velocity: An Airline Passenger Drops a Coin

An airline passenger drops a coin while the plane is moving at 260 m/s. What is the velocity of the coin when it strikes the floor 1.50 m below its point of release: (a) Measured relative to the plane? (b) Measured relative to the Earth?

Strategy

Both problems can be solved with the techniques for falling objects and projectiles. In part (a), the initial velocity of the coin is zero relative to the plane, so the motion is that of a falling object (one-dimensional). In part (b), the initial velocity is 260 m/s horizontal relative to the Earth and gravity is vertical, so this motion is a projectile motion. In both parts, it is best to use a coordinate system with vertical and horizontal axes.

Solution for (a)

Using the given information, we note that the initial velocity and position are zero, and the final position is 1.50 m. The final velocity can be found using the equation:
\[ v_y^2 = v_{0,y}^2 - 2g(y - y_0). \]  

(3.93)

Substituting known values into the equation, we get

\[ v_y^2 = 0^2 - 2(9.80 \text{ m/s}^2)(-1.50 \text{ m} - 0 \text{ m}) = 29.4 \text{ m}^2/\text{s}^2 \]

yielding

\[ v_y = -5.42 \text{ m/s}. \]

(3.95)

We know that the square root of 29.4 has two roots: 5.42 and -5.42. We choose the negative root because we know that the velocity is directed downwards, and we have defined the positive direction to be upwards. There is no initial horizontal velocity relative to the plane and no horizontal acceleration, and so the motion is straight down relative to the plane.

**Solution for (b)**

Because the initial vertical velocity is zero relative to the ground and vertical motion is independent of horizontal motion, the final vertical velocity for the coin relative to the ground is \( v_y = -5.42 \text{ m/s} \), the same as found in part (a). In contrast to part (a), there now is a horizontal component of the velocity. However, since there is no horizontal acceleration, the initial and final horizontal velocities are the same and \( v_x = 260 \text{ m/s} \). The \( x \)- and \( y \)-components of velocity can be combined to find the magnitude of the final velocity:

\[ v = \sqrt{v_x^2 + v_y^2}. \]

(3.96)

Thus,

\[ v = \sqrt{(260 \text{ m/s})^2 + (-5.42 \text{ m/s})^2} \]

yielding

\[ v = 260.06 \text{ m/s}. \]

(3.98)

The direction is given by:

\[ \theta = \tan^{-1}(v_y/v_x) = \tan^{-1}(-5.42/260) \]

(3.99)

so that

\[ \theta = \tan^{-1}(-0.0208) = -1.19^\circ. \]

(3.100)

**Discussion**

In part (a), the final velocity relative to the plane is the same as it would be if the coin were dropped from rest on the Earth and fell 1.50 m. This result fits our experience; objects in a plane fall the same way when the plane is flying horizontally as when it is at rest on the ground. This result is also true in moving cars. In part (b), an observer on the ground sees a much different motion for the coin. The plane is moving so fast horizontally to begin with that its final velocity is barely greater than the initial velocity. Once again, we see that in two dimensions, vectors do not add like ordinary numbers—the final velocity \( v \) in part (b) is not \( (260 - 5.42) \text{ m/s} \); rather, it is \( 260.06 \text{ m/s} \). The velocity's magnitude had to be calculated to five digits to see any difference from that of the airplane. The motions as seen by different observers (one in the plane and one on the ground) in this example are analogous to those discussed for the binoculars dropped from the mast of a moving ship, except that the velocity of the plane is much larger, so that the two observers see very different paths. (See Figure 3.49.) In addition, both observers see the coin fall 1.50 m vertically, but the one on the ground also sees it move forward 144 m (this calculation is left for the reader). Thus, one observer sees a vertical path, the other a nearly horizontal path.

**Making Connections: Relativity and Einstein**

Because Einstein was able to clearly define how measurements are made (some involve light) and because the speed of light is the same for all observers, the outcomes are spectacularly unexpected. Time varies with observer, energy is stored as increased mass, and more surprises await.

---

**Motion in 2D**

Try the "Motion in 2D" simulation to Learn about position, velocity, and acceleration vectors. Move the ball with the mouse or let the simulation move the ball in four types of motion (2 types of linear, simple harmonic, circle).

(This media type is not supported in this reader. Click to open media in browser.) [http://legacy.cnx.org/content/m42045/1.30/#fs-id1167062339457]
Section Summary

3.1 Kinematics in Two Dimensions: An Introduction
- The shortest path between any two points is a straight line. In two dimensions, this path can be represented by a vector with horizontal and vertical components.
- The horizontal and vertical components of a vector are independent of one another. Motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

3.2 Vector Addition and Subtraction: Graphical Methods
- The graphical method of adding vectors $\mathbf{A}$ and $\mathbf{B}$ involves drawing vectors on a graph and adding them using the head-to-tail method. The resultant vector $\mathbf{R}$ is defined such that $\mathbf{A} + \mathbf{B} = \mathbf{R}$. The magnitude and direction of $\mathbf{R}$ are then determined with a ruler and protractor, respectively.
- The graphical method of subtracting vector $\mathbf{B}$ from $\mathbf{A}$ involves adding the opposite of vector $\mathbf{B}$, which is defined as
In this case, \( \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = \mathbf{R} \). Then, the head-to-tail method of addition is followed in the usual way to obtain the resultant vector \( \mathbf{R} \).

- Addition of vectors is **commutative** such that \( \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \).
- The **head-to-tail method** of adding vectors involves drawing the first vector on a graph and then placing the tail of each subsequent vector at the head of the previous vector. The resultant vector is then drawn from the tail of the first vector to the head of the final vector.
- If a vector \( \mathbf{A} \) is multiplied by a scalar quantity \( c \), the magnitude of the product is given by \( c\mathbf{A} \). If \( c \) is positive, the direction of the product points in the same direction as \( \mathbf{A} \); if \( c \) is negative, the direction of the product points in the opposite direction as \( \mathbf{A} \).

### 3.3 Vector Addition and Subtraction: Analytical Methods

- The analytical method of vector addition and subtraction involves using the Pythagorean theorem and trigonometric identities to determine the magnitude and direction of a resultant vector.
- The steps to add vectors \( \mathbf{A} \) and \( \mathbf{B} \) using the analytical method are as follows:
  1. Determine the coordinate system for the vectors. Then, determine the horizontal and vertical components of each vector using the equations
     \[
     A_x = A \cos \theta \\
     B_x = B \cos \theta \\
     A_y = A \sin \theta \\
     B_y = B \sin \theta.
     \]
  2. Add the horizontal and vertical components of each vector to determine the components \( R_x \) and \( R_y \) of the resultant vector, \( \mathbf{R} \):
     \[
     R_x = A_x + B_x \\
     R_y = A_y + B_y.
     \]
  3. Use the Pythagorean theorem to determine the magnitude, \( R \), of the resultant vector \( \mathbf{R} \):
     \[
     R = \sqrt{R_x^2 + R_y^2}.
     \]
  4. Use a trigonometric identity to determine the direction, \( \theta \), of \( \mathbf{R} \):
     \[
     \theta = \tan^{-1}(R_y/R_x).
     \]

### 3.4 Projectile Motion

- Projectile motion is the motion of an object through the air that is subject only to the acceleration of gravity.
- To solve projectile motion problems, perform the following steps:
  1. Determine a coordinate system. Then, resolve the position and/or velocity of the object in the horizontal and vertical components. The components of position \( s \) are given by the quantities \( x \) and \( y \), and the components of the velocity \( v \) are given by \( v_x = v \cos \theta \) and \( v_y = v \sin \theta \), where \( v \) is the magnitude of the velocity and \( \theta \) is its direction.
  2. Analyze the motion of the projectile in the horizontal direction using the following equations:
     \[
     \text{Horizontal motion}(a_x = 0) \\
     x = x_0 + v_x t \\
     v_x = v_{0x}, \text{ velocity is a constant.}
     \]
  3. Analyze the motion of the projectile in the vertical direction using the following equations:
     \[
     \text{Vertical Motion} \quad (\text{assuming positive is up}) \\
     a_y = -g = -9.80 \text{m/s}^2 \\
     y = y_0 + \frac{1}{2}(v_{0y} + v_y)t \\
     v_y = v_{0y} - gt
     \]
\[ y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \]
\[ v_y^2 = v_{0y}^2 - 2g(y - y_0). \]

4. Recombine the horizontal and vertical components of location and/or velocity using the following equations:

\[ s = \sqrt{x^2 + y^2} \]
\[ \theta = \tan^{-1}(y/x) \]
\[ v = \sqrt{v_x^2 + v_y^2} \]
\[ \theta_v = \tan^{-1}(v_y/v_x). \]

- The maximum height \( h \) of a projectile launched with initial vertical velocity \( v_{0y} \) is given by

\[ h = \frac{v_{0y}^2}{2g}. \]

- The maximum horizontal distance traveled by a projectile is called the range. The range \( R \) of a projectile on level ground launched at an angle \( \theta_0 \) above the horizontal with initial speed \( v_0 \) is given by

\[ R = \frac{v_0^2 \sin 2\theta_0}{g}. \]

### 3.5 Addition of Velocities

- Velocities in two dimensions are added using the same analytical vector techniques, which are rewritten as

\[ v_x = v \cos \theta \]
\[ v_y = v \sin \theta \]
\[ v = \sqrt{v_x^2 + v_y^2} \]
\[ \theta = \tan^{-1}(v_y/v_x). \]

- Relative velocity is the velocity of an object as observed from a particular reference frame, and it varies dramatically with reference frame.

- Relative velocity is the study of how different observers measure the same phenomenon, particularly when the observers move relative to one another. Classical relativity is limited to situations where speed is less than about 1% of the speed of light (3000 km/s).

### Conceptual Questions

#### 3.2 Vector Addition and Subtraction: Graphical Methods

1. Which of the following is a vector: a person’s height, the altitude on Mt. Everest, the age of the Earth, the boiling point of water, the cost of this book, the Earth’s population, the acceleration of gravity?

2. Give a specific example of a vector, stating its magnitude, units, and direction.

3. What do vectors and scalars have in common? How do they differ?
4. Two campers in a national park hike from their cabin to the same spot on a lake, each taking a different path, as illustrated below. The total distance traveled along Path 1 is 7.5 km, and that along Path 2 is 8.2 km. What is the final displacement of each camper?

![Figure 3.50](image_url)

5. If an airplane pilot is told to fly 123 km in a straight line to get from San Francisco to Sacramento, explain why he could end up anywhere on the circle shown in Figure 3.51. What other information would he need to get to Sacramento?

![Figure 3.51](image_url)

6. Suppose you take two steps \( \mathbf{A} \) and \( \mathbf{B} \) (that is, two nonzero displacements). Under what circumstances can you end up at your starting point? More generally, under what circumstances can two nonzero vectors add to give zero? Is the maximum distance you can end up from the starting point \( \mathbf{A} + \mathbf{B} \) the sum of the lengths of the two steps?

7. Explain why it is not possible to add a scalar to a vector.

8. If you take two steps of different sizes, can you end up at your starting point? More generally, can two vectors with different magnitudes ever add to zero? Can three or more?

### 3.3 Vector Addition and Subtraction: Analytical Methods

9. Suppose you add two vectors \( \mathbf{A} \) and \( \mathbf{B} \). What relative direction between them produces the resultant with the greatest magnitude? What is the maximum magnitude? What relative direction between them produces the resultant with the smallest magnitude? What is the minimum magnitude?

10. Give an example of a nonzero vector that has a component of zero.

11. Explain why a vector cannot have a component greater than its own magnitude.

12. If the vectors \( \mathbf{A} \) and \( \mathbf{B} \) are perpendicular, what is the component of \( \mathbf{A} \) along the direction of \( \mathbf{B} \)? What is the component of \( \mathbf{B} \) along the direction of \( \mathbf{A} \)?
3.4 Projectile Motion

13. Answer the following questions for projectile motion on level ground assuming negligible air resistance (the initial angle being neither 0º nor 90º): (a) Is the velocity ever zero? (b) When is the velocity a minimum? A maximum? (c) Can the velocity ever be the same as the initial velocity at a time other than at $t = 0$? (d) Can the speed ever be the same as the initial speed at a time other than at $t = 0$?

14. Answer the following questions for projectile motion on level ground assuming negligible air resistance (the initial angle being neither 0º nor 90º): (a) Is the acceleration ever zero? (b) Is the acceleration ever in the same direction as a component of velocity? (c) Is the acceleration ever opposite in direction to a component of velocity?

15. For a fixed initial speed, the range of a projectile is determined by the angle at which it is fired. For all but the maximum, there are two angles that give the same range. Considering factors that might affect the ability of an archer to hit a target, such as wind, explain why the smaller angle (closer to the horizontal) is preferable. When would it be necessary for the archer to use the larger angle? Why does the punter in a football game use the higher trajectory?

16. During a lecture demonstration, a professor places two coins on the edge of a table. She then flicks one of the coins horizontally off the table, simultaneously nudging the other over the edge. Describe the subsequent motion of the two coins, in particular discussing whether they hit the floor at the same time.

3.5 Addition of Velocities

17. What frame or frames of reference do you instinctively use when driving a car? When flying in a commercial jet airplane?

18. A basketball player dribbling down the court usually keeps his eyes fixed on the players around him. He is moving fast. Why doesn't he need to keep his eyes on the ball?

19. If someone is riding in the back of a pickup truck and throws a softball straight backward, is it possible for the ball to fall straight down as viewed by a person standing at the side of the road? Under what condition would this occur? How would the motion of the ball appear to the person who threw it?

20. The hat of a jogger running at constant velocity falls off the back of his head. Draw a sketch showing the path of the hat in the jogger’s frame of reference. Draw its path as viewed by a stationary observer.

21. A clod of dirt falls from the bed of a moving truck. It strikes the ground directly below the end of the truck. What is the direction of its velocity relative to the truck just before it hits? Is this the same as the direction of its velocity relative to ground just before it hits? Explain your answers.
Problems & Exercises

3.2 Vector Addition and Subtraction: Graphical Methods

Use graphical methods to solve these problems. You may assume data taken from graphs is accurate to three digits.

1. Find the following for path A in Figure 3.52: (a) the total distance traveled, and (b) the magnitude and direction of the displacement from start to finish.

   ![Figure 3.52](image)
   The various lines represent paths taken by different people walking in a city. All blocks are 120 m on a side.

2. Find the following for path B in Figure 3.52: (a) the total distance traveled, and (b) the magnitude and direction of the displacement from start to finish.

3. Find the north and east components of the displacement for the hikers shown in Figure 3.50.

4. Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements A and B, as in Figure 3.53, then this problem finds their sum \( \mathbf{R} = \mathbf{A} + \mathbf{B} \).)

   ![Figure 3.53](image)
   The two displacements A and B add to give a total displacement \( \mathbf{R} \) having magnitude \( R \) and direction \( \theta \).

5. Suppose you first walk 12.0 m in a direction \( 20^\circ \) west of north and then 20.0 m in a direction \( 40.0^\circ \) south of west. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements A and B, as in Figure 3.54, then this problem finds their sum \( \mathbf{R} = \mathbf{A} + \mathbf{B} \).)

   ![Figure 3.54](image)

6. Repeat the problem above, but reverse the order of the two legs of the walk: show that you get the same final result. That is, you first walk leg B, which is 20.0 m in a direction exactly \( 40^\circ \) south of west, and then leg A, which is 12.0 m in a direction exactly \( 20^\circ \) west of north. (This problem shows that \( \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \).)

7. (a) Repeat the problem two problems prior, but for the second leg you walk 20.0 m in a direction \( 40.0^\circ \) north of east (which is equivalent to subtracting B from A—that is, to finding \( \mathbf{R}' = \mathbf{A} - \mathbf{B} \)). (b) Repeat the problem two problems prior, but now you first walk 20.0 m in a direction \( 40.0^\circ \) south of west and then 12.0 m in a direction \( 20.0^\circ \) east of south (which is equivalent to subtracting A from B—that is, to finding \( \mathbf{R}'' = \mathbf{B} - \mathbf{A} = -\mathbf{R}' \)). Show that this is the case.

8. Show that the order of addition of three vectors does not affect their sum. Show this property by choosing any three vectors A, B, and C, all having different lengths and directions. Find the sum \( \mathbf{A} + \mathbf{B} + \mathbf{C} \) then find their sum when added in a different order and show the result is the same. (There are five other orders in which A, B, and C can be added; choose only one.)

9. Show that the sum of the vectors discussed in Example 3.2 gives the result shown in Figure 3.24.
10. Find the magnitudes of velocities $v_A$ and $v_B$ in Figure 3.55.

11. Find the components of $v_{tot}$ along the x- and y-axes in Figure 3.55.

12. Find the components of $v_{tot}$ along a set of perpendicular axes rotated 30º counterclockwise relative to those in Figure 3.55.

3.3 Vector Addition and Subtraction: Analytical Methods

13. Find the following for path C in Figure 3.56: (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.

14. Find the following for path D in Figure 3.56: (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.

15. Find the north and east components of the displacement from San Francisco to Sacramento shown in Figure 3.57.

16. Solve the following problem using analytical techniques: Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements $A$ and $B$, as in Figure 3.58, then this problem asks you to find their sum $R = A + B$.)

17. Repeat Exercise 3.16 using analytical techniques, but reverse the order of the two legs of the walk and show that you get the same final result. (This problem shows that adding them in reverse order gives the same result—that is, $B + A = A + B$.) Discuss how taking another path to reach the same point might help to overcome an obstacle blocking your other path.
18. **You drive 7.50 km in a straight line in a direction 15° east of north.** (a) Find the distances you would have to drive straight east and then straight north to arrive at the same point. (This determination is equivalent to find the components of the displacement along the east and north directions.) (b) Show that you still arrive at the same point if the east and north legs are reversed in order.

19. **Do Exercise 3.16 again using analytical techniques and change the second leg of the walk to 25.0 m straight south.** (This is equivalent to subtracting B from A—that is, finding \( R' = A - B \)) (b) Repeat again, but now you first walk 25.0 m north and then 18.0 m east. (This is equivalent to subtract A from B—that is, to find \( A = B + C \). Is that consistent with your result?)

20. **A new landowner has a triangular piece of flat land she wishes to fence. Starting at the west corner, she measures the first side to be 80.0 m long and the next to be 105 m. These sides are represented as displacement vectors A from B in Figure 3.59. She then correctly calculates the length and orientation of the third side C. What is her result?**

```
A + B + C = 0
```

21. **You fly 32.0 km in a straight line in still air in the direction 35.0° south of west.** (a) Find the distances you would have to fly straight south and then straight west to arrive at the same point. (This determination is equivalent to finding the components of the displacement along the south and west directions.) (b) Find the distances you would have to fly first in a direction 45.0° south of west and then in a direction 45.0° west of north. These are the components of the displacement along a different set of axes—one rotated 45°.

```
A = 80 m
B = 105 m
```

22. **A farmer wants to fence off his four-sided plot of flat land. He measures the first three sides, shown as A, B, and C in Figure 3.60, and then correctly calculates the length and orientation of the fourth side D. What is his result?**

```
D = 7.5 km
```

23. **In an attempt to escape his island, Gilligan builds a raft and sets to sea. The wind shifts a great deal during the day, and he is blown along the following straight lines: 2.50 km 45.0° north of west; then 4.70 km 60.0° south of east; then 1.30 km 25.0° south of west; then 5.10 km straight east; then 1.70 km 5.00° east of north; then 7.20 km 55.0° south of west; and finally 2.80 km 10.0° north of east. What is his final position relative to the island?**

```
D = 7.5 km
```

24. **Suppose a pilot flies 40.0 km in a direction 60° north of east and then flies 30.0 km in a direction 15° north of east as shown in Figure 3.61. Find her total distance \( R \) from the starting point and the direction \( \theta \) of the straight-line path to the final position. Discuss qualitatively how this flight would be altered by a wind from the north and how the effect of the wind would depend on both wind speed and the speed of the plane relative to the air mass.**

```
A = 40 km
B = 30 km
```

25. **A projectile is launched at ground level with an initial speed of 50.0 m/s at an angle of 30.0° above the horizontal. It strikes a target above the ground 3.00 seconds later. What are the x and y distances from where the projectile was launched to where it lands?**

**3.4 Projectile Motion**

```
3.4 Projectiles
```
26. A ball is kicked with an initial velocity of 16 m/s in the horizontal direction and 12 m/s in the vertical direction. (a) At what speed does the ball hit the ground? (b) For how long does the ball remain in the air? (c) What maximum height is attained by the ball?

27. A ball is thrown horizontally from the top of a 60.0-m building and lands 100.0 m from the base of the building. Ignore air resistance. (a) How long is the ball in the air? (b) What must have been the initial horizontal component of the velocity? (c) What is the vertical component of the velocity just before the ball hits the ground? (d) What is the velocity (including both the horizontal and vertical components) of the ball just before it hits the ground?

28. (a) A daredevil is attempting to jump his motorcycle over a line of buses parked end to end by driving up a 32º ramp at a speed of 40.0 m/s (144 km/h). How many buses can he clear if the top of the takeoff ramp is at the same height as the bus tops and the buses are 20.0 m long? (b) Discuss what your answer implies about the margin of error in this act—that is, consider how much greater the range is than the horizontal distance he must travel to miss the end of the last bus. (Neglect air resistance.)

29. An archer shoots an arrow at a 75.0 m distant target; the bull's-eye of the target is at same height as the release height of the arrow. (a) At what angle must the arrow be released to hit the bull's-eye if its initial speed is 35.0 m/s? In this part of the problem, explicitly show how you follow the steps involved in solving projectile motion problems. (b) There is a large tree halfway between the archer and the target with an overhanging horizontal branch 3.50 m above the release height of the arrow. Will the arrow go over or under the branch?

30. A rugby player passes the ball 7.00 m across the field, where it is caught at the same height as it left his hand. (a) At what angle was the ball thrown if its initial speed was 12.0 m/s, assuming that the smaller of the two possible angles was used? (b) What other angle gives the same range, and why would it not be used? (c) How long did this pass take?

31. Verify the ranges for the projectiles in Figure 3.40(a) for \( \theta = 45^\circ \) and the given initial velocities.

32. Verify the ranges shown for the projectiles in Figure 3.40(b) for an initial velocity of 50 m/s at the given initial angles.

33. The cannon on a battleship can fire a shell a maximum distance of 32.0 km. (a) Calculate the initial velocity of the shell. (b) What maximum height does it reach? (At its highest, the shell is above 60% of the atmosphere—but air resistance is not really negligible as assumed to make this problem easier.) (c) The ocean is not flat, because the Earth is curved. Assume that the radius of the Earth is \( 6.37 \times 10^3 \) km. How many meters lower will its surface be 32.0 km from the ship along a horizontal line parallel to the surface at the ship? Does your answer imply that error introduced by the assumption of a flat Earth in projectile motion is significant here?

34. An arrow is shot from a height of 1.5 m toward a cliff of height \( H \). It is shot with a velocity of 30 m/s at an angle of \( 60^\circ \) above the horizontal. It lands on the top edge of the cliff 4.0 s later. (a) What is the height of the cliff? (b) What is the maximum height reached by the arrow along its trajectory? (c) What is the arrow's impact speed just before hitting the cliff?

35. In the standing broad jump, one squats and then pushes off with the legs to see how far one can jump. Suppose the extension of the legs from the crouch position is 0.600 m and the acceleration achieved from this position is 1.25 times the acceleration due to gravity, \( g \). How far can they jump? State your assumptions. (Increased range can be achieved by swinging the arms in the direction of the jump.)

36. The world long jump record is 8.95 m (Mike Powell, USA, 1991). Treated as a projectile, what is the maximum range obtainable by a person if he has a take-off speed of 9.5 m/s? State your assumptions.

37. Serving at a speed of 170 km/h, a tennis player hits the ball at a height of 2.5 m and an angle \( \theta \) below the horizontal. The base line is 11.9 m from the net, which is 0.91 m high. What is the angle \( \theta \) such that the ball just crosses the net? Will the ball land in the service box, whose service line is 6.40 m from the net?

38. A football quarterback is moving straight backward at a speed of 2.00 m/s when he throws a pass to a player 18.0 m straight downfield. (a) If the ball is thrown at an angle of \( 25^\circ \) relative to the ground and is caught at the same height as it is released, what is its initial speed relative to the ground? (b) How long does it take to get to the receiver? (c) What is its maximum height above its point of release?

39. Gun sights are adjusted to aim high to compensate for the effect of gravity, effectively making the gun accurate only for a specific range. (a) If a gun is sighted to hit targets that are at the same height as the gun and 100.0 m away, how low will the bullet hit if aimed directly at a target 150.0 m away? The muzzle velocity of the bullet is 275 m/s. (b) Discuss qualitatively how a larger muzzle velocity would affect this problem and what would be the effect of air resistance.

40. An eagle is flying horizontally at a speed of 3.00 m/s when the fish in her talons wiggles loose and falls into the lake 5.00 m below. Calculate the velocity of the fish relative to the water when it hits the water.

41. An owl is carrying a mouse to the chicks in its nest. Its position at that time is 4.00 m west and 12.0 m above the center of the 30.0 cm diameter nest. The owl is flying east at 3.50 m/s at an angle \( 30.0^\circ \) below the horizontal when it accidentally drops the mouse. Is the owl lucky enough to have the mouse hit the nest? To answer this question, calculate the horizontal position of the mouse when it has fallen 12.0 m.

42. Suppose a soccer player kicks the ball from a distance 30 m toward the goal. Find the initial speed of the ball if it just passes over the goal, 2.4 m above the ground, given the initial direction to be \( 40^\circ \) above the horizontal.

43. Can a goalkeeper at her/ his goal kick a soccer ball into the opponent's goal without the ball touching the ground? The distance will be about 95 m. A goalkeeper can give the ball a speed of 30 m/s.

44. The free throw line in basketball is 4.57 m (15 ft) from the basket, which is 3.05 m (10 ft) above the floor. A player standing on the free throw line throws the ball with an initial speed of 8.15 m/s, releasing it at a height of 2.44 m (8 ft) above the floor. At what angle above the horizontal must the ball be thrown to exactly hit the basket? Note that most players will use a large initial angle rather than a flat shot because it allows for a larger margin of error. Explicitly show how you follow the steps involved in solving projectile motion problems.
In 2007, Michael Carter (U.S.) set a world record in the shot put with a throw of 24.77 m. What was the initial speed of the shot if he released it at a height of 2.10 m and threw it at an angle of 38.0° above the horizontal? (Although the maximum distance for a projectile on level ground is achieved at 45° when air resistance is neglected, the actual angle to achieve maximum range is smaller; thus, 38° will give a longer range than 45° in the shot put.)

A basketball player is running at 5.00 m/s directly toward the basket when he jumps into the air to dunk the ball. He maintains his horizontal velocity. (a) What vertical velocity does he need to rise 0.750 m above the floor? (b) How far from the basket (measured in the horizontal direction) must he start his jump to reach his maximum height at the same time as he reaches the basket?

A football player punts the ball at a 45.0° angle. Without an effect from the wind, the ball would travel 60.0 m horizontally. (a) What is the initial speed of the ball? (b) When the ball is near its maximum height it experiences a brief gust of wind that reduces its horizontal velocity by 1.50 m/s. What distance does the ball travel horizontally?

Prove that the trajectory of a projectile is parabolic, having the form \[ y = ax + bx^2. \] To obtain this expression, solve the equation \[ x = v_0x t \] for \( t \) and substitute it into the expression for \( y = v_0y t - \frac{1}{2}gt^2 \). (These equations describe the \( x \) and \( y \) positions of a projectile that starts at the origin.) You should obtain an equation of the form \( y = ax + bx^2 \) where \( a \) and \( b \) are constants.

Derive \[ R = \frac{v_0^2 \sin 2\theta}{g} \] for the range of a projectile on level ground by finding the time \( t \) at which \( y \) becomes zero and substituting this value of \( t \) into the expression for \( x = x_0 \), noting that \( R = x - x_0 \).

Unreasonable Results
(a) Find the maximum range of a super cannon that has a muzzle velocity of 4.0 km/s. (b) What is unreasonable about the range you found? (c) Is the premise unreasonable or is the available equation inapplicable? Explain your answer. (d) If such a muzzle velocity could be obtained, discuss the effects of air resistance, thinning air with altitude, and the curvature of the Earth on the range of the super cannon.

Construct Your Own Problem
Consider a ball tossed over a fence. Construct a problem in which you calculate the ball’s needed initial velocity to just clear the fence. Among the things to determine are; the height of the fence, the distance to the fence from the point of release of the ball, and the height at which the ball is released. You should also consider whether it is possible to choose the initial speed for the ball and just calculate the angle at which it is thrown. Also examine the possibility of multiple solutions given the distances and heights you have chosen.

3.5 Addition of Velocities
Bryan Allen pedaled a human-powered aircraft across the English Channel from the cliffs of Dover to Cap Gris-Nez on June 12, 1979. (a) He flew for 169 min at an average velocity of 3.53 m/s in a direction 45° south of east. What was his total displacement? (b) Allen encountered a headwind averaging 2.00 m/s almost precisely in the opposite direction of his motion relative to the Earth. What was his average velocity relative to the air mass?

A seagull flies at a velocity of 9.00 m/s straight into the wind. (a) If it takes the bird 20.0 min to travel 6.00 km relative to the Earth, what is the velocity of the wind? (b) If the bird turns around and flies with the wind, how long will he take to return 6.00 km? (c) Discuss how the wind affects the total round-trip time compared to what it would be with no wind.

Near the end of a marathon race, the first two runners are separated by a distance of 45.0 m. The front runner has a velocity of 3.50 m/s, and the second a velocity of 4.20 m/s. (a) What is the velocity of the second runner relative to the first? (b) If the front runner is 250 m from the finish line, who will win the race, assuming they run at constant velocity? (c) What distance ahead will the winner be when she crosses the finish line?

Verify that the coin dropped by the airline passenger in the Example 3.8 travels 144 m horizontally while falling 1.50 m in the frame of reference of the Earth.

A football quarterback is moving straight backward at a speed of 2.00 m/s when he throws a pass to a player 18.0 m straight downfield. The ball is thrown at an angle of 25.0° relative to the ground and is caught at the same height as it is released. What is the initial velocity of the ball relative to the quarterback?

A ship sets sail from Rotterdam, The Netherlands, heading due north at 7.00 m/s relative to the water. The local ocean current is 1.50 m/s in a direction 40.0° north of east. What is the velocity of the ship relative to the Earth?

(a) A jet airplane flying from Darwin, Australia, has an air speed of 260 m/s in a direction 5.0° south of west. It is in the jet stream, which is blowing at 35.0 m/s in a direction 15° south of east. What is the velocity of the airplane relative to the Earth? (b) Discuss whether your answers are consistent with your expectations for the effect of the wind on the plane’s path.

In what direction would the ship in Exercise 3.57 have to travel in order to have a velocity straight north relative to the Earth, assuming its speed relative to the water remains 7.00 m/s? (b) What would its speed be relative to the Earth?

Another airplane is flying in a jet stream that is blowing at 45.0 m/s in a direction 20° south of east (as in Exercise 3.58). Its direction of motion relative to the Earth is 45.0° south of west, while its direction of travel relative to the air is 5.0° south of west. What is the airplane’s speed relative to the air mass? (b) What is the airplane’s speed relative to the Earth?
61. A sandal is dropped from the top of a 15.0-m-high mast on a ship moving at 1.75 m/s due south. Calculate the velocity of the sandal when it hits the deck of the ship: (a) relative to the ship and (b) relative to a stationary observer on shore. (c) Discuss how the answers give a consistent result for the position at which the sandal hits the deck.

62. The velocity of the wind relative to the water is crucial to sailboats. Suppose a sailboat is in an ocean current that has a velocity of 2.20 m/s in a direction 30.0º east of north relative to the Earth. It encounters a wind that has a velocity of 4.50 m/s in a direction of 50.0º south of west relative to the Earth. What is the velocity of the wind relative to the water?

63. The great astronomer Edwin Hubble discovered that all distant galaxies are receding from our Milky Way Galaxy with velocities proportional to their distances. It appears to an observer on the Earth that we are at the center of an expanding universe. Figure 3.62 illustrates this for five galaxies lying along a straight line, with the Milky Way Galaxy at the center. Using the data from the figure, calculate the velocities: (a) relative to galaxy 2 and (b) relative to galaxy 5. The results mean that observers on all galaxies will see themselves at the center of the expanding universe, and they would likely be aware of relative velocities, concluding that it is not possible to locate the center of expansion with the given information.

64. (a) Use the distance and velocity data in Figure 3.62 to find the rate of expansion as a function of distance. (b) If you extrapolate back in time, how long ago would all of the galaxies have been at approximately the same position? The two parts of this problem give you some idea of how the Hubble constant for universal expansion and the time back to the Big Bang are determined, respectively.

65. An athlete crosses a 25-m-wide river by swimming perpendicular to the water current at a speed of 0.5 m/s relative to the water. He reaches the opposite side at a distance 40 m downstream from his starting point. How fast is the water in the river flowing with respect to the ground? What is the speed of the swimmer with respect to a friend at rest on the ground?

66. A ship sailing in the Gulf Stream is heading 25.0º west of north at a speed of 4.00 m/s relative to the water. Its velocity relative to the Earth is 4.80 m/s 5.00º west of north. What is the velocity of the Gulf Stream? (The velocity obtained is typical for the Gulf Stream a few hundred kilometers off the east coast of the United States.)

67. An ice hockey player is moving at 8.00 m/s when he hits the puck toward the goal. The speed of the puck relative to the player is 29.0 m/s. The plane between the center of the goal and the player makes a 90.0º angle relative to his path as shown in Figure 3.63. What angle must the puck’s velocity make relative to the player (in his frame of reference) to hit the center of the goal?

68. Unreasonable Results

Suppose you wish to shoot supplies straight up to astronauts in an orbit 36,000 km above the surface of the Earth. (a) At what velocity must the supplies be launched? (b) What is unreasonable about this velocity? (c) Is there a problem with the relative velocity between the supplies and the astronauts when the supplies reach their maximum height? (d) Is the premise unreasonable or is the available equation inapplicable? Explain your answer.

69. Unreasonable Results

A commercial airplane has an air speed of 280 m/s due east and flies with a strong tailwind. It travels 3000 km in a direction 5º south of east in 1.50 h. (a) What was the velocity of the plane relative to the ground? (b) Calculate the magnitude and direction of the tailwind’s velocity. (c) What is unreasonable about both of these velocities? (d) Which premise is unreasonable?

70. Construct Your Own Problem

Consider an airplane headed for a runway in a cross wind. Construct a problem in which you calculate the angle the airplane must fly relative to the air mass in order to have a velocity parallel to the runway. Among the things to consider are the direction of the runway, the wind speed and direction (its velocity) and the speed of the plane relative to the air mass. Also calculate the speed of the airplane relative to the ground. Discuss any last minute maneuvers the pilot might have to perform in order for the plane to land with its wheels pointing straight down the runway.
4 DYNAMICS: FORCE AND NEWTON'S LAWS OF MOTION

Introduction to Dynamics: Newton's Laws of Motion

Motion draws our attention. Motion itself can be beautiful, causing us to marvel at the forces needed to achieve spectacular motion, such as that of a dolphin jumping out of the water, or a pole vaulter, or the flight of a bird, or the orbit of a satellite. The study of motion is kinematics, but kinematics only describes the way objects move—their velocity and their acceleration. Dynamics considers the forces that affect the motion of moving objects and systems. Newton's laws of motion are the foundation of dynamics. These laws provide an example of the breadth and simplicity of principles under which nature functions.
They are also universal laws in that they apply to similar situations on Earth as well as in space.

Isaac Newton’s (1642–1727) laws of motion were just one part of the monumental work that has made him legendary. The development of Newton’s laws marks the transition from the Renaissance into the modern era. This transition was characterized by a revolutionary change in the way people thought about the physical universe. For many centuries natural philosophers had debated the nature of the universe based largely on certain rules of logic with great weight given to the thoughts of earlier classical philosophers such as Aristotle (384–322 BC). Among the many great thinkers who contributed to this change were Newton and Galileo.

Galileo was instrumental in establishing observation as the absolute determinant of truth, rather than “logical” argument. Galileo’s use of the telescope was his most notable achievement in demonstrating the importance of observation. He discovered moons orbiting Jupiter and made other observations that were inconsistent with certain ancient ideas and religious dogma. For this reason, and because of the manner in which he dealt with those in authority, Galileo was tried by the Inquisition and punished. He spent the final years of his life under a form of house arrest. Because others before Galileo had also made discoveries by observing the nature of the universe, and because repeated observations verified those of Galileo, his work could not be suppressed or denied. After his death, his work was verified by others, and his ideas were eventually accepted by the church and scientific communities.

Galileo also contributed to the formation of what is now called Newton’s first law of motion. Newton made use of the work of his predecessors, which enabled him to develop laws of motion, discover the law of gravity, invent calculus, and make great contributions to the theories of light and color. It is amazing that many of these developments were made with Newton working alone, without the benefit of the usual interactions that take place among scientists today.

It was not until the advent of modern physics early in the 20th century that it was discovered that Newton’s laws of motion produce a good approximation to motion only when the objects are moving at speeds much, much less than the speed of light and when those objects are larger than the size of most molecules (about \(10^{-9}\) m in diameter). These constraints define the realm of classical mechanics, as discussed in Introduction to the Nature of Science and Physics. At the beginning of the 20th century, Albert Einstein (1879–1955) developed the theory of relativity and, along with many other scientists, developed quantum theory. This theory does not have the constraints present in classical physics. All of the situations we consider in this chapter, and all those preceding the introduction of relativity in Special Relativity, are in the realm of classical physics.

**Making Connections: Past and Present Philosophy**

The importance of observation and the concept of cause and effect were not always so entrenched in human thinking. This realization was a part of the evolution of modern physics from natural philosophy. The achievements of Galileo, Newton, Einstein, and others were key milestones in the history of scientific thought. Most of the scientific theories that are described in this book descended from the work of these scientists.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42129/1.12/#concept-trailer-newtons-laws)

### 4.1 Development of Force Concept

**Dynamics** is the study of the forces that cause objects and systems to move. To understand this, we need a working definition of force. Our intuitive definition of force—that is, a push or a pull—is a good place to start. We know that a push or pull has both magnitude and direction (therefore, it is a vector quantity) and can vary considerably in each regard. For example, a cannon
exerts a strong force on a cannonball that is launched into the air. In contrast, Earth exerts only a tiny downward pull on a flea. Our everyday experiences also give us a good idea of how multiple forces add. If two people push in different directions on a third person, as illustrated in Figure 4.3, we might expect the total force to be in the direction shown. Since force is a vector, it adds just like other vectors, as illustrated in Figure 4.3(a) for two ice skaters. Forces, like other vectors, are represented by arrows and can be added using the familiar head-to-tail method or by trigonometric methods. These ideas were developed in Two-Dimensional Kinematics.

Figure 4.3 Part (a) shows an overhead view of two ice skaters pushing on a third. Forces are vectors and add like other vectors, so the total force on the third skater is in the direction shown. In part (b), we see a free-body diagram representing the forces acting on the third skater.

A more quantitative definition of force can be based on some standard force, just as distance is measured in units relative to a standard distance. One possibility is to stretch a spring a certain fixed distance, as illustrated in Figure 4.4, and use the force it exerts to pull itself back to its relaxed shape—called a restoring force—as a standard. The magnitude of all other forces can be stated as multiples of this standard unit of force. Many other possibilities exist for standard forces. (One that we will encounter in Magnetism is the magnetic force between two wires carrying electric current.) Some alternative definitions of force will be given later in this chapter.

![Figure 4.3](image)

Figure 4.3(b) is our first example of a free-body diagram, which is a technique used to illustrate all the external forces acting on a body. The body is represented by a single isolated point (or free body), and only those forces acting on the body from the outside (external forces) are shown. (These forces are the only ones shown, because only external forces acting on the body affect its motion. We can ignore any internal forces within the body.) Free-body diagrams are very useful in analyzing forces acting on a system and are employed extensively in the study and application of Newton's laws of motion.

4.2 Newton's First Law of Motion: Inertia

Experience suggests that an object at rest will remain at rest if left alone, and that an object in motion tends to slow down and stop unless some effort is made to keep it moving. What Newton's first law of motion states, however, is the following:
Newton's First Law of Motion

A body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force.

Note the repeated use of the verb “remains.” We can think of this law as preserving the status quo of motion.

Rather than contradicting our experience, Newton's first law of motion states that there must be a cause (which is a net external force) for there to be any change in velocity (either a change in magnitude or direction). We will define net external force in the next section. An object sliding across a table or floor slows down due to the net force of friction acting on the object. If friction disappeared, would the object still slow down?

The idea of cause and effect is crucial in accurately describing what happens in various situations. For example, consider what happens to an object sliding along a rough horizontal surface. The object quickly grinds to a halt. If we spray the surface with talcum powder to make the surface smoother, the object slides farther. If we make the surface even smoother by rubbing lubricating oil on it, the object slides farther yet. Extrapolating to a frictionless surface, we can imagine the object sliding in a straight line indefinitely. Friction is thus the cause of the slowing (consistent with Newton's first law). The object would not slow down at all if friction were completely eliminated. Consider an air hockey table. When the air is turned off, the puck slides only a short distance before friction slows it to a stop. However, when the air is turned on, it creates a nearly frictionless surface, and the puck glides long distances without slowing down. Additionally, if we know enough about the friction, we can accurately predict how quickly the object will slow down. Friction is an external force.

Newton's first law is completely general and can be applied to anything from an object sliding on a table to a satellite in orbit to blood pumped from the heart. Experiments have thoroughly verified that any change in velocity (speed or direction) must be caused by an external force. The idea of generally applicable or universal laws is important not only here—it is a basic feature of all laws of physics. Identifying these laws is like recognizing patterns in nature from which further patterns can be discovered. The genius of Galileo, who first developed the idea for the first law, and Newton, who clarified it, was to ask the fundamental question, “What is the cause?” Thinking in terms of cause and effect is a worldview fundamentally different from the typical ancient Greek approach when questions such as "Why does a tiger have stripes?" would have been answered in Aristotelian fashion, “That is the nature of the beast.” True perhaps, but not a useful insight.

Mass

The property of a body to remain at rest or to remain in motion with constant velocity is called inertia. Newton's first law is often called the law of inertia. As we know from experience, some objects have more inertia than others. It is obviously more difficult to change the motion of a large boulder than that of a basketball, for example. The inertia of an object is measured by its mass. Roughly speaking, mass is a measure of the amount of “stuff” (or matter) in something. The quantity or amount of matter in an object is determined by the numbers of atoms and molecules of various types it contains. Unlike weight, mass does not vary with location. The mass of an object is the same on Earth, in orbit, or on the surface of the Moon. In practice, it is very difficult to count and identify all of the atoms and molecules in an object, so masses are not often determined in this manner. Operationally, the masses of objects are determined by comparison with the standard kilogram.

Check Your Understanding

Which has more mass: a kilogram of cotton balls or a kilogram of gold?

Solution

They are equal. A kilogram of one substance is equal in mass to a kilogram of another substance. The quantities that might differ between them are volume and density.

4.3 Newton's Second Law of Motion: Concept of a System

Newton's second law of motion is closely related to Newton's first law of motion. It mathematically states the cause and effect relationship between force and changes in motion. Newton's second law of motion is more quantitative and is used extensively to calculate what happens in situations involving a force. Before we can write down Newton's second law as a simple equation giving the exact relationship of force, mass, and acceleration, we need to sharpen some ideas that have already been mentioned.

First, what do we mean by a change in motion? The answer is that a change in motion is equivalent to a change in velocity. A change in velocity means, by definition, that there is an acceleration. Newton's first law says that a net external force causes a change in motion; thus, we see that a net external force causes acceleration.

Another question immediately arises. What do we mean by an external force? An intuitive notion of external is correct—an external force acts from outside the system of interest. For example, in Figure 4.5(a) the system of interest is the wagon plus the child in it. The two forces exerted by the other children are external forces. An internal force acts between elements of the system. Again looking at Figure 4.5(a), the force the child in the wagon exerts to hang onto the wagon is an internal force between elements of the system of interest. Only external forces affect the motion of a system, according to Newton's first law. (The internal forces actually cancel, as we shall see in the next section.) You must define the boundaries of the system before you can determine which forces are external. Sometimes the system is obvious, whereas other times identifying the boundaries of a system is more subtle. The concept of a system is fundamental to many areas of physics, as is the correct application of Newton's laws. This concept will be revisited many times on our journey through physics.
Different forces exerted on the same mass produce different accelerations. (a) Two children push a wagon with a child in it. Arrows representing all external forces are shown. The system of interest is the wagon and its rider. The weight \( w \) of the system and the support of the ground \( N \) are also shown for completeness and are assumed to cancel. The vector \( f \) represents the friction acting on the wagon, and it acts to the left, opposing the motion of the wagon. (b) All of the external forces acting on the system add together to produce a net force, \( F_{\text{net}} \). The free-body diagram shows all of the forces acting on the system of interest. The dot represents the center of mass of the system. Each force vector extends from this dot. Because there are two forces acting to the right, we draw the vectors collinearly. (c) A larger net external force produces a larger acceleration \( a' > a \) when an adult pushes the child.

Now, it seems reasonable that acceleration should be directly proportional to and in the same direction as the net (total) external force acting on a system. This assumption has been verified experimentally and is illustrated in Figure 4.5. In part (a), a smaller force causes a smaller acceleration than the larger force illustrated in part (c). For completeness, the vertical forces are also shown; they are assumed to cancel since there is no acceleration in the vertical direction. The vertical forces are the weight \( w \) and the support of the ground \( N \), and the horizontal force \( f \) represents the force of friction. These will be discussed in more detail in later sections. For now, we will define friction as a force that opposes the motion past each other of objects that are touching. Figure 4.5(b) shows how vectors representing the external forces add together to produce a net force, \( F_{\text{net}} \).

To obtain an equation for Newton’s second law, we first write the relationship of acceleration and net external force as the proportionality

\[ a \propto F_{\text{net}}. \]

where the symbol \( \propto \) means “proportional to,” and \( F_{\text{net}} \) is the net external force. (The net external force is the vector sum of all external forces and can be determined graphically, using the head-to-tail method, or analytically, using components. The techniques are the same as for the addition of other vectors, and are covered in Two-Dimensional Kinematics.) This proportionality states what we have said in words—acceleration is directly proportional to the net external force. Once the system of interest is chosen, it is important to identify the external forces and ignore the internal ones. It is a tremendous simplification not to have to consider the numerous internal forces acting between objects within the system, such as muscular forces within the child’s body, let alone the myriad of forces between atoms in the objects, but by doing so, we can easily solve some very complex problems with only minimal error due to our simplification.

Now, it also seems reasonable that acceleration should be inversely proportional to the mass of the system. In other words, the larger the mass (the inertia), the smaller the acceleration produced by a given force. And indeed, as illustrated in Figure 4.6, the same net external force applied to a car produces a much smaller acceleration than when applied to a basketball. The proportionality is written as

\[ a \propto \frac{1}{m}, \]

where \( m \) is the mass of the system. Experiments have shown that acceleration is exactly inversely proportional to mass, just as it is exactly linearly proportional to the net external force.
It has been found that the acceleration of an object depends only on the net external force and the mass of the object. Combining the two proportionalities just given yields Newton's second law of motion.

Newton's Second Law of Motion

The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass.

In equation form, Newton's second law of motion is

\[ a = \frac{F_{\text{net}}}{m}. \]  

(4.3)

This is often written in the more familiar form

\[ F_{\text{net}} = ma. \]  

(4.4)

When only the magnitude of force and acceleration are considered, this equation is simply

\[ F_{\text{net}} = ma. \]  

(4.5)

Although these last two equations are really the same, the first gives more insight into what Newton's second law means. The law is a cause and effect relationship among three quantities that is not simply based on their definitions. The validity of the second law is completely based on experimental verification.

Units of Force

\( F_{\text{net}} = ma \) is used to define the units of force in terms of the three basic units for mass, length, and time. The SI unit of force is called the newton (abbreviated N) and is the force needed to accelerate a 1-kg system at the rate of \( 1 \text{ m/s}^2 \). That is, since \( F_{\text{net}} = ma \),

\[ 1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2. \]  

(4.6)

While almost the entire world uses the newton for the unit of force, in the United States the most familiar unit of force is the pound (lb), where \( 1 \text{ N} = 0.225 \text{ lb} \).

Weight and the Gravitational Force

When an object is dropped, it accelerates toward the center of Earth. Newton's second law states that a net force on an object is responsible for its acceleration. If air resistance is negligible, the net force on a falling object is the gravitational force, commonly called its weight \( w \). Weight can be denoted as a vector \( \mathbf{w} \) because it has a direction; down is, by definition, the direction of gravity, and hence weight is a downward force. The magnitude of weight is denoted as \( w \). Galileo was instrumental in showing that, in the absence of air resistance, all objects fall with the same acceleration \( g \). Using Galileo's result and Newton's second law, we can derive an equation for weight.

Consider an object with mass \( m \) falling downward toward Earth. It experiences only the downward force of gravity, which has magnitude \( w \). Newton's second law states that the magnitude of the net external force on an object is \( F_{\text{net}} = ma \).

Since the object experiences only the downward force of gravity, \( F_{\text{net}} = w \). We know that the acceleration of an object due to gravity is \( g \), or \( a = g \). Substituting these into Newton's second law gives
Weight

This is the equation for weight—the gravitational force on a mass \( m \):

\[
w = mg. \tag{4.7}
\]

Since \( g = 9.80 \text{ m/s}^2 \) on Earth, the weight of a 1.0 kg object on Earth is 9.8 N, as we see:

\[
w = mg = (1.0 \text{ kg})(9.80 \text{ m/s}^2) = 9.8 \text{ N}. \tag{4.8}
\]

Recall that \( g \) can take a positive or negative value, depending on the positive direction in the coordinate system. Be sure to take this into consideration when solving problems with weight.

When the net external force on an object is its weight, we say that it is in free-fall. That is, the only force acting on the object is the force of gravity. In the real world, when objects fall downward toward Earth, they are never truly in free-fall because there is always some upward force from the air acting on the object.

The acceleration due to gravity \( g \) varies slightly over the surface of Earth, so that the weight of an object depends on location and is not an intrinsic property of the object. Weight varies dramatically if one leaves Earth’s surface. On the Moon, for example, the acceleration due to gravity is only \( 1.625 \text{ m/s}^2 \). A 1.0-kg mass thus has a weight of 9.8 N on Earth and only about 1.7 N on the Moon.

The broadest definition of weight in this sense is that the weight of an object is the gravitational force on it from the nearest large body, such as Earth, the Moon, the Sun, and so on. This is the most common and useful definition of weight in physics. It differs dramatically, however, from the definition of weight used by NASA and the popular media in relation to space travel and exploration. When they speak of “weightlessness” and “microgravity,” they are really referring to the phenomenon we call “free-fall” in physics. We shall use the above definition of weight, and we will make careful distinctions between free-fall and actual weightlessness.

It is important to be aware that weight and mass are very different physical quantities, although they are closely related. Mass is the quantity of matter (how much “stuff”) and does not vary in classical physics, whereas weight is the gravitational force and does vary depending on gravity. It is tempting to equate the two, since most of our examples take place on Earth, where the weight of an object only varies a little with the location of the object. Furthermore, the terms mass and weight are used interchangeably in everyday language; for example, our medical records often show our “weight” in kilograms, but never in the correct units of newtons.

Common Misconceptions: Mass vs. Weight

Mass and weight are often used interchangeably in everyday language. However, in science, these terms are distinctly different from one another. Mass is a measure of how much matter is in an object. The typical measure of mass is the kilogram (or the “slug” in English units). Weight, on the other hand, is a measure of the force of gravity acting on an object. Weight is equal to the mass of an object \( (m) \) multiplied by the acceleration due to gravity \( (g) \). Like any other force, weight is measured in terms of newtons (or pounds in English units).

Assuming the mass of an object is kept intact, it will remain the same, regardless of its location. However, because weight depends on the acceleration due to gravity, the weight of an object can change when the object enters into a region with stronger or weaker gravity. For example, the acceleration due to gravity on the Moon is \( 1.625 \text{ m/s}^2 \) (which is much less than the acceleration due to gravity on Earth, \( 9.80 \text{ m/s}^2 \)). If you measured your weight on Earth and then measured your weight on the Moon, you would find that you “weigh” much less, even though you do not look any skinner. This is because the force of gravity is weaker on the Moon. In fact, when people say that they are “losing weight,” they really mean that they are losing “mass” (which in turn causes them to weigh less).

Take-Home Experiment: Mass and Weight

What do bathroom scales measure? When you stand on a bathroom scale, what happens to the scale? It depresses slightly. The scale contains springs that compress in proportion to your weight—similar to rubber bands expanding when pulled. The springs provide a measure of your weight (for an object which is not accelerating). This is a force in newtons (or pounds). In most countries, the measurement is divided by 9.80 to give a reading in mass units of kilograms. The scale measures weight but is calibrated to provide information about mass. While standing on a bathroom scale, push down on a table next to you. What happens to the reading? Why? Would your scale measure the same “mass” on Earth as on the Moon?

Example 4.1 What Acceleration Can a Person Produce when Pushing a Lawn Mower?

Suppose that the net external force (push minus friction) exerted on a lawn mower is 51 N (about 11 lb) parallel to the ground. The mass of the mower is 24 kg. What is its acceleration?
The net force on a lawn mower is 51 N to the right. At what rate does the lawn mower accelerate to the right?

**Strategy**

Since \( F_{\text{net}} \) and \( m \) are given, the acceleration can be calculated directly from Newton's second law as stated in \( F_{\text{net}} = ma \).

**Solution**

The magnitude of the acceleration \( a \) is \( a = \frac{F_{\text{net}}}{m} \). Entering known values gives

\[
a = \frac{51 \text{ N}}{24 \text{ kg}}
\]

Substituting the units \( \text{kg} \cdot \text{m/s}^2 \) for N yields

\[
a = \frac{51 \text{ kg} \cdot \text{m/s}^2}{24 \text{ kg}} = 2.1 \text{ m/s}^2.
\]

**Discussion**

The direction of the acceleration is the same direction as that of the net force, which is parallel to the ground. There is no information given in this example about the individual external forces acting on the system, but we can say something about their relative magnitudes. For example, the force exerted by the person pushing the mower must be greater than the friction opposing the motion (since we know the mower moves forward), and the vertical forces must cancel if there is to be no acceleration in the vertical direction (the mower is moving only horizontally). The acceleration found is small enough to be reasonable for a person pushing a mower. Such an effort would not last too long because the person's top speed would soon be reached.

**Example 4.2 What Rocket Thrust Accelerates This Sled?**

Prior to space flights carrying astronauts, rocket sleds were used to test aircraft, missile equipment, and physiological effects on human subjects at high speeds. They consisted of a platform that was mounted on one or two rails and propelled by several rockets. Calculate the magnitude of force exerted by each rocket, called its thrust \( T \), for the four-rocket propulsion system shown in Figure 4.8. The sled's initial acceleration is \( 49 \text{ m/s}^2 \), the mass of the system is 2100 kg, and the force of friction opposing the motion is known to be 650 N.
Figure 4.8 A sled experiences a rocket thrust that accelerates it to the right. Each rocket creates an identical thrust \( T \). As in other situations where there is only horizontal acceleration, the vertical forces cancel. The ground exerts an upward force \( N \) on the system that is equal in magnitude and opposite in direction to its weight, \( w \). The system here is the sled, its rockets, and rider, so none of the forces between these objects are considered. The arrow representing friction \( f \) is drawn larger than scale.

**Strategy**

Although there are forces acting vertically and horizontally, we assume the vertical forces cancel since there is no vertical acceleration. This leaves us with only horizontal forces and a simpler one-dimensional problem. Directions are indicated with plus or minus signs, with right taken as the positive direction. See the free-body diagram in the figure.

**Solution**

Since acceleration, mass, and the force of friction are given, we start with Newton's second law and look for ways to find the thrust of the engines. Since we have defined the direction of the force and acceleration as acting "to the right," we need to consider only the magnitudes of these quantities in the calculations. Hence we begin with

\[
F_{\text{net}} = ma, \tag{4.11}
\]

where \( F_{\text{net}} \) is the net force along the horizontal direction. We can see from Figure 4.8 that the engine thrusts add, while friction opposes the thrust. In equation form, the net external force is

\[
F_{\text{net}} = 4T - f. \tag{4.12}
\]

Substituting this into Newton's second law gives

\[
F_{\text{net}} = ma = 4T - f. \tag{4.13}
\]

Using a little algebra, we solve for the total thrust \( 4T \):

\[
4T = ma + f. \tag{4.14}
\]

Substituting known values yields

\[
4T = ma + f = (2100 \text{ kg})(49 \text{ m/s}^2) + 650 \text{ N}. \tag{4.15}
\]

So the total thrust is

\[
4T = 1.0 \times 10^5 \text{ N}, \tag{4.16}
\]

and the individual thrusts are

\[
T = \frac{1.0 \times 10^5 \text{ N}}{4} = 2.6 \times 10^4 \text{ N}. \tag{4.17}
\]

**Discussion**

The numbers are quite large, so the result might surprise you. Experiments such as this were performed in the early 1960s to test the limits of human endurance and the setup designed to protect human subjects in jet fighter emergency ejections. Speeds of 1000 km/h were obtained, with accelerations of 45 \text{ g}'s. (Recall that \text{ g}, the acceleration due to gravity, is
9.80 m/s\(^2\). When we say that an acceleration is 45 \( g \)'s, it is \( 45 \times 9.80 \text{ m/s}^2 \), which is approximately \( 440 \text{ m/s}^2 \). While living subjects are not used any more, land speeds of 10,000 km/h have been obtained with rocket sleds. In this example, as in the preceding one, the system of interest is obvious. We will see in later examples that choosing the system of interest is crucial—and the choice is not always obvious.

Newton's second law of motion is more than a definition; it is a relationship among acceleration, force, and mass. It can help us make predictions. Each of those physical quantities can be defined independently, so the second law tells us something basic and universal about nature. The next section introduces the third and final law of motion.

### 4.4 Newton's Third Law of Motion: Symmetry in Forces

Baseball relief pitcher Mariano Rivera was so highly regarded that during his retirement year, opposing teams conducted farewell presentations when he played at their stadiums. The Minnesota Twins offered a unique gift: A chair made of broken bats. Any pitch can break a bat, but with Rivera's signature pitch—known as a cutter—the ball and the bat frequently came together at a point that shattered the hardwood. Typically, we think of a baseball or softball hitter exerting a force on the incoming ball, and baseball analysts now focus on the resulting “exit velocity” as a key statistic. But the force of the ball can do its own damage. This is exactly what happens whenever one body exerts a force on another—the first also experiences a force (equal in magnitude and opposite in direction). Numerous common experiences, such as stubbing a toe or pushing off the floor during a jump, confirm this. It is precisely stated in **Newton's third law of motion**.

**Newton's Third Law of Motion**

Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts.

This law represents a certain symmetry in nature: Forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself. We sometimes refer to this law loosely as “action-reaction,” where the force exerted is the action and the force experienced as a consequence is the reaction. Newton’s third law has practical uses in analyzing the origin of forces and understanding which forces are external to a system.

We can readily see Newton’s third law at work by taking a look at how people move about. Consider a swimmer pushing off from the side of a pool, as illustrated in Figure 4.9. She pushes against the pool wall with her feet and accelerates in the direction opposite to that of her push. The wall has exerted an equal and opposite force back on the swimmer. You might think that two equal and opposite forces would cancel, but they do not because they act on different systems. In this case, there are two systems that we could investigate: the swimmer or the wall. If we select the swimmer to be the system of interest, as in the figure, then \( \mathbf{F}_{\text{wall on feet}} \) is an external force on this system and affects its motion. The swimmer moves in the direction of \( \mathbf{F}_{\text{wall on feet}} \). In contrast, the force \( \mathbf{F}_{\text{feet on wall}} \) acts on the wall and not on our system of interest. Thus \( \mathbf{F}_{\text{feet on wall}} \) does not directly affect the motion of the system and does not cancel \( \mathbf{F}_{\text{wall on feet}} \). Note that the swimmer pushes in the direction opposite to that in which she wishes to move. The reaction to her push is thus in the desired direction.
When the swimmer exerts a force $F_{\text{feet on wall}}$ on the wall, she accelerates in the direction opposite to that of her push. This means the net external force on her is in the direction opposite to $F_{\text{feet on wall}}$. This opposition occurs because, in accordance with Newton's third law of motion, the wall exerts a force $F_{\text{wall on feet}}$ on her, equal in magnitude but in the direction opposite to the one she exerts on it. The line around the swimmer indicates the system of interest. Note that $F_{\text{feet on wall}}$ does not act on this system (the swimmer) and, thus, does not cancel $F_{\text{wall on feet}}$. Thus the free-body diagram shows only $F_{\text{wall on feet}}$, $w$, the gravitational force, and $BF$, the buoyant force of the water supporting the swimmer’s weight. The vertical forces $w$ and $BF$ cancel since there is no vertical motion.

Other examples of Newton’s third law are easy to find. As a professor paces in front of a whiteboard, she exerts a force backward on the floor. The floor exerts a reaction force forward on the professor that causes her to accelerate forward. Similarly, a car accelerates because the ground pushes forward on the drive wheels in reaction to the drive wheels pushing backward on the ground. You can see evidence of the wheels pushing backward when tires spin on a gravel road and throw rocks backward. In another example, rockets move forward by expelling gas backward at high velocity. This means the rocket exerts a large backward force on the gas in the rocket combustion chamber, and the gas therefore exerts a large reaction force forward on the rocket. This reaction force is called thrust. It is a common misconception that rockets propel themselves by pushing on the ground or on the air behind them. They actually work better in a vacuum, where they can more readily expel the exhaust gases. Helicopters similarly create lift by pushing air down, thereby experiencing an upward reaction force. Birds and airplanes also fly by exerting force on air in a direction opposite to that of whatever force they need. For example, the wings of a bird force air downward and backward in order to get lift and move forward. An octopus propels itself in the water by ejecting water through a funnel from its body, similar to a jet ski. In a situation similar to Sancho’s, professional cage fighters experience reaction forces when they punch, sometimes breaking their hand by hitting an opponent’s body.

**Example 4.3 Getting Up To Speed: Choosing the Correct System**

A physics professor pushes a cart of demonstration equipment to a lecture hall, as seen in Figure 4.10. Her mass is 65.0 kg, the cart’s is 12.0 kg, and the equipment’s is 7.0 kg. Calculate the acceleration produced when the professor exerts a backward force of 150 N on the floor. All forces opposing the motion, such as friction on the cart’s wheels and air resistance, total 24.0 N.
A professor pushes a cart of demonstration equipment. The lengths of the arrows are proportional to the magnitudes of the forces (except for \( f \), since it is too small to draw to scale). Different questions are asked in each example; thus, the system of interest must be defined differently for each. System 1 is appropriate for this example, since it asks for the acceleration of the entire group of objects. Only \( F_{\text{foot}} \) and \( f \) are external forces acting on System 1 along the line of motion. All other forces either cancel or act on the outside world. System 2 is chosen for Example 4.4 so that \( F_{\text{prof}} \) will be an external force and enter into Newton’s second law. Note that the free-body diagrams, which allow us to apply Newton’s second law, vary with the system chosen.

**Strategy**

Since they accelerate as a unit, we define the system to be the professor, cart, and equipment. This is System 1 in Figure 4.10. The professor pushes backward with a force \( F_{\text{foot}} \) of 150 N. According to Newton’s third law, the floor exerts a forward reaction force \( F_{\text{floor}} \) of 150 N on System 1. Because all motion is horizontal, we can assume there is no net force in the vertical direction. The problem is therefore one-dimensional along the horizontal direction. As noted, \( f \) opposes the motion and is thus in the opposite direction of \( F_{\text{floor}} \). Note that we do not include the forces \( F_{\text{prof}} \) or \( F_{\text{cart}} \) because these are internal forces, and we do not include \( F_{\text{foot}} \) because it acts on the floor, not on the system. There are no other significant forces acting on System 1. If the net external force can be found from all this information, we can use Newton’s second law to find the acceleration as requested. See the free-body diagram in the figure.

**Solution**

Newton’s second law is given by

\[
a = \frac{F_{\text{net}}}{m}.
\]

The net external force on System 1 is deduced from Figure 4.10 and the discussion above to be

\[
F_{\text{net}} = F_{\text{foot}} - f = 150 \text{ N} - 24.0 \text{ N} = 126 \text{ N}.
\]

The mass of System 1 is

\[
m = (65.0 + 12.0 + 7.0) \text{ kg} = 84 \text{ kg}.
\]

These values of \( F_{\text{net}} \) and \( m \) produce an acceleration of

\[
a = \frac{F_{\text{net}}}{m} = \frac{126 \text{ N}}{84 \text{ kg}} = 1.5 \text{ m/s}^2.
\]

**Discussion**

None of the forces between components of System 1, such as between the professor’s hands and the cart, contribute to the
net external force because they are internal to System 1. Another way to look at this is to note that forces between components of a system cancel because they are equal in magnitude and opposite in direction. For example, the force exerted by the professor on the cart results in an equal and opposite force back on her. In this case both forces act on the same system and, therefore, cancel. Thus internal forces (between components of a system) cancel. Choosing System 1 was crucial to solving this problem.

**Example 4.4 Force on the Cart—Choosing a New System**

Calculate the force the professor exerts on the cart in Figure 4.10 using data from the previous example if needed.

**Strategy**

If we now define the system of interest to be the cart plus equipment (System 2 in Figure 4.10), then the net external force on System 2 is the force the professor exerts on the cart minus friction. The force she exerts on the cart, $F_{\text{prof}}$, is an external force acting on System 2. $F_{\text{prof}}$ was internal to System 1, but it is external to System 2 and will enter Newton’s second law for System 2.

**Solution**

Newton’s second law can be used to find $F_{\text{prof}}$. Starting with

$$a = \frac{F_{\text{net}}}{m} \quad (4.22)$$

and noting that the magnitude of the net external force on System 2 is

$$F_{\text{net}} = F_{\text{prof}} - f, \quad (4.23)$$

we solve for $F_{\text{prof}}$, the desired quantity:

$$F_{\text{prof}} = F_{\text{net}} + f. \quad (4.24)$$

The value of $f$ is given, so we must calculate net $F_{\text{net}}$. That can be done since both the acceleration and mass of System 2 are known. Using Newton’s second law we see that

$$F_{\text{net}} = ma, \quad (4.25)$$

where the mass of System 2 is 19.0 kg ($m = 12.0 \text{ kg} + 7.0 \text{ kg}$) and its acceleration was found to be $a = 1.5 \text{ m/s}^2$ in the previous example. Thus,

$$F_{\text{net}} = ma, \quad (4.26)$$

$$F_{\text{net}} = (19.0 \text{ kg})(1.5 \text{ m/s}^2) = 29 \text{ N}. \quad (4.27)$$

Now we can find the desired force:

$$F_{\text{prof}} = F_{\text{net}} + f, \quad (4.28)$$

$$F_{\text{prof}} = 29 \text{ N} + 24.0 \text{ N} = 53 \text{ N}. \quad (4.29)$$

**Discussion**

It is interesting that this force is significantly less than the 150-N force the professor exerted backward on the floor. Not all of that 150-N force is transmitted to the cart; some of it accelerates the professor.

The choice of a system is an important analytical step both in solving problems and in thoroughly understanding the physics of the situation (which is not necessarily the same thing).

**Gravity Force Lab**

Visualize the gravitational force that two objects exert on each other. Change properties of the objects in order to see how it changes the gravity force.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42074/1.17//Gravity_Force_Lab)
4.5 Normal, Tension, and Other Examples of Forces

Forces are given many names, such as push, pull, thrust, lift, weight, friction, and tension. Traditionally, forces have been grouped into several categories and given names relating to their source, how they are transmitted, or their effects. The most important of these categories are discussed in this section, together with some interesting applications. Further examples of forces are discussed later in this text.

Normal Force

**Weight** (also called force of gravity) is a pervasive force that acts at all times and must be counteracted to keep an object from falling. You definitely notice that you must support the weight of a heavy object by pushing up on it when you hold it stationary, as illustrated in Figure 4.12(a). But how do inanimate objects like a table support the weight of a mass placed on them, such as shown in Figure 4.12(b)? When the bag of dog food is placed on the table, the table actually sags slightly under the load. This would be noticeable if the load were placed on a card table, but even rigid objects deform when a force is applied to them. Unless the object is deformed beyond its limit, it will exert a restoring force much like a deformed spring (or trampoline or diving board). The greater the deformation, the greater the restoring force. So when the load is placed on the table, the table sags until the restoring force becomes as large as the weight of the load. At this point the net external force on the load is zero. That is the situation when the load is stationary on the table. The table sags quickly, and the sag is slight so we do not notice it. But it is similar to the sagging of a trampoline when you climb onto it.

![Free-body diagrams](image)

Figure 4.12 (a) The person holding the bag of dog food must supply an upward force \(F_{\text{hand}}\) equal in magnitude and opposite in direction to the weight of the food \(w\). (b) The card table sags when the dog food is placed on it, much like a stiff trampoline. Elastic restoring forces in the table grow as it sags until they supply a force \(N\) equal in magnitude and opposite in direction to the weight of the load.

We must conclude that whatever supports a load, be it animate or not, must supply an upward force equal to the weight of the load, as we assumed in a few of the previous examples. If the force supporting a load is perpendicular to the surface of contact between the load and its support, this force is defined to be a **normal force** and here is given the symbol \(N\). (This is not the unit for force \(N\).) The word *normal* means perpendicular to a surface. The normal force can be less than the object’s weight if the object is on an incline, as you will see in the next example.

**Common Misconception: Normal Force (N) vs. Newton (N)**

In this section we have introduced the quantity normal force, which is represented by the variable \(N\). This should not be confused with the symbol for the newton, which is also represented by the letter \(N\). These symbols are particularly important to distinguish because the units of a normal force (\(N\)) happen to be newtons (N). For example, the normal force \(N\) that the floor exerts on a chair might be \(N = 100\) N. One important difference is that normal force is a vector, while the newton is simply a unit. Be careful not to confuse these letters in your calculations! You will encounter more similarities among
variables and units as you proceed in physics. Another example of this is the quantity work \( W \) and the unit watts (W).

**Example 4.5 Weight on an Incline, a Two-Dimensional Problem**

Consider the skier on a slope shown in Figure 4.13. Her mass including equipment is 60.0 kg. (a) What is her acceleration if friction is negligible? (b) What is her acceleration if friction is known to be 45.0 N?

![Free-body diagram](image)

Figure 4.13 Since motion and friction are parallel to the slope, it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier). \( N \) is perpendicular to the slope and \( f \) is parallel to the slope, but \( w \) has components along both axes, namely \( w_{\perp} \) and \( w_{\parallel} \). \( N \) is equal in magnitude to \( w_{\perp} \), so that there is no motion perpendicular to the slope, but \( f \) is less than \( w_{\parallel} \), so that there is a downslope acceleration (along the parallel axis).

**Strategy**

This is a two-dimensional problem, since the forces on the skier (the system of interest) are not parallel. The approach we have used in two-dimensional kinematics also works very well here. Choose a convenient coordinate system and project the vectors onto its axes, creating two connected one-dimensional problems to solve. The most convenient coordinate system for motion on an incline is one that has one coordinate parallel to the slope and one perpendicular to the slope. (Remember that motions along mutually perpendicular axes are independent.) We use the symbols \( \perp \) and \( \parallel \) to represent perpendicular and parallel, respectively. This choice of axes simplifies this type of problem, because there is no motion perpendicular to the slope and because friction is always parallel to the surface between two objects. The only external forces acting on the system are the skier’s weight, friction, and the support of the slope, respectively labeled \( w \), \( f \), and \( N \) in Figure 4.13. \( N \) is always perpendicular to the slope, and \( f \) is parallel to it. But \( w \) is not in the direction of either axis, and so the first step we take is to project it into components along the chosen axes, defining \( w_{\parallel} \) to be the component of weight parallel to the slope and \( w_{\perp} \) the component of weight perpendicular to the slope. Once this is done, we can consider the two separate problems of forces parallel to the slope and forces perpendicular to the slope.

**Solution**

The magnitude of the component of the weight parallel to the slope is \( w_{\parallel} = w \sin (25\degree) = mg \sin (25\degree) \), and the magnitude of the component of the weight perpendicular to the slope is \( w_{\perp} = w \cos (25\degree) = mg \cos (25\degree) \).

(a) Neglecting friction. Since the acceleration is parallel to the slope, we need only consider forces parallel to the slope. (Forces perpendicular to the slope add to zero, since there is no acceleration in that direction.) The forces parallel to the slope are the amount of the skier’s weight parallel to the slope \( w_{\parallel} \) and friction \( f \). Using Newton’s second law, with subscripts to denote quantities parallel to the slope,

\[
a_{\parallel} = \frac{F_{\text{net}}_{\parallel}}{m}
\]

where \( F_{\text{net}}_{\parallel} = w_{\parallel} = mg \sin (25\degree) \), assuming no friction for this part, so that

\[
a_{\parallel} = \frac{F_{\text{net}}_{\parallel}}{m} = \frac{mg \sin (25\degree)}{m} = g \sin (25\degree)
\]

\[
(9.80 \text{ m/s}^2)(0.4226) = 4.14 \text{ m/s}^2
\]

is the acceleration.

(b) Including friction. We now have a given value for friction, and we know its direction is parallel to the slope and it opposes
motion between surfaces in contact. So the net external force is now
\[ F_{\text{net}} || = w || - f, \]  
(4.33)

and substituting this into Newton's second law, \( a || = \frac{F_{\text{net}} ||}{m} \), gives
\[ a || = \frac{F_{\text{net}} ||}{m} = \frac{w || - f}{m} = \frac{mg \sin(25^\circ) - f}{m}. \]  
(4.34)

We substitute known values to obtain
\[ a || = \frac{(60.0 \text{ kg})(9.80 \text{ m/s}^2)(0.4226) - 45.0 \text{ N}}{60.0 \text{ kg}}, \]  
(4.35)

which yields \( a || = 3.39 \text{ m/s}^2 \),
(4.36)

which is the acceleration parallel to the incline when there is 45.0 N of opposing friction.

**Discussion**

Since friction always opposes motion between surfaces, the acceleration is smaller when there is friction than when there is none. In fact, it is a general result that if friction on an incline is negligible, then the acceleration down the incline is \( a = g \sin \theta \), regardless of mass. This is related to the previously discussed fact that all objects fall with the same acceleration in the absence of air resistance. Similarly, all objects, regardless of mass, slide down a frictionless incline with the same acceleration (if the angle is the same).

---

**Resolving Weight into Components**

![Diagram of weight components](image)

Figure 4.14 An object rests on an incline that makes an angle \( \theta \) with the horizontal.

When an object rests on an incline that makes an angle \( \theta \) with the horizontal, the force of gravity acting on the object is divided into two components: a force acting perpendicular to the plane, \( w_\perp \), and a force acting parallel to the plane, \( w_\parallel \). The perpendicular force of weight, \( w_\perp \), is typically equal in magnitude and opposite in direction to the normal force, \( N \).

The force acting parallel to the plane, \( w_\parallel \), causes the object to accelerate down the incline. The force of friction, \( f \), opposes the motion of the object, so it acts upward along the plane.

It is important to be careful when resolving the weight of the object into components. If the angle of the incline is at an angle \( \theta \) to the horizontal, then the magnitudes of the weight components are
\[ w_\parallel = w \sin(\theta) = mg \sin(\theta) \]  
(4.37)

and
\[ w_\perp = w \cos(\theta) = mg \cos(\theta). \]  
(4.38)

Instead of memorizing these equations, it is helpful to be able to determine them from reason. To do this, draw the right triangle formed by the three weight vectors. Notice that the angle \( \theta \) of the incline is the same as the angle formed between \( w \) and \( w_\perp \). Knowing this property, you can use trigonometry to determine the magnitude of the weight components:
\begin{align}
\cos(\theta) &= \frac{w_\perp}{w} \\
     &= w \cos(\theta) = mg \cos(\theta) \\
\sin(\theta) &= \frac{w_\parallel}{w} \\
     &= w \sin(\theta) = mg \sin(\theta)
\end{align}

**Take-Home Experiment: Force Parallel**

To investigate how a force parallel to an inclined plane changes, find a rubber band, some objects to hang from the end of the rubber band, and a board you can position at different angles. How much does the rubber band stretch when you hang the object from the end of the board? Now place the board at an angle so that the object slides off when placed on the board. How much does the rubber band extend if it is lined up parallel to the board and used to hold the object stationary on the board? Try two more angles. What does this show?

**Tension**

A tension is a force along the length of a medium, especially a force carried by a flexible medium, such as a rope or cable. The word “tension” comes from a Latin word meaning “to stretch.” Not coincidentally, the flexible cords that carry muscle forces to other parts of the body are called tendons. Any flexible connector, such as a string, rope, chain, wire, or cable, can exert pulls only parallel to its length; thus, a force carried by a flexible connector is a tension with direction parallel to the connector. It is important to understand that tension is a pull in a connector. In contrast, consider the phrase: “You can't push a rope.” The tension force pulls outward along the two ends of a rope.

Consider a person holding a mass on a rope as shown in Figure 4.15.

![Figure 4.15](image)

**Figure 4.15** When a perfectly flexible connector (one requiring no force to bend it) such as this rope transmits a force $T$, that force must be parallel to the length of the rope, as shown. The pull such a flexible connector exerts is a tension. Note that the rope pulls with equal force but in opposite directions on the hand and the supported mass (neglecting the weight of the rope). This is an example of Newton’s third law. The rope is the medium that carries the equal and opposite forces between the two objects. The tension anywhere in the rope between the hand and the mass is equal. Once you have determined the tension in one location, you have determined the tension at all locations along the rope.

Tension in the rope must equal the weight of the supported mass, as we can prove using Newton’s second law. If the 5.00-kg mass in the figure is stationary, then its acceleration is zero, and thus $F_{\text{net}} = 0$. The only external forces acting on the mass are its weight $w$ and the tension $T$ supplied by the rope. Thus,

$$F_{\text{net}} = T - w = 0,$$

where $T$ and $w$ are the magnitudes of the tension and weight and their signs indicate direction, with up being positive here. Thus, just as you would expect, the tension equals the weight of the supported mass:

$$T = w = mg.$$  

For a 5.00-kg mass, then (neglecting the mass of the rope) we see that

$$T = mg = (5.00 \text{ kg})(9.80 \text{ m/s}^2) = 49.0 \text{ N}.$$
If we cut the rope and insert a spring, the spring would extend a length corresponding to a force of 49.0 N, providing a direct observation and measure of the tension force in the rope.

Flexible connectors are often used to transmit forces around corners, such as in a hospital traction system, a finger joint, or a bicycle brake cable. If there is no friction, the tension is transmitted undiminished. Only its direction changes, and it is always parallel to the flexible connector. This is illustrated in Figure 4.16 (a) and (b).

![Figure 4.16](image)

**Figure 4.16** (a) Tendons in the finger carry force $T$ from the muscles to other parts of the finger, usually changing the force’s direction, but not its magnitude (the tendons are relatively friction free). (b) The brake cable on a bicycle carries the tension $T$ from the handlebars to the brake mechanism. Again, the direction but not the magnitude of $T$ is changed.

**Example 4.6 What Is the Tension in a Tightrope?**

Calculate the tension in the wire supporting the 70.0-kg tightrope walker shown in Figure 4.17.

![Figure 4.17](image)

**Figure 4.17** The weight of a tightrope walker causes a wire to sag by 5.0 degrees. The system of interest here is the point in the wire at which the tightrope walker is standing.

**Strategy**

As you can see in the figure, the wire is not perfectly horizontal (it cannot be!), but is bent under the person's weight. Thus, the tension on either side of the person has an upward component that can support his weight. As usual, forces are vectors represented pictorially by arrows having the same directions as the forces and lengths proportional to their magnitudes. The system is the tightrope walker, and the only external forces acting on him are his weight $w$ and the two tensions $T_L$ (left tension) and $T_R$ (right tension), as illustrated. It is reasonable to neglect the weight of the wire itself. The net external force is zero since the system is stationary. A little trigonometry can now be used to find the tensions. One conclusion is possible at the outset—we can see from part (b) of the figure that the magnitudes of the tensions $T_L$ and $T_R$ must be equal. This is
because there is no horizontal acceleration in the rope, and the only forces acting to the left and right are $T_L$ and $T_R$.

Thus, the magnitude of those forces must be equal so that they cancel each other out.

Whenever we have two-dimensional vector problems in which no two vectors are parallel, the easiest method of solution is to pick a convenient coordinate system and project the vectors onto its axes. In this case the best coordinate system has one axis horizontal and the other vertical. We call the horizontal the $x$-axis and the vertical the $y$-axis.

**Solution**

First, we need to resolve the tension vectors into their horizontal and vertical components. It helps to draw a new free-body diagram showing all of the horizontal and vertical components of each force acting on the system.

Consider the horizontal components of the forces (denoted with a subscript $x$):

$$F_{\text{net}} = T_{Lx} - T_{Rx}. \quad (4.44)$$

The net external horizontal force $F_{\text{net}} = 0$, since the person is stationary. Thus,

$$F_{\text{net}} = 0 = T_{Lx} - T_{Rx} \quad (4.45)$$

$$T_{Lx} = T_{Rx}.$$

Now, observe **Figure 4.18**. You can use trigonometry to determine the magnitude of $T_L$ and $T_R$. Notice that:

$$\cos(5.0^\circ) = \frac{T_{Lx}}{T_L} \quad (4.46)$$

$$T_{Lx} = T_L \cos(5.0^\circ)$$

$$\cos(5.0^\circ) = \frac{T_{Rx}}{T_R} \quad (4.47)$$

$$T_{Rx} = T_R \cos(5.0^\circ).$$

Equating $T_{Lx}$ and $T_{Rx}$:

$$T_L \cos(5.0^\circ) = T_R \cos(5.0^\circ) \quad (4.48)$$

Thus,

$$T_L = T_R = T.$$

as predicted. Now, considering the vertical components (denoted by a subscript $y$), we can solve for $T$. Again, since the person is stationary, Newton’s second law implies that net $F_y = 0$. Thus, as illustrated in the free-body diagram in **Figure 4.18**,

$$F_{\text{net}} = T_{Ly} + T_{Ry} - w = 0. \quad (4.49)$$

Observing **Figure 4.18**, we can use trigonometry to determine the relationship between $T_{Ly}$, $T_{Ry}$, and $T$. As we determined from the analysis in the horizontal direction, $T_L = T_R = T$: 
\[
\sin (5.0^\circ) = \frac{T_{Ly}}{T_L} \\
T_{Ly} = T_L \sin (5.0^\circ) = T \sin (5.0^\circ) \\
\sin (5.0^\circ) = \frac{T_{Ry}}{T_R} \\
T_{Ry} = T_R \sin (5.0^\circ) = T \sin (5.0^\circ).
\]

Now, we can substitute the values for \( T_{Ly} \) and \( T_{Ry} \), into the net force equation in the vertical direction:

\[
F_{nety} = T_{Ly} + T_{Ry} - w = 0 \\
F_{nety} = T \sin (5.0^\circ) + T \sin (5.0^\circ) - w = 0 \\
2 T \sin (5.0^\circ) - w = 0 \\
2 T \sin (5.0^\circ) = w
\]

and

\[
T = \frac{w}{2 \sin (5.0^\circ)} = \frac{mg}{2 \sin (5.0^\circ)}.
\]

so that

\[
T = \frac{(70.0 \text{ kg})(9.80 \text{ m/s}^2)}{2(0.0872)},
\]

and the tension is

\[
T = 3900 \text{ N}.
\]

**Discussion**

Note that the vertical tension in the wire acts as a normal force that supports the weight of the tightrope walker. The tension is almost six times the 686-N weight of the tightrope walker. Since the wire is nearly horizontal, the vertical component of its tension is only a small fraction of the tension in the wire. The large horizontal components are in opposite directions and cancel, and so most of the tension in the wire is not used to support the weight of the tightrope walker.

If we wish to create a very large tension, all we have to do is exert a force perpendicular to a flexible connector, as illustrated in Figure 4.19. As we saw in the last example, the weight of the tightrope walker acted as a force perpendicular to the rope. We saw that the tension in the rope related to the weight of the tightrope walker in the following way:

\[
T = \frac{w}{2 \sin (\theta)}.
\]

We can extend this expression to describe the tension \( T \) created when a perpendicular force (\( F_\perp \)) is exerted at the middle of a flexible connector:

\[
T = \frac{F_\perp}{2 \sin (\theta)}.
\]

Note that \( \theta \) is the angle between the horizontal and the bent connector. In this case, \( T \) becomes very large as \( \theta \) approaches zero. Even the relatively small weight of any flexible connector will cause it to sag, since an infinite tension would result if it were horizontal (i.e., \( \theta = 0 \) and \( \sin \theta = 0 \)). (See Figure 4.19.)

Figure 4.19 We can create a very large tension in the chain by pushing on it perpendicular to its length, as shown. Suppose we wish to pull a car out of the mud when no tow truck is available. Each time the car moves forward, the chain is tightened to keep it as nearly straight as possible. The tension in the chain is given by \( T = \frac{F_\perp}{2 \sin (\theta)} \); since \( \theta \) is small, \( T \) is very large. This situation is analogous to the tightrope walker shown in Figure 4.17, except that the tensions shown here are those transmitted to the car and the tree rather than those acting at the point where \( F_\perp \) is applied.
Extended Topic: Real Forces and Inertial Frames

There is another distinction among forces in addition to the types already mentioned. Some forces are real, whereas others are not. Real forces are those that have some physical origin, such as the gravitational pull. Contrastingly, fictitious forces are those that arise simply because an observer is in an accelerating frame of reference, such as one that rotates (like a merry-go-round) or undergoes linear acceleration (like a car slowing down). For example, if a satellite is heading due north above Earth’s northern hemisphere, then to an observer on Earth it will appear to experience a force to the west that has no physical origin. Of course, what is happening here is that Earth is rotating toward the east and moves east under the satellite. In Earth’s frame this looks like a westward force on the satellite, or it can be interpreted as a violation of Newton’s first law (the law of inertia). An inertial frame of reference is one in which all forces are real and, equivalently, one in which Newton's laws have the simple forms given in this chapter.

Earth’s rotation is slow enough that Earth is nearly an inertial frame. You ordinarily must perform precise experiments to observe fictitious forces and the slight departures from Newton’s laws, such as the effect just described. On the large scale, such as for the rotation of weather systems and ocean currents, the effects can be easily observed.

The crucial factor in determining whether a frame of reference is inertial is whether it accelerates or rotates relative to a known inertial frame. Unless stated otherwise, all phenomena discussed in this text are considered in inertial frames.

All the forces discussed in this section are real forces, but there are a number of other real forces, such as lift and thrust, that are not discussed in this section. They are more specialized, and it is not necessary to discuss every type of force. It is natural, however, to ask where the basic simplicity we seek to find in physics is in the long list of forces. Are some more basic than others? Are some different manifestations of the same underlying force? The answer to both questions is yes, as will be seen in the next (extended) section and in the treatment of modern physics later in the text.

4.6 Problem-Solving Strategies

Success in problem solving is obviously necessary to understand and apply physical principles, not to mention the more immediate need of passing exams. The basics of problem solving, presented earlier in this text, are followed here, but specific strategies useful in applying Newton’s laws of motion are emphasized. These techniques also reinforce concepts that are useful in many other areas of physics. Many problem-solving strategies are stated outright in the worked examples, and so the following techniques should reinforce skills you have already begun to develop.

Problem-Solving Strategy for Newton’s Laws of Motion

Step 1. As usual, it is first necessary to identify the physical principles involved. Once it is determined that Newton’s laws of motion are involved (if the problem involves forces), it is particularly important to draw a careful sketch of the situation. Such a sketch is shown in Figure 4.21(a). Then, as in Figure 4.21(b), use arrows to represent all forces, label them carefully, and make their lengths and directions correspond to the forces they represent (whenever sufficient information exists).
Step 2. Identify what needs to be determined and what is known or can be inferred from the problem as stated. That is, make a list of knowns and unknowns. Then carefully determine the system of interest. This decision is a crucial step, since Newton’s second law involves only external forces. Once the system of interest has been identified, it becomes possible to determine which forces are external and which are internal, a necessary step to employ Newton’s second law. (See Figure 4.21(c).)

Newton’s third law may be used to identify whether forces are exerted between components of a system (internal) or between the system and something outside (external). As illustrated earlier in this chapter, the system of interest depends on what question we need to answer. This choice becomes easier with practice, eventually developing into an almost unconscious process. Skill in clearly defining systems will be beneficial in later chapters as well.

A diagram showing the system of interest and all of the external forces is called a free-body diagram. Only forces are shown on free-body diagrams, not acceleration or velocity. We have drawn several of these in worked examples. Figure 4.21(c) shows a free-body diagram for the system of interest. Note that no internal forces are shown in a free-body diagram.

Step 3. Once a free-body diagram is drawn, Newton’s second law can be applied to solve the problem. This is done in Figure 4.21(d) for a particular situation. In general, once external forces are clearly identified in free-body diagrams, it should be a straightforward task to put them into equation form and solve for the unknown, as done in all previous examples. If the problem is one-dimensional—that is, if all forces are parallel—then they add like scalars. If the problem is two-dimensional, then it must be broken down into a pair of one-dimensional problems. This is done by projecting the force vectors onto a set of axes chosen for convenience. As seen in previous examples, the choice of axes can simplify the problem. For example, when an incline is involved, a set of axes with one axis parallel to the incline and one perpendicular to it is most convenient. It is almost always convenient to make one axis parallel to the direction of motion, if this is known.

Applying Newton’s Second Law

Before you write net force equations, it is critical to determine whether the system is accelerating in a particular direction. If the acceleration is zero in a particular direction, then the net force is zero in that direction. Similarly, if the acceleration is nonzero in a particular direction, then the net force is described by the equation: \( F_{\text{net}} = ma \).

For example, if the system is accelerating in the horizontal direction, but it is not accelerating in the vertical direction, then you will have the following conclusions:

\[
F_{\text{net},x} = ma, \quad (4.57)
\]
$F_{\text{net}, y} = 0. \quad (4.58)$

You will need this information in order to determine unknown forces acting in a system.

Step 4. As always, check the solution to see whether it is reasonable. In some cases, this is obvious. For example, it is reasonable to find that friction causes an object to slide down an incline more slowly than when no friction exists. In practice, intuition develops gradually through problem solving, and with experience it becomes progressively easier to judge whether an answer is reasonable. Another way to check your solution is to check the units. If you are solving for force and end up with units of m/s, then you have made a mistake.

### 4.7 Further Applications of Newton's Laws of Motion

There are many interesting applications of Newton's laws of motion, a few more of which are presented in this section. These serve also to illustrate some further subtleties of physics and to help build problem-solving skills.

#### Example 4.7 Drag Force on a Barge

Suppose two tugboats push on a barge at different angles, as shown in Figure 4.22. The first tugboat exerts a force of $2.7 \times 10^5$ N in the $x$-direction, and the second tugboat exerts a force of $3.6 \times 10^5$ N in the $y$-direction.

![Figure 4.22](image)

(a) A view from above of two tugboats pushing on a barge. (b) The free-body diagram for the ship contains only forces acting in the plane of the water. It omits the two vertical forces—the weight of the barge and the buoyant force of the water supporting it cancel and are not shown. Since the applied forces are perpendicular, the $x$- and $y$-axes are in the same direction as $F_x$ and $F_y$. The problem quickly becomes a one-dimensional problem along the direction of $F_{\text{app}}$, since friction is in the direction opposite to $F_{\text{app}}$.

If the mass of the barge is $5.0 \times 10^6$ kg and its acceleration is observed to be $7.5 \times 10^{-2}$ m/s$^2$ in the direction shown, what is the drag force of the water on the barge resisting the motion? (Note: drag force is a frictional force exerted by fluids, such as air or water. The drag force opposes the motion of the object.)

**Strategy**

The directions and magnitudes of acceleration and the applied forces are given in Figure 4.22(a). We will define the total force of the tugboats on the barge as $F_{\text{app}}$ so that:

$$F_{\text{app}} = F_x + F_y \quad (4.59)$$

Since the barge is flat bottomed, the drag of the water $F_D$ will be in the direction opposite to $F_{\text{app}}$, as shown in the free-body diagram in Figure 4.22(b). The system of interest here is the barge, since the forces on it are given as well as its acceleration. Our strategy is to find the magnitude and direction of the net applied force $F_{\text{app}}$, and then apply Newton's second law to solve for the drag force $F_D$.

**Solution**

Since $F_x$ and $F_y$ are perpendicular, the magnitude and direction of $F_{\text{app}}$ are easily found. First, the resultant magnitude is given by the Pythagorean theorem:
\[ F_{\text{app}} = \sqrt{F_x^2 + F_y^2} \]

The angle is given by

\[ \theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) \]

\[ \theta = \tan^{-1}\left(\frac{3.6 \times 10^5 \text{ N}}{2.7 \times 10^5 \text{ N}}\right) = 53^\circ, \]

which we know, because of Newton’s first law, is the same direction as the acceleration. \( \mathbf{F}_D \) is in the opposite direction of \( \mathbf{F}_{\text{app}} \), since it acts to slow down the acceleration. Therefore, the net external force is in the same direction as \( \mathbf{F}_{\text{app}} \), but its magnitude is slightly less than \( \mathbf{F}_{\text{app}} \). The problem is now one-dimensional. From Figure 4.22(b), we can see that

\[ F_{\text{net}} = F_{\text{app}} - F_D. \]

But Newton’s second law states that

\[ F_{\text{net}} = ma. \]

Thus,

\[ F_{\text{app}} - F_D = ma. \]

This can be solved for the magnitude of the drag force of the water \( \mathbf{F}_D \) in terms of known quantities:

\[ F_D = F_{\text{app}} - ma. \]

Substituting known values gives

\[ F_D = (4.5 \times 10^5 \text{ N}) - (5.0 \times 10^6 \text{ kg})(7.5 \times 10^{-2} \text{ m/s}^2) = 7.5 \times 10^4 \text{ N}. \]

The direction of \( \mathbf{F}_D \) has already been determined to be in the direction opposite to \( \mathbf{F}_{\text{app}} \), or at an angle of 53° south of west.

**Discussion**

The numbers used in this example are reasonable for a moderately large barge. It is certainly difficult to obtain larger accelerations with tugboats, and small speeds are desirable to avoid running the barge into the docks. Drag is relatively small for a well-designed hull at low speeds, consistent with the answer to this example, where \( \mathbf{F}_D \) is less than 1/600th of the weight of the ship.

In the earlier example of a tightrope walker we noted that the tensions in wires supporting a mass were equal only because the angles on either side were equal. Consider the following example, where the angles are not equal; slightly more trigonometry is involved.

**Example 4.8 Different Tensions at Different Angles**

Consider the traffic light (mass 15.0 kg) suspended from two wires as shown in Figure 4.23. Find the tension in each wire, neglecting the masses of the wires.
Figure 4.23 A traffic light is suspended from two wires. (b) Some of the forces involved. (c) Only forces acting on the system are shown here. The free-body diagram for the traffic light is also shown. (d) The forces projected onto vertical ($y$) and horizontal ($x$) axes. The horizontal components of the tensions must cancel, and the sum of the vertical components of the tensions must equal the weight of the traffic light. (e) The free-body diagram shows the vertical and horizontal forces acting on the traffic light.

Strategy

The system of interest is the traffic light, and its free-body diagram is shown in Figure 4.23(c). The three forces involved are not parallel, and so they must be projected onto a coordinate system. The most convenient coordinate system has one axis vertical and one horizontal, and the vector projections on it are shown in part (d) of the figure. There are two unknowns in this problem ($T_1$ and $T_2$), so two equations are needed to find them. These two equations come from applying Newton’s second law along the vertical and horizontal axes, noting that the net external force is zero along each axis because acceleration is zero.

Solution

First consider the horizontal or $x$-axis:

$$F_{netx} = T_{2x} - T_{1x} = 0.$$  \hspace{1cm} (4.67)

Thus, as you might expect,
\[ T_{1x} = T_{2x}. \]  

(4.68)

This gives us the following relationship between \( T_1 \) and \( T_2 \):

\[ T_1 \cos(30º) = T_2 \cos(45º). \]  

(4.69)

Thus,

\[ T_2 = (1.225)T_1. \]  

(4.70)

Note that \( T_1 \) and \( T_2 \) are not equal in this case, because the angles on either side are not equal. It is reasonable that \( T_2 \) ends up being greater than \( T_1 \), because it is exerted more vertically than \( T_1 \).

Now consider the force components along the vertical or \( y \)-axis:

\[ F_{\text{net}y} = T_{1y} + T_{2y} - w = 0. \]  

(4.71)

This implies

\[ T_{1y} + T_{2y} = w. \]  

(4.72)

Substituting the expressions for the vertical components gives

\[ T_1 \sin(30º) + T_2 \sin(45º) = w. \]  

(4.73)

There are two unknowns in this equation, but substituting the expression for \( T_2 \) in terms of \( T_1 \) reduces this to one equation with one unknown:

\[ T_1(0.500) + (1.225T_1)(0.707) = w = mg. \]  

(4.74)

which yields

\[ (1.366)T_1 = (15.0 \text{ kg})(9.80 \text{ m/s}^2). \]  

(4.75)

Solving this last equation gives the magnitude of \( T_1 \) to be

\[ T_1 = 108 \text{ N}. \]  

(4.76)

Finally, the magnitude of \( T_2 \) is determined using the relationship between them, \( T_2 = 1.225 T_1 \), found above. Thus we obtain

\[ T_2 = 132 \text{ N}. \]  

(4.77)

Discussion

Both tensions would be larger if both wires were more horizontal, and they will be equal if and only if the angles on either side are the same (as they were in the earlier example of a tightrope walker).

The bathroom scale is an excellent example of a normal force acting on a body. It provides a quantitative reading of how much it must push upward to support the weight of an object. But can you predict what you would see on the dial of a bathroom scale if you stood on it during an elevator ride? Will you see a value greater than your weight when the elevator starts up? What about when the elevator moves upward at a constant speed: will the scale still read more than your weight at rest? Consider the following example.

**Example 4.9 What Does the Bathroom Scale Read in an Elevator?**

**Figure 4.24** shows a 75.0-kg man (weight of about 165 lb) standing on a bathroom scale in an elevator. Calculate the scale reading: (a) if the elevator accelerates upward at a rate of \( 1.20 \text{ m/s}^2 \), and (b) if the elevator moves upward at a constant speed of 1 m/s.
Figure 4.24 (a) The various forces acting when a person stands on a bathroom scale in an elevator. The arrows are approximately correct for when the elevator is accelerating upward—broken arrows represent forces too large to be drawn to scale. $T$ is the tension in the supporting cable, $w$ is the weight of the person, $w_s$ is the weight of the scale, $w_e$ is the weight of the elevator, $F_p$ is the force of the person on the scale, $F_s$ is the force of the scale on the person, $F_t$ is the force of the scale on the floor of the elevator, and $N$ is the force of the floor upward on the scale. (b) The free-body diagram shows only the external forces acting on the designated system of interest—the person.

**Strategy**

If the scale is accurate, its reading will equal $F_p$, the magnitude of the force the person exerts downward on it. Figure 4.24(a) shows the numerous forces acting on the elevator, scale, and person. It makes this one-dimensional problem look much more formidable than if the person is chosen to be the system of interest and a free-body diagram is drawn as in Figure 4.24(b). Analysis of the free-body diagram using Newton’s laws can produce answers to both parts (a) and (b) of this example, as well as some other questions that might arise. The only forces acting on the person are his weight $w$ and the upward force of the scale $F_s$. According to Newton’s third law $F_p$ and $F_s$ are equal in magnitude and opposite in direction, so that we need to find $F_s$ in order to find what the scale reads. We can do this, as usual, by applying Newton’s second law,

$$F_{net} = ma. \tag{4.78}$$

From the free-body diagram we see that $F_{net} = F_s - w$, so that

$$F_s - w = ma. \tag{4.79}$$

Solving for $F_s$ gives an equation with only one unknown:

$$F_s = ma + w, \tag{4.80}$$

or, because $w = mg$, simply

$$F_s = ma + mg. \tag{4.81}$$

No assumptions were made about the acceleration, and so this solution should be valid for a variety of accelerations in addition to the ones in this exercise.

**Solution for (a)**
In this part of the problem, \( a = 1.20 \text{ m/s}^2 \), so that
\[
F_s = (75.0 \text{ kg})(1.20 \text{ m/s}^2) + (75.0 \text{ kg})(9.80 \text{ m/s}^2),
\]
yielding
\[
F_s = 825 \text{ N}.
\]

**Discussion for (a)**
This is about 185 lb. What would the scale have read if he were stationary? Since his acceleration would be zero, the force of the scale would be equal to his weight:
\[
F_{net} = ma = 0 = F_s - w
\]
\[
F_s = w = mg
\]
\[
F_s = (75.0 \text{ kg})(9.80 \text{ m/s}^2)
\]
\[
F_s = 735 \text{ N}.
\]
So, the scale reading in the elevator is greater than his 735-N (165 lb) weight. This means that the scale is pushing up on the person with a force greater than his weight, as it must in order to accelerate him upward. Clearly, the greater the acceleration of the elevator, the greater the scale reading, consistent with what you feel in rapidly accelerating versus slowly accelerating elevators.

**Solution for (b)**
Now, what happens when the elevator reaches a constant upward velocity? Will the scale still read more than his weight?
For any constant velocity—up, down, or stationary—acceleration is zero because \( a = \frac{\Delta v}{\Delta t} \), and \( \Delta v = 0 \).

Thus,
\[
F_s = ma + mg = 0 + mg.
\]

Now
\[
F_s = (75.0 \text{ kg})(9.80 \text{ m/s}^2),
\]
which gives
\[
F_s = 735 \text{ N}.
\]

**Discussion for (b)**
The scale reading is 735 N, which equals the person's weight. This will be the case whenever the elevator has a constant velocity—moving up, moving down, or stationary.

The solution to the previous example also applies to an elevator accelerating downward, as mentioned. When an elevator accelerates downward, \( a \) is negative, and the scale reading is less than the weight of the person, until a constant downward velocity is reached, at which time the scale reading again becomes equal to the person's weight. If the elevator is in free-fall and accelerating downward at \( g \), then the scale reading will be zero and the person will appear to be weightless.

**Integrating Concepts: Newton’s Laws of Motion and Kinematics**

Physics is most interesting and most powerful when applied to general situations that involve more than a narrow set of physical principles. Newton’s laws of motion can also be integrated with other concepts that have been discussed previously in this text to solve problems of motion. For example, forces produce accelerations, a topic of kinematics, and hence the relevance of earlier chapters. When approaching problems that involve various types of forces, acceleration, velocity, and/or position, use the following steps to approach the problem:

**Problem-Solving Strategy**

1. **Identify which physical principles are involved.** Listing the givens and the quantities to be calculated will allow you to identify the principles involved.
2. **Solve the problem using strategies outlined in the text.** If these are available for the specific topic, you should refer to them. You should also refer to the sections of the text that deal with a particular topic. The following worked example illustrates how these strategies are applied to an integrated concept problem.

**Example 4.10 What Force Must a Soccer Player Exert to Reach Top Speed?**

A soccer player starts from rest and accelerates forward, reaching a velocity of 8.00 m/s in 2.50 s. (a) What was his average acceleration? (b) What average force did he exert backward on the ground to achieve this acceleration? The player’s mass
is 70.0 kg, and air resistance is negligible.

**Strategy**

1. To solve an integrated concept problem, we must first identify the physical principles involved and identify the chapters in which they are found. Part (a) of this example considers acceleration along a straight line. This is a topic of kinematics. Part (b) deals with force, a topic of dynamics found in this chapter.

2. The following solutions to each part of the example illustrate how the specific problem-solving strategies are applied. These involve identifying knowns and unknowns, checking to see if the answer is reasonable, and so forth.

**Solution for (a)**

We are given the initial and final velocities (zero and 8.00 m/s forward); thus, the change in velocity is $\Delta v = 8.00 \text{ m/s}$. We are given the elapsed time, and so $\Delta t = 2.50 \text{ s}$. The unknown is acceleration, which can be found from its definition:

$$a = \frac{\Delta v}{\Delta t}.$$  \hspace{1cm} (4.88)

Substituting the known values yields

$$a = \frac{8.00 \text{ m/s}}{2.50 \text{ s}} = 3.20 \text{ m/s}^2.$$  \hspace{1cm} (4.89)

**Discussion for (a)**

This is an attainable acceleration for an athlete in good condition.

**Solution for (b)**

Here we are asked to find the average force the player exerts backward to achieve this forward acceleration. Neglecting air resistance, this would be equal in magnitude to the net external force on the player, since this force causes his acceleration. Since we now know the player’s acceleration and are given his mass, we can use Newton’s second law to find the force exerted. That is,

$$F_{\text{net}} = ma.$$  \hspace{1cm} (4.90)

Substituting the known values of $m$ and $a$ gives

$$F_{\text{net}} = (70.0 \text{ kg})(3.20 \text{ m/s}^2) = 224 \text{ N}.$$  \hspace{1cm} (4.91)

**Discussion for (b)**

This is about 50 pounds, a reasonable average force.

This worked example illustrates how to apply problem-solving strategies to situations that include topics from different chapters. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknown using familiar problem-solving strategies. These strategies are found throughout the text, and many worked examples show how to use them for single topics. You will find these techniques for integrated concept problems useful in applications of physics outside of a physics course, such as in your profession, in other science disciplines, and in everyday life. The following problems will build your skills in the broad application of physical principles.

**4.8 Extended Topic: The Four Basic Forces—An Introduction**

One of the most remarkable simplifications in physics is that only four distinct forces account for all known phenomena. In fact, nearly all of the forces we experience directly are due to only one basic force, called the electromagnetic force. (The gravitational force is the only force we experience directly that is not electromagnetic.) This is a tremendous simplification of the myriad of apparently different forces we can list, only a few of which were discussed in the previous section. As we will see, the basic forces are all thought to act through the exchange of microscopic carrier particles, and the characteristics of the basic forces are determined by the types of particles exchanged. Action at a distance, such as the gravitational force of Earth on the Moon, is explained by the existence of a force field rather than by “physical contact.”

The four basic forces are the gravitational force, the electromagnetic force, the weak nuclear force, and the strong nuclear force. Their properties are summarized in Table 4.1. Since the weak and strong nuclear forces act over an extremely short range, the size of a nucleus or less, we do not experience them directly, although they are crucial to the very structure of matter. These forces determine which nuclei are stable and which decay, and they are the basis of the release of energy in certain nuclear reactions. Nuclear forces determine not only the stability of nuclei, but also the relative abundance of elements in nature. The properties of the nucleus of an atom determine the number of electrons it has and, thus, indirectly determine the chemistry of the atom. More will be said of all of these topics in later chapters.
The four basic forces will be encountered in more detail as you progress through the text. The gravitational force is defined in Uniform Circular Motion and Gravitation, electric force in Electric Charge and Electric Field, magnetic force in Magnetism, and nuclear forces in Radioactivity and Nuclear Physics. On a macroscopic scale, electromagnetism and gravity are the basis for all forces. The nuclear forces are vital to the substructure of matter, but they are not directly experienced on the macroscopic scale.

<table>
<thead>
<tr>
<th>Force</th>
<th>Approximate Relative Strengths</th>
<th>Range</th>
<th>Attraction/Repulsion</th>
<th>Carrier Particle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravitational</td>
<td>$10^{-38}$</td>
<td>$\infty$</td>
<td>only</td>
<td>Graviton</td>
</tr>
<tr>
<td>Electromagnetic</td>
<td>$10^{-2}$</td>
<td>$\infty$</td>
<td>attractive and repulsive</td>
<td>Photon</td>
</tr>
<tr>
<td>Weak nuclear</td>
<td>$10^{-13}$</td>
<td>$&lt;10^{-18}$ m</td>
<td>attractive and repulsive</td>
<td>$W^+, W^-, Z^0$</td>
</tr>
<tr>
<td>Strong nuclear</td>
<td>1</td>
<td>$&lt;10^{-15}$ m</td>
<td>attractive and repulsive</td>
<td>gluons</td>
</tr>
</tbody>
</table>

The gravitational force is surprisingly weak—it is only because gravity is always attractive that we notice it at all. Our weight is the gravitational force due to the entire Earth acting on us. On the very large scale, as in astronomical systems, the gravitational force is the dominant force determining the motions of moons, planets, stars, and galaxies. The gravitational force also affects the nature of space and time. As we shall see later in the study of general relativity, space is curved in the vicinity of very massive bodies, such as the Sun, and time actually slows down near massive bodies.

Electromagnetic forces can be either attractive or repulsive. They are long-range forces, which act over extremely large distances, and they nearly cancel for macroscopic objects. (Remember that it is the net external force that is important.) If they did not cancel, electromagnetic forces would completely overwhelm the gravitational force. The electromagnetic force is a combination of electrical forces (such as those that cause static electricity) and magnetic forces (such as those that affect a compass needle). These two forces were thought to be quite distinct until early in the 19th century, when scientists began to discover that they are different manifestations of the same force. This discovery is a classical case of the unification of forces. Similarly, friction, tension, and all of the other classes of forces we experience directly (except gravity, of course) are due to electromagnetic interactions of atoms and molecules. It is still convenient to consider these forces separately in specific applications, however, because of the ways they manifest themselves.

Physicists are now exploring whether the four basic forces are in some way related. Attempts to unify all forces into one come under the rubric of Grand Unified Theories (GUTs), with which there has been some success in recent years. It is now known that under conditions of extremely high density and temperature, such as existed in the early universe, the electromagnetic and weak nuclear forces are indistinguishable. They can now be considered to be different manifestations of one force, called the electroweak force. So the list of four has been reduced in a sense to only three. Further progress in unifying all forces is proving difficult—especially the inclusion of the gravitational force, which has the special characteristics of affecting the space and time in which the other forces exist.

While the unification of forces will not affect how we discuss forces in this text, it is fascinating that such underlying simplicity exists in the face of the overt complexity of the universe. There is no reason that nature must be simple—it simply is.

**Concept Connections: Unifying Forces**

Attempts to unify the four basic forces are discussed in relation to elementary particles later in this text. By “unify” we mean finding connections between the forces that show that they are different manifestations of a single force. Even if such unification is achieved, the forces will retain their separate characteristics on the macroscopic scale and may be identical only under extreme conditions such as those existing in the early universe.

**Concept Connections: The Four Basic Forces**

The four basic forces will be encountered in more detail as you progress through the text. The gravitational force is defined in Uniform Circular Motion and Gravitation, electric force in Electric Charge and Electric Field, magnetic force in Magnetism, and nuclear forces in Radioactivity and Nuclear Physics. On a macroscopic scale, electromagnetism and gravity are the basis for all forces. The nuclear forces are vital to the substructure of matter, but they are not directly experienced on the macroscopic scale. The four basic forces will be encountered in more detail as you progress through the text. The gravitational force is defined in Uniform Circular Motion and Gravitation, electric force in Electric Charge and Electric Field, magnetic force in Magnetism, and nuclear forces in Radioactivity and Nuclear Physics. On a macroscopic scale, electromagnetism and gravity are the basis for all forces. The nuclear forces are vital to the substructure of matter, but they are not directly experienced on the macroscopic scale. The four basic forces will be encountered in more detail as you progress through the text. The gravitational force is defined in Uniform Circular Motion and Gravitation, electric force in Electric Charge and Electric Field, magnetic force in Magnetism, and nuclear forces in Radioactivity and Nuclear Physics. On a macroscopic scale, electromagnetism and gravity are the basis for all forces. The nuclear forces are vital to the substructure of matter, but they are not directly experienced on the macroscopic scale. The four basic forces will be encountered in more detail as you progress through the text. The gravitational force is defined in Uniform Circular Motion and Gravitation, electric force in Electric Charge and Electric Field, magnetic force in Magnetism, and nuclear forces in Radioactivity and Nuclear Physics. On a macroscopic scale, electromagnetism and gravity are the basis for all forces. The nuclear forces are vital to the substructure of matter, but they are not directly experienced on the macroscopic scale. The four basic forces will be encountered in more detail as you progress through the text. The gravitational force is defined in Uniform Circular Motion and Gravitation, electric force in Electric Charge and Electric Field, magnetic force in Magnetism, and nuclear forces in Radioactivity and Nuclear Physics. On a macroscopic scale, electromagnetism and gravity are the basis for all forces. The nuclear forces are vital to the substructure of matter, but they are not directly experienced on the macroscopic scale. The four basic forces will be encountered in more detail as you progress through the text. The gravitational force is defined in Uniform Circular Motion and Gravitation, electric force in Electric Charge and Electric Field, magnetic force in Magnetism, and nuclear forces in Radioactivity and Nuclear Physics. On a macroscopic scale, electromagnetism and gravity are the basis for all forces. The nuclear forces are vital to the substructure of matter, but they are not directly experienced on the macroscopic scale.
placed in it. Earth’s gravitational field, for example, is a function of the mass of Earth and the distance from its center, independent of the presence of other masses. The concept of a field is useful because equations can be written for force fields surrounding objects (for gravity, this yields $w = mg$ at Earth’s surface), and motions can be calculated from these equations. (See Figure 4.25.)

![Figure 4.25](image)

Figure 4.25 The electric force field between a positively charged particle and a negatively charged particle. When a positive test charge is placed in the field, the charge will experience a force in the direction of the force field lines.

**Concept Connections: Force Fields**

The concept of a force field is also used in connection with electric charge and is presented in Electric Charge and Electric Field. It is also a useful idea for all the basic forces, as will be seen in Particle Physics. Fields help us to visualize forces and how they are transmitted, as well as to describe them with precision and to link forces with subatomic carrier particles.

The field concept has been applied very successfully; we can calculate motions and describe nature to high precision using field equations. As useful as the field concept is, however, it leaves unanswered the question of what carries the force. It has been proposed in recent decades, starting in 1935 with Hideki Yukawa’s (1907–1981) work on the strong nuclear force, that all forces are transmitted by the exchange of elementary particles. We can visualize particle exchange as analogous to macroscopic phenomena such as two people passing a basketball back and forth, thereby exerting a repulsive force without touching one another. (See Figure 4.26.)
Figure 4.26 The exchange of masses resulting in repulsive forces. (a) The person throwing the basketball exerts a force $F_{p1}$ on it toward the other person and feels a reaction force $F_B$ away from the second person. (b) The person catching the basketball exerts a force $F_{p2}$ on it to stop the ball and feels a reaction force $F'_B$ away from the first person. (c) The analogous exchange of a meson between a proton and a neutron carries the strong nuclear forces $F_{exch}$ and $F'_{exch}$ between them. An attractive force can also be exerted by the exchange of a mass—if person 2 pulled the basketball away from the first person as he tried to retain it, then the force between them would be attractive.

This idea of particle exchange deepens rather than contradicts field concepts. It is more satisfying philosophically to think of something physical actually moving between objects acting at a distance. Table 4.1 lists the exchange or carrier particles, both observed and proposed, that carry the four forces. But the real fruit of the particle-exchange proposal is that searches for Yukawa’s proposed particle found it and a number of others that were completely unexpected, stimulating yet more research. All of this research eventually led to the proposal of quarks as the underlying substructure of matter, which is a basic tenet of GUTs. If successful, these theories will explain not only forces, but also the structure of matter itself. Yet physics is an experimental science, so the test of these theories must lie in the domain of the real world. As of this writing, scientists at the CERN laboratory in Switzerland are starting to test these theories using the world’s largest particle accelerator: the Large Hadron Collider. This accelerator (27 km in circumference) allows two high-energy proton beams, traveling in opposite directions, to collide. An energy of 14 trillion electron volts will be available. It is anticipated that some new particles, possibly force carrier particles, will be found. (See Figure 4.27.) One of the force carriers of high interest that researchers hope to detect is the Higgs boson. The observation of its properties might tell us why different particles have different masses.

Figure 4.27 The world’s largest particle accelerator spans the border between Switzerland and France. Two beams, traveling in opposite directions close to the speed of light, collide in a tube similar to the central tube shown here. External magnets determine the beam’s path. Special detectors will analyze particles created in these collisions. Questions as broad as what is the origin of mass and what was matter like the first few seconds of our universe will be explored. This accelerator began preliminary operation in 2008. (credit: Frank Hommes)
Tiny particles also have wave-like behavior, something we will explore more in a later chapter. To better understand force-carrier particles from another perspective, let us consider gravity. The search for gravitational waves has been going on for a number of years. Almost 100 years ago, Einstein predicted the existence of these waves as part of his general theory of relativity. Gravitational waves are created during the collision of massive stars, in black holes, or in supernova explosions—like shock waves. These gravitational waves will travel through space from such sites much like a pebble dropped into a pond sends out ripples—except these waves move at the speed of light. A detector apparatus has been built in the U.S., consisting of two large installations nearly 3000 km apart—one in Washington state and one in Louisiana! The facility is called the Laser Interferometer Gravitational-Wave Observatory (LIGO). Each installation is designed to use optical lasers to examine any slight shift in the relative positions of two masses due to the effect of gravity waves. The two sites allow simultaneous measurements of these small effects to be separated from other natural phenomena, such as earthquakes. Initial operation of the detectors began in 2002, and work is proceeding on increasing their sensitivity. Similar installations have been built in Italy (VIRGO), Germany (GEO600), and Japan (TAMA300) to provide a worldwide network of gravitational wave detectors.

International collaboration in this area is moving into space with the joint EU/US project LISA (Laser Interferometer Space Antenna). Earthquakes and other earthly noises will be no problem for these monitoring spacecraft. LISA will complement LIGO by looking at much more massive black holes through the observation of gravitational-wave sources emitting much larger wavelengths. Three satellites will be placed in space above Earth in an equilateral triangle (with 5,000,000-km sides) (Figure 4.28). The system will measure the relative positions of each satellite to detect passing gravitational waves. Accuracy to within 10% of the size of an atom will be needed to detect any waves. The launch of this project might be as early as 2018.

“I’m sure LIGO will tell us something about the universe that we didn’t know before. The history of science tells us that any time you go where you haven’t been before, you usually find something that really shakes the scientific paradigms of the day. Whether gravitational wave astrophysics will do that, only time will tell.” —David Reitze, LIGO Input Optics Manager, University of Florida

The ideas presented in this section are but a glimpse into topics of modern physics that will be covered in much greater depth in later chapters.

Glossary

**acceleration**: the rate at which an object’s velocity changes over a period of time

**carrier particle**: a fundamental particle of nature that is surrounded by a characteristic force field; photons are carrier particles of the electromagnetic force

**dynamics**: the study of how forces affect the motion of objects and systems

**external force**: a force acting on an object or system that originates outside of the object or system

**force**: a push or pull on an object with a specific magnitude and direction; can be represented by vectors; can be expressed as a multiple of a standard force

**force field**: a region in which a test particle will experience a force

**free-body diagram**: a sketch showing all of the external forces acting on an object or system; the system is represented by a dot, and the forces are represented by vectors extending outward from the dot

**free-fall**: a situation in which the only force acting on an object is the force due to gravity
friction: a force past each other of objects that are touching; examples include rough surfaces and air resistance

inertia: the tendency of an object to remain at rest or remain in motion

inertial frame of reference: a coordinate system that is not accelerating; all forces acting in an inertial frame of reference are real forces, as opposed to fictitious forces that are observed due to an accelerating frame of reference

law of inertia: see Newton’s first law of motion

mass: the quantity of matter in a substance; measured in kilograms

net external force: the vector sum of all external forces acting on an object or system; causes a mass to accelerate

Newton’s first law of motion: a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force; also known as the law of inertia

Newton’s second law of motion: the net external force \( F_{\text{net}} \) on an object with mass \( m \) is proportional to and in the same direction as the acceleration of the object, \( a \), and inversely proportional to the mass; defined mathematically as

\[
a = \frac{F_{\text{net}}}{m}
\]

Newton’s third law of motion: whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts

normal force: the force that a surface applies to an object to support the weight of the object; acts perpendicular to the surface on which the object rests

system: defined by the boundaries of an object or collection of objects being observed; all forces originating from outside of the system are considered external forces

tension: the pulling force that acts along a medium, especially a stretched flexible connector, such as a rope or cable; when a rope supports the weight of an object, the force on the object due to the rope is called a tension force

thrust: a reaction force that pushes a body forward in response to a backward force; rockets, airplanes, and cars are pushed forward by a thrust reaction force

weight: the force \( w \) due to gravity acting on an object of mass \( m \); defined mathematically as: \( w = mg \), where \( g \) is the magnitude and direction of the acceleration due to gravity

**Section Summary**

4.1 Development of Force Concept

- **Dynamics** is the study of how forces affect the motion of objects.
- **Force** is a push or pull that can be defined in terms of various standards, and it is a vector having both magnitude and direction.
- **External forces** are any outside forces that act on a body. A **free-body diagram** is a drawing of all external forces acting on a body.

4.2 Newton’s First Law of Motion: Inertia

- **Newton’s first law of motion** states that a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force. This is also known as the **law of inertia**.
- **Inertia** is the tendency of an object to remain at rest or remain in motion. Inertia is related to an object’s mass.
- **Mass** is the quantity of matter in a substance.

4.3 Newton’s Second Law of Motion: Concept of a System

- Acceleration, \( a \), is defined as a change in velocity, meaning a change in its magnitude or direction, or both.
- An external force is one acting on a system from outside the system, as opposed to internal forces, which act between components within the system.
- Newton’s second law of motion states that the acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass.

- In equation form, Newton’s second law of motion is \( a = \frac{F_{\text{net}}}{m} \).
- This is often written in the more familiar form: \( F_{\text{net}} = ma \).
- The weight \( w \) of an object is defined as the force of gravity acting on an object of mass \( m \). The object experiences an acceleration due to gravity \( g \):
w = mg.

- If the only force acting on an object is due to gravity, the object is in free fall.
- Friction is a force that opposes the motion past each other of objects that are touching.

### 4.4 Newton's Third Law of Motion: Symmetry in Forces

- **Newton's third law of motion** represents a basic symmetry in nature. It states: Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts.
- A **thrust** is a reaction force that pushes a body forward in response to a backward force. Rockets, airplanes, and cars are pushed forward by a thrust reaction force.

### 4.5 Normal, Tension, and Other Examples of Forces

- When objects rest on a surface, the surface applies a force to the object that supports the weight of the object. This supporting force acts perpendicular to and away from the surface. It is called a normal force, \( \mathbf{N} \).
- When objects rest on a non-accelerating horizontal surface, the magnitude of the normal force is equal to the weight of the object:
  \[
  N = mg.
  \]
- When objects rest on an inclined plane that makes an angle \( \theta \) with the horizontal surface, the weight of the object can be resolved into components that act perpendicular (\( w_\perp \)) and parallel (\( w_\parallel \)) to the surface of the plane. These components can be calculated using:
  \[
  w_\parallel = w \sin(\theta) = mg \sin(\theta)
  \]
  \[
  w_\perp = w \cos(\theta) = mg \cos(\theta).
  \]
- The pulling force that acts along a stretched flexible connector, such as a rope or cable, is called tension, \( \mathbf{T} \). When a rope supports the weight of an object that is at rest, the tension in the rope is equal to the weight of the object:
  \[
  T = mg.
  \]
- In any inertial frame of reference (one that is not accelerated or rotated), Newton’s laws have the simple forms given in this chapter and all forces are real forces having a physical origin.

### 4.6 Problem-Solving Strategies

- To solve problems involving Newton’s laws of motion, follow the procedure described:
  1. Draw a sketch of the problem.
  2. Identify known and unknown quantities, and identify the system of interest. Draw a free-body diagram, which is a sketch showing all of the forces acting on an object. The object is represented by a dot, and the forces are represented by vectors extending in different directions from the dot. If vectors act in directions that are not horizontal or vertical, resolve the vectors into horizontal and vertical components and draw them on the free-body diagram.
  3. Write Newton’s second law in the horizontal and vertical directions and add the forces acting on the object. If the object does not accelerate in a particular direction (for example, the \( x \)-direction) then \( F_{\text{net}}x = 0 \). If the object does accelerate in that direction, \( F_{\text{net}}x = ma \).
  4. Check your answer. Is the answer reasonable? Are the units correct?

### 4.7 Further Applications of Newton’s Laws of Motion

- Newton’s laws of motion can be applied in numerous situations to solve problems of motion. Some problems will contain multiple force vectors acting in different directions on an object. Be sure to draw diagrams, resolve all force vectors into horizontal and vertical components, and draw a free-body diagram. Always analyze the direction in which an object accelerates so that you can determine whether \( F_{\text{net}} = ma \) or \( F_{\text{net}} = 0 \).
- The normal force on an object is not always equal in magnitude to the weight of the object. If an object is accelerating, the normal force will be less than or greater than the weight of the object. Also, if the object is on an inclined plane, the normal force will always be less than the full weight of the object.
- Some problems will contain various physical quantities, such as forces, acceleration, velocity, or position. You can apply concepts from kinematics and dynamics in order to solve these problems of motion.

### 4.8 Extended Topic: The Four Basic Forces—An Introduction

- The various types of forces that are categorized for use in many applications are all manifestations of the **four basic forces** in nature.
- The properties of these forces are summarized in **Table 4.1**.
- Everything we experience directly without sensitive instruments is due to either electromagnetic forces or gravitational forces. The nuclear forces are responsible for the submicroscopic structure of matter, but they are not directly sensed because of their short ranges. Attempts are being made to show all four forces are different manifestations of a single unified force.
A force field surrounds an object creating a force and is the carrier of that force.

### Conceptual Questions

#### 4.1 Development of Force Concept

1. Propose a force standard different from the example of a stretched spring discussed in the text. Your standard must be capable of producing the same force repeatedly.
2. What properties do forces have that allow us to classify them as vectors?

#### 4.2 Newton's First Law of Motion: Inertia

3. How are inertia and mass related?
4. What is the relationship between weight and mass? Which is an intrinsic, unchanging property of a body?

#### 4.3 Newton's Second Law of Motion: Concept of a System

5. Which statement is correct? (a) Net force causes motion. (b) Net force causes change in motion. Explain your answer and give an example.
6. Why can we neglect forces such as those holding a body together when we apply Newton's second law of motion?
7. Explain how the choice of the “system of interest” affects which forces must be considered when applying Newton's second law of motion.
8. Describe a situation in which the net external force on a system is not zero, yet its speed remains constant.
9. A system can have a nonzero velocity while the net external force on it is zero. Describe such a situation.
10. A rock is thrown straight up. What is the net external force acting on the rock when it is at the top of its trajectory?
11. (a) Give an example of different net external forces acting on the same system to produce different accelerations. (b) Give an example of the same net external force acting on systems of different masses, producing different accelerations. (c) What law accurately describes both effects? State it in words and as an equation.
12. If the acceleration of a system is zero, are no external forces acting on it? What about internal forces? Explain your answers.
13. If a constant, nonzero force is applied to an object, what can you say about the velocity and acceleration of the object?
14. The gravitational force on the basketball in Figure 4.6 is ignored. When gravity is taken into account, what is the direction of the net external force on the basketball—above horizontal, below horizontal, or still horizontal?

#### 4.4 Newton's Third Law of Motion: Symmetry in Forces

15. When you take off in a jet aircraft, there is a sensation of being pushed back into the seat. Explain why you move backward in the seat—is there really a force backward on you? (The same reasoning explains whiplash injuries, in which the head is apparently thrown backward.)
16. A device used since the 1940s to measure the kick or recoil of the body due to heart beats is the “ballistocardiograph.” What physics principle(s) are involved here to measure the force of cardiac contraction? How might we construct such a device?
17. Describe a situation in which one system exerts a force on another and, as a consequence, experiences a force that is equal in magnitude and opposite in direction. Which of Newton's laws of motion apply?
18. Why does an ordinary rifle recoil (kick backward) when fired? The barrel of a recoilless rifle is open at both ends. Describe how Newton's third law applies when one is fired. Can you safely stand close behind one when it is fired?
19. An American football lineman reasons that it is senseless to try to out-push the opposing player, since no matter how hard he pushes he will experience an equal and opposite force from the other player. Use Newton's laws and draw a free-body diagram of an appropriate system to explain how he can still out-push the opposition if he is strong enough.
20. Newton's third law of motion tells us that forces always occur in pairs of equal and opposite magnitude. Explain how the choice of the “system of interest” affects whether one such pair of forces cancels.
4.5 Normal, Tension, and Other Examples of Forces

21. If a leg is suspended by a traction setup as shown in Figure 4.29, what is the tension in the rope?

22. In a traction setup for a broken bone, with pulleys and rope available, how might we be able to increase the force along the tibia using the same weight? (See Figure 4.29.) (Note that the tibia is the shin bone shown in this image.)

4.7 Further Applications of Newton’s Laws of Motion

23. To simulate the apparent weightlessness of space orbit, astronauts are trained in the hold of a cargo aircraft that is accelerating downward at \( g \). Why will they appear to be weightless, as measured by standing on a bathroom scale, in this accelerated frame of reference? Is there any difference between their apparent weightlessness in orbit and in the aircraft?

24. A cartoon shows the toupee coming off the head of an elevator passenger when the elevator rapidly stops during an upward ride. Can this really happen without the person being tied to the floor of the elevator? Explain your answer.

4.8 Extended Topic: The Four Basic Forces—An Introduction

25. Explain, in terms of the properties of the four basic forces, why people notice the gravitational force acting on their bodies if it is such a comparatively weak force.

26. What is the dominant force between astronomical objects? Why are the other three basic forces less significant over these very large distances?

27. Give a detailed example of how the exchange of a particle can result in an attractive force. (For example, consider one child pulling a toy out of the hands of another.)
Problems & Exercises

4.3 Newton’s Second Law of Motion: Concept of a System

You may assume data taken from illustrations is accurate to three digits.

1. A 63.0-kg sprinter starts a race with an acceleration of 4.20 m/s². What is the net external force on him?

2. If the sprinter from the previous problem accelerates at that rate for 20 m, and then maintains that velocity for the remainder of the 100-m dash, what will be his time for the race?

3. A cleaner pushes a 4.50-kg laundry cart in such a way that the net external force on it is 60.0 N. Calculate the magnitude of its acceleration.

4. Since astronauts in orbit are apparently weightless, a clever method of measuring their masses is needed to monitor their mass gains or losses to adjust diets. One way to do this is to exert a known force on an astronaut and measure the acceleration produced. Suppose a net external force of 50.0 N is exerted and the astronaut’s acceleration is measured to be 0.893 m/s². (a) Calculate her mass. (b) By exerting a force on the astronaut, the vehicle in which they orbit experiences an equal and opposite force. Discuss how this would affect the measurement of the astronaut’s acceleration. Propose a method in which recoil of the vehicle is avoided.

5. In Figure 4.7, the net external force on the 24-kg mower is stated to be 51 N. If the force of friction opposing the motion is 24 N, what force F (in newtons) is the person exerting on the mower? Suppose the mower is moving at 1.5 m/s when the force F is removed. How far will the mower go before stopping?

6. The same rocket sled drawn in Figure 4.30 is decelerated at a rate of 196 m/s². What force is necessary to produce this deceleration? Assume that the rockets are off. The mass of the system is 2100 kg.

7. (a) If the rocket sled shown in Figure 4.31 starts with only one rocket burning, what is the magnitude of its acceleration? Assume that the mass of the system is 2100 kg, the thrust T is 2.4x10⁴ N, and the force of friction opposing the motion is known to be 650 N. (b) Why is the acceleration not one-fourth of what it is with all rockets burning?

8. What is the deceleration of the rocket sled if it comes to rest in 1.1 s from a speed of 1000 km/h? (Such deceleration caused one test subject to black out and have temporary blindness.)

9. Suppose two children push horizontally, but in exactly opposite directions, on a third child in a wagon. The first child exerts a force of 75.0 N, the second a force of 90.0 N, friction is 12.0 N, and the mass of the third child plus wagon is 23.0 kg.
   a. What is the system of interest if the acceleration of the child in the wagon is to be calculated?
   b. Draw a free-body diagram, including all forces acting on the system.
   c. Calculate the acceleration.
   d. What would the acceleration be if friction were 15.0 N?

10. A powerful motorcycle can produce an acceleration of 3.50 m/s² while traveling at 90.0 km/h. At that speed the forces resisting motion, including friction and air resistance, total 400 N. (Air resistance is analogous to air friction. It always opposes the motion of an object.) What is the magnitude of the force the motorcycle exerts backward on the ground to produce its acceleration if the mass of the motorcycle with rider is 245 kg?

11. The rocket sled shown in Figure 4.32 accelerates at a rate of 49.0 m/s². Its passenger has a mass of 75.0 kg. (a) Calculate the horizontal component of the force the seat exerts against his body. Compare this with his weight by using a ratio. (b) Calculate the direction and magnitude of the total force the seat exerts against his body.

12. Repeat the previous problem for the situation in which the rocket sled decelerates at a rate of 201 m/s². In this problem, the forces are exerted by the seat and restraining belts.

13. The weight of an astronaut plus his space suit on the Moon is only 250 N. How much do they weigh on Earth? What is the mass on the Moon? On Earth?

14. Suppose the mass of a fully loaded module in which astronauts take off from the Moon is 10,000 kg. The thrust of its engines is 30,000 N. (a) Calculate its magnitude of acceleration in a vertical takeoff from the Moon. (b) Could it lift off from Earth? If not, why not? If it could, calculate the magnitude of its acceleration.
4.4 Newton’s Third Law of Motion: Symmetry in Forces

15. What net external force is exerted on a 1100-kg artillery shell fired from a battleship if the shell is accelerated at $2.40 \times 10^4 \text{ m/s}^2$? What is the magnitude of the force exerted on the ship by the artillery shell?

16. A brave but inadequate rugby player is being pushed backward by an opposing player who is exerting a force of 800 N on him. The mass of the losing player plus equipment is 90.0 kg, and he is accelerating at $1.20 \text{ m/s}^2$ backward. (a) What is the force of friction between the losing player’s feet and the grass? (b) What force does the winning player exert on the ground to move forward if his mass plus equipment is 110 kg? (c) Draw a sketch of the situation showing the system of interest used to solve each part. For this situation, draw a free-body diagram and write the net force equation.

4.5 Normal, Tension, and Other Examples of Forces

17. Two teams of nine members each engage in a tug of war. Each of the first team’s members has an average mass of 68 kg and exerts an average force of 1350 N horizontally. Each of the second team’s members has an average mass of 73 kg and exerts an average force of 1365 N horizontally. (a) What is magnitude of the acceleration of the two teams? (b) What is the tension in the section of rope between the teams?

18. What force does a trampoline have to apply to a 45.0-kg gymnast to accelerate her straight up at $7.50 \text{ m/s}^2$? Note that the answer is independent of the velocity of the gymnast—she can be moving either up or down, or be stationary.

19. (a) Calculate the tension in a vertical strand of spider web if a spider of mass $8.00 \times 10^{-5} \text{ kg}$ hangs motionless on it. (b) Calculate the tension in a horizontal strand of spider web if the same spider sits motionless in the middle of it much like the tightrope walker in Figure 4.17. The strand sags at an angle of $12^\circ$ below the horizontal. Compare this with the tension in the vertical strand (find their ratio).

20. Suppose a 60.0-kg gymnast climbs a rope. (a) What is the tension in the rope if he climbs at a constant speed? (b) What is the tension in the rope if he accelerates upward at a rate of $1.50 \text{ m/s}^2$?

21. Show that, as stated in the text, a force $F_\perp$ exerted on a flexible medium at its center and perpendicular to its length (such as on the tightrope wire in Figure 4.17) gives rise to a tension of magnitude $T = \frac{F_\perp}{2 \sin(\theta)}$.

22. Consider the baby being weighed in Figure 4.33. (a) What is the mass of the child and basket if a scale reading of 55 N is observed? (b) What is the tension $T_1$ in the cord attaching the baby to the scale? (c) What is the tension $T_2$ in the cord attaching the scale to the ceiling, if the scale has a mass of 0.500 kg? (d) Draw a sketch of the situation indicating the system of interest used to solve each part. The masses of the cords are negligible.

4.6 Problem-Solving Strategies

23. A $5.00 \times 10^5 \text{-kg}$ rocket is accelerating straight up. Its engines produce $1.250 \times 10^7 \text{ N}$ of thrust, and air resistance is $4.50 \times 10^6 \text{ N}$. What is the rocket’s acceleration? Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton’s laws of motion.

24. The wheels of a midsize car exert a force of 2100 N backward on the road to accelerate the car in the forward direction. If the force of friction including air resistance is 250 N and the acceleration of the car is $1.80 \text{ m/s}^2$, what is the mass of the car plus its occupants? Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton’s laws of motion. For this situation, draw a free-body diagram and write the net force equation.

25. Calculate the force a 70.0-kg high jumper must exert on the ground to produce an upward acceleration 4.00 times the acceleration due to gravity. Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton’s laws of motion.

26. When landing after a spectacular somersault, a 40.0-kg gymnast decelerates by pushing straight down on the mat. Calculate the force she must exert if her deceleration is 7.00 times the acceleration due to gravity. Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton’s laws of motion.
27. A freight train consists of two 8.00 \times 10^4 \text{ kg} engines and 45 cars with average masses of 5.50 \times 10^4 \text{ kg}. (a) What force must each engine exert backward on the track to accelerate the train at a rate of 5.00 \times 10^{-2} \text{ m/s}^2 if the force of friction is 7.50 \times 10^5 \text{ N}, assuming the engines exert identical forces? This is not a large frictional force for such a massive system. Rolling friction for trains is small, and consequently trains are very energy-efficient transportation systems. (b) What is the force in the coupling between the 37th and 38th cars (this is the force each exerts on the other), assuming all cars have the same mass and that friction is evenly distributed among all of the cars and engines?

28. Commercial airplanes are sometimes pushed out of the passenger loading area by a tractor. (a) An 1800-kg tractor exerts a force of 1.75 \times 10^4 \text{ N} backward on the pavement, and the system experiences forces resisting motion that total 2400 N. If the acceleration is 0.150 \text{ m/s}^2, what is the mass of the airplane? (b) Calculate the force exerted by the tractor on the airplane, assuming 2200 N of the friction is experienced by the airplane. (c) Draw two sketches showing the systems of interest used to solve each part, including the free-body diagrams for each.

29. A 1100-kg car pulls a boat on a trailer. (a) What total force resists the motion of the car, boat, and trailer, if the car exerts a 1900-N force on the road and produces an acceleration of 0.550 \text{ m/s}^2? The mass of the boat plus trailer is 700 kg. (b) What is the force in the hitch between the car and the trailer if 80% of the resisting forces are experienced by the boat and trailer?

30. (a) Find the magnitudes of the forces \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) that add to give the total force \( \mathbf{F}_{\text{tot}} \) shown in Figure 4.34. This may be done either graphically or by using trigonometry. (b) Show graphically that the same total force is obtained independent of the order of addition of \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \). (c) Find the direction and magnitude of some other pair of vectors that add to give \( \mathbf{F}_{\text{tot}} \). Draw these to scale on the same drawing used in part (b) or a similar picture.

31. Two children pull a third child on a snow saucer sled exerting forces \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) as shown from above in Figure 4.35. Find the acceleration of the 49.00-kg sled and child system. Note that the direction of the frictional force is unspecified; it will be in the opposite direction of the sum of \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \).

32. Suppose your car was mired deeply in the mud and you wanted to use the method illustrated in Figure 4.36 to pull it out. (a) What force would you have to exert perpendicular to the center of the rope to produce a force of 12,000 N on the car if the angle is 2.00°? In this part, explicitly show how you follow the steps in the Problem-Solving Strategy for Newton’s laws of motion. (b) Real ropes stretch under such forces. What force would be exerted on the car if the angle increases to 7.00° and you still apply the force found in part (a) to its center?

33. What force is exerted on the tooth in Figure 4.37 if the tension in the wire is 25.0 N? Note that the force applied to the tooth is smaller than the tension in the wire, but this is necessitated by practical considerations of how force can be applied in the mouth. Explicitly show how you follow steps in the Problem-Solving Strategy for Newton’s laws of motion.
34. **Figure 4.38** shows Superhero and Trusty Sidekick hanging motionless from a rope. Superhero’s mass is 90.0 kg, while Trusty Sidekick’s is 55.0 kg, and the mass of the rope is negligible. (a) Draw a free-body diagram of the situation showing all forces acting on Superhero, Trusty Sidekick, and the rope. (b) Find the tension in the rope above Superhero. (c) Find the tension in the rope between Superhero and Trusty Sidekick. Indicate on your free-body diagram the system of interest used to solve each part.

35. **A nurse pushes a cart by exerting a force on the handle at a downward angle 35.0° below the horizontal. The loaded cart has a mass of 28.0 kg, and the force of friction is 60.0 N.** (a) Draw a free-body diagram for the system of interest. (b) What force must the nurse exert to move at a constant velocity?

36. **Construct Your Own Problem**

Consider the tension in an elevator cable during the time the elevator starts from rest and accelerates its load upward to some cruising velocity. Taking the elevator and its load to be the system of interest, draw a free-body diagram. Then calculate the tension in the cable. Among the things to consider are the mass of the elevator and its load, the final velocity, and the time taken to reach that velocity.

37. **Construct Your Own Problem**

Consider two people pushing a toboggan with four children on it up a snow-covered slope. Construct a problem in which you calculate the acceleration of the toboggan and its load. Include a free-body diagram of the appropriate system of interest as the basis for your analysis. Show vector forces and their components and explain the choice of coordinates. Among the things to be considered are the forces exerted by those pushing, the angle of the slope, and the masses of the toboggan and children.

38. **Unreasonable Results**

(a) Repeat Exercise 4.29, but assume an acceleration of 1.20 m/s² is produced. (b) What is unreasonable about the result? (c) Which premise is unreasonable, and why is it unreasonable?

39. **Unreasonable Results**

(a) What is the initial acceleration of a rocket that has a mass of 1.50×10^6 kg at takeoff, the engines of which produce a thrust of 2.00×10^6 N? Do not neglect gravity. (b) What is unreasonable about the result? (This result has been unintentionally achieved by several real rockets.) (c) Which premise is unreasonable, or which premises are inconsistent? (You may find it useful to compare this problem to the rocket problem earlier in this section.)

4.7 Further Applications of Newton’s Laws of Motion

40. **A flea jumps by exerting a force of 1.20×10⁻⁵ N straight down on the ground. A breeze blowing on the flea parallel to the ground exerts a force of 0.500×10⁻⁶ N on the flea. Find the direction and magnitude of the acceleration of the flea if its mass is 6.00×10⁻⁷ kg. Do not neglect the gravitational force.**

41. **Two muscles in the back of the leg pull upward on the Achilles tendon, as shown in Figure 4.39. (These muscles are called the medial and lateral heads of the gastrocnemius muscle.) Find the magnitude and direction of the total force on the Achilles tendon. What type of movement could be caused by this force?**
42. A 76.0-kg person is being pulled away from a burning building as shown in Figure 4.40. Calculate the tension in the two ropes if the person is momentarily motionless. Include a free-body diagram in your solution.

43. Integrated Concepts
A 35.0-kg dolphin decelerates from 12.0 to 7.50 m/s in 2.30 s to join another dolphin in play. What average force was exerted to slow him if he was moving horizontally? (The gravitational force is balanced by the buoyant force of the water.)

44. Integrated Concepts
When starting a foot race, a 70.0-kg sprinter exerts an average force of 650 N backward on the ground for 0.800 s. (a) What is his final speed? (b) How far does he travel?

45. Integrated Concepts
A large rocket has a mass of $2.00 \times 10^6$ kg at takeoff, and its engines produce a thrust of $3.50 \times 10^7$ N. (a) Find its initial acceleration if it takes off vertically. (b) How long does it take to reach a velocity of 120 km/h straight up, assuming constant mass and thrust? (c) In reality, the mass of a rocket decreases significantly as its fuel is consumed. Describe qualitatively how this affects the acceleration and time for this motion.

46. Integrated Concepts
A basketball player jumps straight up for a ball. To do this, he lowers his body 0.300 m and then accelerates through this distance by forcefully straightening his legs. This player leaves the floor with a vertical velocity sufficient to carry him 0.900 m above the floor. (a) Calculate his velocity when he leaves the floor. (b) Calculate his acceleration while he is straightening his legs. He goes from zero to the velocity found in part (a) in a distance of 0.300 m. (c) Calculate the force he exerts on the floor to do this, given that his mass is 110 kg.

47. Integrated Concepts
A 2.50-kg fireworks shell is fired straight up from a mortar and reaches a height of 110 m. (a) Neglecting air resistance (a poor assumption, but we will make it for this example), calculate the shell’s velocity when it leaves the mortar. (b) The mortar itself is a tube 0.450 m long. Calculate the average acceleration of the shell in the tube as it goes from zero to the velocity found in (a). (c) What is the average force on the shell in the mortar? Express your answer in newtons and as a ratio to the weight of the shell.

48. Integrated Concepts
Repeat Exercise 4.47 for a shell fired at an angle $10.0^\circ$ from the vertical.

49. Integrated Concepts
An elevator filled with passengers has a mass of 1700 kg. (a) The elevator accelerates upward from rest at a rate of $1.20 \text{ m/s}^2$ for 1.50 s. Calculate the tension in the cable supporting the elevator. (b) The elevator continues upward at constant velocity for 8.50 s. What is the tension in the cable during this time? (c) The elevator decelerates at a rate of $0.600 \text{ m/s}^2$ for 3.00 s. What is the tension in the cable during deceleration? (d) How high has the elevator moved above its original starting point, and what is its final velocity?

50. Unreasonable Results
(a) What is the final velocity of a car originally traveling at 50.0 km/h that decelerates at a rate of $0.400 \text{ m/s}^2$ for 50.0 s? (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?
51. **Unreasonable Results**

A 75.0-kg man stands on a bathroom scale in an elevator that accelerates from rest to 30.0 m/s in 2.00 s. (a) Calculate the scale reading in newtons and compare it with his weight. (The scale exerts an upward force on him equal to its reading.) (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

4.8 **Extended Topic: The Four Basic Forces—An Introduction**

52. (a) What is the strength of the weak nuclear force relative to the strong nuclear force? (b) What is the strength of the weak nuclear force relative to the electromagnetic force? Since the weak nuclear force acts at only very short distances, such as inside nuclei, where the strong and electromagnetic forces also act, it might seem surprising that we have any knowledge of it at all. We have such knowledge because the weak nuclear force is responsible for beta decay, a type of nuclear decay not explained by other forces.

53. (a) What is the ratio of the strength of the gravitational force to that of the strong nuclear force? (b) What is the ratio of the strength of the gravitational force to that of the weak nuclear force? (c) What is the ratio of the strength of the gravitational force to that of the electromagnetic force? What do your answers imply about the influence of the gravitational force on atomic nuclei?

54. What is the ratio of the strength of the strong nuclear force to that of the electromagnetic force? Based on this ratio, you might expect that the strong force dominates the nucleus, which is true for small nuclei. Large nuclei, however, have sizes greater than the range of the strong nuclear force. At these sizes, the electromagnetic force begins to affect nuclear stability. These facts will be used to explain nuclear fusion and fission later in this text.
Figure 5.1 Total hip replacement surgery has become a common procedure. The head (or ball) of the patient’s femur fits into a cup that has a hard plastic-like inner lining. (credit: National Institutes of Health, via Wikimedia Commons)

### Chapter Outline

#### 5.1. Friction
- Discuss the general characteristics of friction.
- Describe the various types of friction.
- Calculate the magnitude of static and kinetic friction.

#### 5.2. Drag Forces
- Express mathematically the drag force.
- Discuss the applications of drag force.
- Define terminal velocity.
- Determine the terminal velocity given mass.

#### 5.3. Elasticity: Stress and Strain
- State Hooke’s law.
- Explain Hooke’s law using graphical representation between deformation and applied force.
- Discuss the three types of deformations such as changes in length, sideways shear and changes in volume.
- Describe with examples the young’s modulus, shear modulus and bulk modulus.
- Determine the change in length given mass, length and radius.

### Introduction: Further Applications of Newton’s Laws

Describe the forces on the hip joint. What means are taken to ensure that this will be a good movable joint? From the photograph (for an adult) in Figure 5.1, estimate the dimensions of the artificial device.

It is difficult to categorize forces into various types (aside from the four basic forces discussed in previous chapter). We know that a net force affects the motion, position, and shape of an object. It is useful at this point to look at some particularly interesting and common forces that will provide further applications of Newton’s laws of motion. We have in mind the forces of friction, air or liquid drag, and deformation.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42138/1.9/#concept-trailer-terminal-velocity)
5.1 Friction

Friction is a force that is around us all the time that opposes relative motion between surfaces in contact but also allows us to move (which you have discovered if you have ever tried to walk on ice). While a common force, the behavior of friction is actually very complicated and is still not completely understood. We have to rely heavily on observations for whatever understandings we can gain. However, we can still deal with its more elementary general characteristics and understand the circumstances in which it behaves.

One of the simpler characteristics of friction is that it is parallel to the contact surface between surfaces and always in a direction that opposes motion or attempted motion of the systems relative to each other. If two surfaces are in contact and moving relative to one another, then the friction between them is called kinetic friction. For example, friction slows a hockey puck sliding on ice. But when objects are stationary, static friction can act between them; the static friction is usually greater than the kinetic friction between the surfaces.

Kinetic Friction

If two surfaces are in contact and moving relative to one another, then the friction between them is called kinetic friction.

Imagine, for example, trying to slide a heavy crate across a concrete floor—you may push harder and harder on the crate and not move it at all. This means that the static friction responds to what you do—it increases to be equal to and in the opposite direction of your push. But if you finally push hard enough, the crate seems to slip suddenly and starts to move. Once in motion it is easier to keep it in motion than it was to get it started, indicating that the kinetic friction force is less than the static friction force. If you add mass to the crate, say by placing a box on top of it, you need to push even harder to get it started and also to keep it moving. Furthermore, if you oiled the concrete you would find it to be easier to get the crate started and keep it going (as you might expect).

Figure 5.2 is a crude pictorial representation of how friction occurs at the interface between two objects. Close-up inspection of these surfaces shows them to be rough. So when you push to get an object moving (in this case, a crate), you must raise the object until it can skip along with just the tips of the surface hitting, break off the points, or do both. A considerable force can be resisted by friction with no apparent motion. The harder the surfaces are pushed together (such as if another box is placed on the crate), the more force is needed to move them. Part of the friction is due to adhesive forces between the surface molecules of the two objects, which explain the dependence of friction on the nature of the substances. Adhesion varies with substances in contact and is a complicated aspect of surface physics. Once an object is moving, there are fewer points of contact (fewer molecules adhering), so less force is required to keep the object moving. At small but nonzero speeds, friction is nearly independent of speed.

![Diagram of friction forces](image)

Figure 5.2 Frictional forces, such as $f$, always oppose motion or attempted motion between surfaces in contact. Friction arises in part because of the roughness of the surfaces in contact, as seen in the expanded view. In order for the object to move, it must rise to where the peaks can skip along the bottom surface. Thus a force is required just to set the object in motion. Some of the peaks will be broken off, also requiring a force to maintain motion. Much of the friction is actually due to attractive forces between molecules making up the two objects, so that even perfectly smooth surfaces are not friction-free. Such adhesive forces also depend on the substances the surfaces are made of, explaining, for example, why rubber-soled shoes slip less than those with leather soles.

The magnitude of the frictional force has two forms: one for static situations (static friction), the other for when there is motion (kinetic friction).

When there is no motion between the objects, the magnitude of static friction $f_s$ is

$$f_s \leq \mu_s N,$$

where $\mu_s$ is the coefficient of static friction and $N$ is the magnitude of the normal force (the force perpendicular to the surface).
Magnitude of Static Friction

Magnitude of static friction \( f_s \) is

\[ f_s \leq \mu_s N, \]  \hspace{1cm} (5.2)

where \( \mu_s \) is the coefficient of static friction and \( N \) is the magnitude of the normal force.

The symbol \( \leq \) means less or equal to, implying that static friction can have a minimum and a maximum value of \( \mu_s N \). Static friction is a responsive force that increases to be equal and opposite to whatever force is exerted, up to its maximum limit. Once the applied force exceeds \( f_{s(max)} \), the object will move. Thus

\[ f_{s(max)} = \mu_s N. \]  \hspace{1cm} (5.3)

Once an object is moving, the magnitude of kinetic friction \( f_k \) is given by

\[ f_k = \mu_k N, \]  \hspace{1cm} (5.4)

where \( \mu_k \) is the coefficient of kinetic friction. A system in which \( f_k = \mu_k N \) is described as a system in which friction behaves simply.

Magnitude of Kinetic Friction

The magnitude of kinetic friction \( f_k \) is given by

\[ f_k = \mu_k N, \]  \hspace{1cm} (5.5)

where \( \mu_k \) is the coefficient of kinetic friction.

As seen in Table 5.1, the coefficients of kinetic friction are less than their static counterparts. That values of \( \mu \) in Table 5.1 are stated to only one or, at most, two digits is an indication of the approximate description of friction given by the above two equations.

### Table 5.1 Coefficients of Static and Kinetic Friction

<table>
<thead>
<tr>
<th>System</th>
<th>Static friction ( \mu_s )</th>
<th>Kinetic friction ( \mu_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rubber on dry concrete</td>
<td>1.0</td>
<td>0.7</td>
</tr>
<tr>
<td>Rubber on wet concrete</td>
<td>0.7</td>
<td>0.5</td>
</tr>
<tr>
<td>Wood on wood</td>
<td>0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>Waxed wood on wet snow</td>
<td>0.14</td>
<td>0.1</td>
</tr>
<tr>
<td>Metal on wood</td>
<td>0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>Steel on steel (dry)</td>
<td>0.6</td>
<td>0.3</td>
</tr>
<tr>
<td>Steel on steel (oiled)</td>
<td>0.05</td>
<td>0.03</td>
</tr>
<tr>
<td>Teflon on steel</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>Bone lubricated by synovial fluid</td>
<td>0.016</td>
<td>0.015</td>
</tr>
<tr>
<td>Shoes on wood</td>
<td>0.9</td>
<td>0.7</td>
</tr>
<tr>
<td>Shoes on ice</td>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>Ice on ice</td>
<td>0.1</td>
<td>0.03</td>
</tr>
<tr>
<td>Steel on ice</td>
<td>0.04</td>
<td>0.02</td>
</tr>
</tbody>
</table>

The equations given earlier include the dependence of friction on materials and the normal force. The direction of friction is always opposite that of motion, parallel to the surface between objects, and perpendicular to the normal force. For example, if the crate you try to push (with a force parallel to the floor) has a mass of 100 kg, then the normal force would be equal to its weight, \( W = mg = (100 \text{ kg})(9.80 \text{ m/s}^2) = 980 \text{ N} \), perpendicular to the floor. If the coefficient of static friction is 0.45, you would have to exert a force parallel to the floor greater than \( f_{s(max)} = \mu_s N = (0.45)(980 \text{ N}) = 440 \text{ N} \) to move the crate.

Once there is motion, friction is less and the coefficient of kinetic friction might be 0.30, so that a force of only 290 N (
\( f_k = \mu_k N = (0.30)(980 \text{ N}) = 290 \text{ N} \) would keep it moving at a constant speed. If the floor is lubricated, both coefficients are considerably less than they would be without lubrication. Coefficient of friction is a unitless quantity with a magnitude usually between 0 and 1.0. The coefficient of the friction depends on the two surfaces that are in contact.

**Take-Home Experiment**

Find a small plastic object (such as a food container) and slide it on a kitchen table by giving it a gentle tap. Now spray water on the table, simulating a light shower of rain. What happens now when you give the object the same-sized tap? Now add a few drops of (vegetable or olive) oil on the surface of the water and give the same tap. What happens now? This latter situation is particularly important for drivers to note, especially after a light rain shower. Why?

Many people have experienced the slipperiness of walking on ice. However, many parts of the body, especially the joints, have much smaller coefficients of friction—often three or four times less than ice. A joint is formed by the ends of two bones, which are connected by thick tissues. The knee joint is formed by the lower leg bone (the tibia) and the thighbone (the femur). The hip is a ball (at the end of the femur) and socket (part of the pelvis) joint. The ends of the bones in the joint are covered by cartilage, which provides a smooth, almost glassy surface. The joints also produce a fluid (synovial fluid) that reduces friction and wear. A damaged or arthritic joint can be replaced by an artificial joint (Figure 5.3). These replacements can be made of metals (stainless steel or titanium) or plastic (polyethylene), also with very small coefficients of friction.

![Figure 5.3](https://openstax.org-lab/xray/knee.jpg) Artificial knee replacement is a procedure that has been performed for more than 20 years. In this figure, we see the post-op X-rays of the right knee joint replacement. (credit: Mike Baird, Flickr)

Other natural lubricants include saliva produced in our mouths to aid in the swallowing process, and the slippery mucus found between organs in the body, allowing them to move freely past each other during heartbeats, during breathing, and when a person moves. Artificial lubricants are also common in hospitals and doctor’s clinics. For example, when ultrasonic imaging is carried out, the gel that couples the transducer to the skin also serves to lubricate the surface between the transducer and the skin—thereby reducing the coefficient of friction between the two surfaces. This allows the transducer to move freely over the skin.

**Example 5.1 Skiing Exercise**

A skier with a mass of 62 kg is sliding down a snowy slope. Find the coefficient of kinetic friction for the skier if friction is known to be 45.0 N.

**Strategy**

The magnitude of kinetic friction was given in to be 45.0 N. Kinetic friction is related to the normal force \( N \) as \( f_k = \mu_k N \); thus, the coefficient of kinetic friction can be found if we can find the normal force of the skier on a slope. The normal force is always perpendicular to the surface, and since there is no motion perpendicular to the surface, the normal force should...
equal the component of the skier’s weight perpendicular to the slope. (See the skier and free-body diagram in Figure 5.4.)

\[ N = w_{\perp} = w \cos 25^\circ = mg \cos 25^\circ. \] (5.6)

Substituting this into our expression for kinetic friction, we get

\[ f_k = \mu_k mg \cos 25^\circ, \] (5.7)

which can now be solved for the coefficient of kinetic friction \( \mu_k \).

**Solution**

Solving for \( \mu_k \) gives

\[ \mu_k = \frac{f_k}{N} = \frac{f_k}{w \cos 25^\circ} = \frac{f_k}{mg \cos 25^\circ}. \] (5.8)

Substituting known values on the right-hand side of the equation,

\[ \mu_k = \frac{45.0 \text{ N}}{(62 \text{ kg})(9.80 \text{ m/s}^2)(0.906)} = 0.082. \] (5.9)

**Discussion**

This result is a little smaller than the coefficient listed in Table 5.1 for waxed wood on snow, but it is still reasonable since values of the coefficients of friction can vary greatly. In situations like this, where an object of mass \( m \) slides down a slope that makes an angle \( \theta \) with the horizontal, friction is given by \( f_k = \mu_k mg \cos \theta \). All objects will slide down a slope with constant acceleration under these circumstances. Proof of this is left for this chapter’s Problems and Exercises.

**Take-Home Experiment**

An object will slide down an inclined plane at a constant velocity if the net force on the object is zero. We can use this fact to measure the coefficient of kinetic friction between two objects. As shown in Example 5.1, the kinetic friction on a slope \( f_k = \mu_k mg \cos \theta \). The component of the weight down the slope is equal to \( mg \sin \theta \) (see the free-body diagram in Figure 5.4). These forces act in opposite directions, so when they have equal magnitude, the acceleration is zero. Writing these out:

\[ f_k = mg \sin \theta \]

\[ \mu_k mg \cos \theta = mg \sin \theta. \] (5.11)

Solving for \( \mu_k \), we find that

\[ \mu_k = \frac{mg \sin \theta}{mg \cos \theta} = \tan \theta. \] (5.12)
Put a coin on a book and tilt it until the coin slides at a constant velocity down the book. You might need to tap the book lightly to get the coin to move. Measure the angle of tilt relative to the horizontal and find \( \mu_k \). Note that the coin will not start to slide at all until an angle greater than \( \theta \) is attained, since the coefficient of static friction is larger than the coefficient of kinetic friction. Discuss how this may affect the value for \( \mu_k \) and its uncertainty.

We have discussed that when an object rests on a horizontal surface, there is a normal force supporting it equal in magnitude to its weight. Furthermore, simple friction is always proportional to the normal force.

**Making Connections: Submicroscopic Explanations of Friction**

The simpler aspects of friction dealt with so far are its macroscopic (large-scale) characteristics. Great strides have been made in the atomic-scale explanation of friction during the past several decades. Researchers are finding that the atomic nature of friction seems to have several fundamental characteristics. These characteristics not only explain some of the simpler aspects of friction—they also hold the potential for the development of nearly friction-free environments that could save hundreds of billions of dollars in energy which is currently being converted (unnecessarily) to heat.

**Figure 5.5** illustrates one macroscopic characteristic of friction that is explained by microscopic (small-scale) research. We have noted that friction is proportional to the normal force, but not to the area in contact, a somewhat counterintuitive notion. When two rough surfaces are in contact, the actual contact area is a tiny fraction of the total area since only high spots touch. When a greater normal force is exerted, the actual contact area increases, and it is found that the friction is proportional to this area.

![Diagram of two rough surfaces with small and large normal forces](image)

**Figure 5.5** Two rough surfaces in contact have a much smaller area of actual contact than their total area. When there is a greater normal force as a result of a greater applied force, the area of actual contact increases as does friction.

But the atomic-scale view promises to explain far more than the simpler features of friction. The mechanism for how heat is generated is now being determined. In other words, why do surfaces get warmer when rubbed? Essentially, atoms are linked with one another to form lattices. When surfaces rub, the surface atoms adhere and cause atomic lattices to vibrate—essentially creating sound waves that penetrate the material. The sound waves diminish with distance and their energy is converted into heat. Chemical reactions that are related to frictional wear can also occur between atoms and molecules on the surfaces. **Figure 5.6** shows how the tip of a probe drawn across another material is deformed by atomic-scale friction. The force needed to drag the tip can be measured and is found to be related to shear stress, which will be discussed later in this chapter. The variation in shear stress is remarkable (more than a factor of \( 10^{12} \)) and difficult to predict theoretically, but shear stress is yielding a fundamental understanding of a large-scale phenomenon known since ancient times—friction.
Figure 5.6 The tip of a probe is deformed sideways by frictional force as the probe is dragged across a surface. Measurements of how the force varies for different materials are yielding fundamental insights into the atomic nature of friction.

Forces and Motion

Explore the forces at work when you try to push a filing cabinet. Create an applied force and see the resulting friction force and total force acting on the cabinet. Charts show the forces, position, velocity, and acceleration vs. time. Draw a free-body diagram of all the forces (including gravitational and normal forces).

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42139/1.23/#fs-id1167067199007)

5.2 Drag Forces

Another interesting force in everyday life is the force of drag on an object when it is moving in a fluid (either a gas or a liquid). You feel the drag force when you move your hand through water. You might also feel it if you move your hand during a strong wind. The faster you move your hand, the harder it is to move. You feel a smaller drag force when you tilt your hand so only the side goes through the air—you have decreased the area of your hand that faces the direction of motion. Like friction, the drag force always opposes the motion of an object. Unlike simple friction, the drag force is proportional to some function of the velocity of the object in that fluid. This functionality is complicated and depends upon the shape of the object, its size, its velocity, and the fluid it is in. For most large objects such as bicyclists, cars, and baseballs not moving too slowly, the magnitude of the drag force is found to be proportional to the square of the speed of the object. We can write this relationship mathematically as $F_D \propto v^2$. When taking into account other factors, this relationship becomes

$$F_D = \frac{1}{2} C \rho A v^2,$$

where $C$ is the drag coefficient, $A$ is the area of the object facing the fluid, and $\rho$ is the density of the fluid. (Recall that density is mass per unit volume.) This equation can also be written in a more generalized fashion as $F_D = b v^2$, where $b$ is a constant equivalent to $0.5 C \rho A$. We have set the exponent for these equations as 2 because, when an object is moving at high velocity through air, the magnitude of the drag force is proportional to the square of the speed. As we shall see in a few pages on fluid dynamics, for small particles moving at low speeds in a fluid, the exponent is equal to 1.

Drag Force

Drag force $F_D$ is found to be proportional to the square of the speed of the object. Mathematically

$$F_D \propto v^2$$

$$F_D = \frac{1}{2} C \rho A v^2,$$

where $C$ is the drag coefficient, $A$ is the area of the object facing the fluid, and $\rho$ is the density of the fluid.

Athletes as well as car designers seek to reduce the drag force to lower their race times. (See Figure 5.7). “Aerodynamic”
shaping of an automobile can reduce the drag force and so increase a car’s gas mileage.

![Figure 5.7](image)

*Figure 5.7* From racing cars to bobsled racers, aerodynamic shaping is crucial to achieving top speeds. Bobsleds are designed for speed. They are shaped like a bullet with tapered fins. (credit: U.S. Army, via Wikimedia Commons)

The value of the drag coefficient, $C$, is determined empirically, usually with the use of a wind tunnel. (See *Figure 5.8*).

![Figure 5.8](image)

*Figure 5.8* NASA researchers test a model plane in a wind tunnel. (credit: NASA/Ames)

The drag coefficient can depend upon velocity, but we will assume that it is a constant here. Table 5.2 lists some typical drag coefficients for a variety of objects. Notice that the drag coefficient is a dimensionless quantity. At highway speeds, over 50% of the power of a car is used to overcome air drag. The most fuel-efficient cruising speed is about 70–80 km/h (about 45–50 mi/h). For this reason, during the 1970s oil crisis in the United States, maximum speeds on highways were set at about 90 km/h (55 mi/h).
### Table 5.2 Drag Coefficient Values

<table>
<thead>
<tr>
<th>Object</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Airfoil</td>
<td>0.05</td>
</tr>
<tr>
<td>Toyota Camry</td>
<td>0.28</td>
</tr>
<tr>
<td>Ford Focus</td>
<td>0.32</td>
</tr>
<tr>
<td>Honda Civic</td>
<td>0.36</td>
</tr>
<tr>
<td>Ferrari Testarossa</td>
<td>0.37</td>
</tr>
<tr>
<td>Dodge Ram pickup</td>
<td>0.43</td>
</tr>
<tr>
<td>Sphere</td>
<td>0.45</td>
</tr>
<tr>
<td>Hummer H2 SUV</td>
<td>0.64</td>
</tr>
<tr>
<td>Skydiver (feet first)</td>
<td>0.70</td>
</tr>
<tr>
<td>Bicycle</td>
<td>0.90</td>
</tr>
<tr>
<td>Skydiver (horizontal)</td>
<td>1.0</td>
</tr>
<tr>
<td>Circular flat plate</td>
<td>1.12</td>
</tr>
</tbody>
</table>

Substantial research is under way in the sporting world to minimize drag. The dimples on golf balls are being redesigned as are the clothes that athletes wear. Bicycle racers and some swimmers and runners wear full bodysuits. Australian Cathy Freeman wore a full body suit in the 2000 Sydney Olympics, and won the gold medal for the 400 m race. Many swimmers in the 2008 Beijing Olympics wore (Speedo) body suits; it might have made a difference in breaking many world records (See Figure 5.9). Most elite swimmers (and cyclists) shave their body hair. Such innovations can have the effect of slicing away milliseconds in a race, sometimes making the difference between a gold and a silver medal. One consequence is that careful and precise guidelines must be continuously developed to maintain the integrity of the sport.

Some interesting situations connected to Newton's second law occur when considering the effects of drag forces upon a moving object. For instance, consider a skydiver falling through air under the influence of gravity. The two forces acting on him are the force of gravity and the drag force (ignoring the buoyant force). The downward force of gravity remains constant regardless of the velocity at which the person is moving. However, as the person's velocity increases, the magnitude of the drag force increases until the magnitude of the drag force is equal to the gravitational force, thus producing a net force of zero. A zero net force means that there is no acceleration, as given by Newton's second law. At this point, the person's velocity remains constant and we say that the person has reached his terminal velocity ($v_t$). Since $F_D$ is proportional to the speed, a heavier skydiver must go faster for $F_D$ to equal his weight. Let's see how this works out more quantitatively.

At the terminal velocity,

$$F_{\text{net}} = mg - F_D = ma = 0.$$  \hspace{1cm} (5.16)

Thus,

$$mg = F_D.$$  \hspace{1cm} (5.17)

Using the equation for drag force, we have
\[ mg = \frac{1}{2} \rho CA v^2. \quad (5.18) \]

Solving for the velocity, we obtain
\[ v = \sqrt{\frac{2mg}{\rho CA}}. \quad (5.19) \]

Assume the density of air is \( \rho = 1.21 \text{ kg/m}^3 \). A 75-kg skydiver descending head first will have an area approximately \( A = 0.18 \text{ m}^2 \) and a drag coefficient of approximately \( C = 0.70 \). We find that
\[ v = \sqrt{\frac{2(75 \text{ kg})(9.80 \text{ m/s}^2)}{(1.21 \text{ kg/m}^3)(0.70)(0.18 \text{ m}^2)}} \]
\[ = 98 \text{ m/s} \]
\[ = 350 \text{ km/h}. \]

This means a skydiver with a mass of 75 kg achieves a maximum terminal velocity of about 350 km/h while traveling in a headfirst position, minimizing the area and his drag. In a spread-eagle position, that terminal velocity may decrease to about 200 km/h as the area increases. This terminal velocity becomes much smaller after the parachute opens.

### Take-Home Experiment

This interesting activity examines the effect of weight upon terminal velocity. Gather together some nested coffee filters. Leaving them in their original shape, measure the time it takes for one, two, three, four, and five nested filters to fall to the floor from the same height (roughly 2 m). (Note that, due to the way the filters are nested, drag is constant and only mass varies.) They obtain terminal velocity quite quickly, so find this velocity as a function of mass. Plot the terminal velocity \( v \) versus mass. Also plot \( v^2 \) versus mass. Which of these relationships is more linear? What can you conclude from these graphs?

### Example 5.2 A Terminal Velocity

Find the terminal velocity of an 85-kg skydiver falling in a spread-eagle position.

**Strategy**

At terminal velocity, \( F_{\text{net}} = 0 \). Thus the drag force on the skydiver must equal the force of gravity (the person’s weight).

Using the equation of drag force, we find \( mg = \frac{1}{2} \rho CA v^2 \).

Thus the terminal velocity \( v_t \) can be written as
\[ v_t = \sqrt{\frac{2mg}{\rho CA}}. \quad (5.21) \]

**Solution**

All quantities are known except the person’s projected area. This is an adult (85 kg) falling spread eagle. We can estimate the frontal area as
\[ A = (2 \text{ m})(0.35 \text{ m}) = 0.70 \text{ m}^2. \quad (5.22) \]

Using our equation for \( v_t \), we find that
\[ v_t = \sqrt{\frac{2(85 \text{ kg})(9.80 \text{ m/s}^2)}{(1.21 \text{ kg/m}^3)(1.0)(0.70 \text{ m}^2)}} \]
\[ = 44 \text{ m/s}. \quad (5.23) \]

**Discussion**

This result is consistent with the value for \( v_t \) mentioned earlier. The 75-kg skydiver going feet first had a \( v = 98 \text{ m/s} \). He weighed less but had a smaller frontal area and so a smaller drag due to the air.

The size of the object that is falling through air presents another interesting application of air drag. If you fall from a 5-m high branch of a tree, you will likely get hurt—possibly fracturing a bone. However, a small squirrel does this all the time, without getting hurt. You don’t reach a terminal velocity in such a short distance, but the squirrel does.
The following interesting quote on animal size and terminal velocity is from a 1928 essay by a British biologist, J.B.S. Haldane, titled “On Being the Right Size.”

To the mouse and any smaller animal, [gravity] presents practically no dangers. You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away, provided that the ground is fairly soft. A rat is killed, a man is broken, and a horse splashes. For the resistance presented to movement by the air is proportional to the surface of the moving object. Divide an animal's length, breadth, and height each by ten; its weight is reduced to a thousandth, but its surface only to a hundredth. So the resistance to falling in the case of the small animal is relatively ten times greater than the driving force.

The above quadratic dependence of air drag upon velocity does not hold if the object is very small, is going very slow, or is in a denser medium than air. Then we find that the drag force is proportional just to the velocity. This relationship is given by Stokes' law, which states that

\[ F_s = 6\pi r \eta v, \]  

(5.24)

where \( r \) is the radius of the object, \( \eta \) is the viscosity of the fluid, and \( v \) is the object's velocity.

Good examples of this law are provided by microorganisms, pollen, and dust particles. Because each of these objects is so small, we find that many of these objects travel unaided only at a constant (terminal) velocity. Terminal velocities for bacteria (size about 1 μm) can be about 2 μm/s. To move at a greater speed, many bacteria swim using flagella (organelles shaped like little tails) that are powered by little motors embedded in the cell. Sediment in a lake can move at a greater terminal velocity (about 5 μm/s), so it can take days to reach the bottom of the lake after being deposited on the surface.

If we compare animals living on land with those in water, you can see how drag has influenced evolution. Fishes, dolphins, and even massive whales are streamlined in shape to reduce drag forces. Birds are streamlined and migratory species that fly large distances often have particular features such as long necks. Flocks of birds fly in the shape of a spear head as the flock forms a streamlined pattern (see Figure 5.10). In humans, one important example of streamlining is the shape of sperm, which need to be efficient in their use of energy.

---

**Stokes’ Law**

\[ F_s = 6\pi r \eta v, \]  

(5.25)

where \( r \) is the radius of the object, \( \eta \) is the viscosity of the fluid, and \( v \) is the object's velocity.

---

If we compare animals living on land with those in water, you can see how drag has influenced evolution. Fishes, dolphins, and even massive whales are streamlined in shape to reduce drag forces. Birds are streamlined and migratory species that fly large distances often have particular features such as long necks. Flocks of birds fly in the shape of a spear head as the flock forms a streamlined pattern (see Figure 5.10). In humans, one important example of streamlining is the shape of sperm, which need to be efficient in their use of energy.

---

**Galileo’s Experiment**

Galileo is said to have dropped two objects of different masses from the Tower of Pisa. He measured how long it took each to reach the ground. Since stopwatches weren't readily available, how do you think he measured their fall time? If the objects were the same size, but with different masses, what do you think he should have observed? Would this result be different if done on the Moon?

---

**5.3 Elasticity: Stress and Strain**

We now move from consideration of forces that affect the motion of an object (such as friction and drag) to those that affect an object's shape. If a bulldozer pushes a car into a wall, the car will not move but it will noticeably change shape. A change in shape due to the application of a force is a deformation. Even very small forces are known to cause some deformation. For small deformations, two important characteristics are observed. First, the object returns to its original shape when the force is removed—that is, the deformation is elastic for small deformations. Second, the size of the deformation is proportional to the...
force—that is, for small deformations, Hooke’s law is obeyed. In equation form, Hooke’s law is given by

\[ F = k \Delta L, \]  

(5.26)

where \( \Delta L \) is the amount of deformation (the change in length, for example) produced by the force \( F \), and \( k \) is a proportionality constant that depends on the shape and composition of the object and the direction of the force. Note that this force is a function of the deformation \( \Delta L \)—it is not constant as a kinetic friction force is. Rearranging this to

\[ \Delta L = \frac{F}{k}, \]  

(5.27)

makes it clear that the deformation is proportional to the applied force. **Figure 5.11** shows the Hooke’s law relationship between the extension \( \Delta L \) of a spring or of a human bone. For metals or springs, the straight line region in which Hooke’s law pertains is much larger. Bones are brittle and the elastic region is small and the fracture abrupt. Eventually a large enough stress to the material will cause it to break or fracture. **Tensile strength** is the breaking stress that will cause permanent deformation or fracture of a material.

<table>
<thead>
<tr>
<th>Hooke’s Law</th>
</tr>
</thead>
</table>
| \[ F = k \Delta L, \]  

(5.28) |
| where \( \Delta L \) is the amount of deformation (the change in length, for example) produced by the force \( F \), and \( k \) is a proportionality constant that depends on the shape and composition of the object and the direction of the force. |
| \[ \Delta L = \frac{F}{k}, \]  

(5.29) |

**Figure 5.11** A graph of deformation \( \Delta L \) versus applied force \( F \). The straight segment is the linear region where Hooke’s law is obeyed. The slope of the straight region is \( \frac{1}{k} \). For larger forces, the graph is curved but the deformation is still elastic—\( \Delta L \) will return to zero if the force is removed. Still greater forces permanently deform the object until it finally fractures. The shape of the curve near fracture depends on several factors, including how the force \( F \) is applied. Note that in this graph the slope increases just before fracture, indicating that a small increase in \( F \) is producing a large increase in \( L \) near the fracture.

The proportionality constant \( k \) depends upon a number of factors for the material. For example, a guitar string made of nylon stretches when it is tightened, and the elongation \( \Delta L \) is proportional to the force applied (at least for small deformations). Thicker nylon strings and ones made of steel stretch less for the same applied force, implying they have a larger \( k \) (see **Figure 5.12**). Finally, all three strings return to their normal lengths when the force is removed, provided the deformation is small. Most materials will behave in this manner if the deformation is less than about 0.1% or about 1 part in \( 10^3 \).
The same force, in this case a weight \( w \), applied to three different guitar strings of identical length produces the three different deformations shown as shaded segments. The string on the left is thin nylon, the one in the middle is thicker nylon, and the one on the right is steel.

**Stretch Yourself a Little**

How would you go about measuring the proportionality constant \( k \) of a rubber band? If a rubber band stretched 3 cm when a 100-g mass was attached to it, then how much would it stretch if two similar rubber bands were attached to the same mass—even if put together in parallel or alternatively if tied together in series?

We now consider three specific types of deformations: changes in length (tension and compression), sideways shear (stress), and changes in volume. All deformations are assumed to be small unless otherwise stated.

**Changes in Length—Tension and Compression: Elastic Modulus**

A change in length \( \Delta L \) is produced when a force is applied to a wire or rod parallel to its length \( L_0 \), either stretching it (a tension) or compressing it. (See Figure 5.13.)

Experiments have shown that the change in length (\( \Delta L \)) depends on only a few variables. As already noted, \( \Delta L \) is proportional to the force \( F \) and depends on the substance from which the object is made. Additionally, the change in length is proportional to the original length \( L_0 \) and inversely proportional to the cross-sectional area of the wire or rod. For example, a long guitar string will stretch more than a short one, and a thick string will stretch less than a thin one. We can combine all these factors into one equation for \( \Delta L \):

\[ \Delta L = \frac{kF}{A} \]

Figure 5.12 The same force, in this case a weight \( w \), applied to three different guitar strings of identical length produces the three different deformations shown as shaded segments. The string on the left is thin nylon, the one in the middle is thicker nylon, and the one on the right is steel.
where $\Delta L$ is the change in length, $F$ the applied force, $Y$ is a factor, called the elastic modulus or Young’s modulus, that depends on the substance, $A$ is the cross-sectional area, and $L_0$ is the original length. Table 5.3 lists values of $Y$ for several materials—those with a large $Y$ are said to have a large tensile stiffness because they deform less for a given tension or compression.

Table 5.3 Elastic Moduli\(^\text{[1]}\)

<table>
<thead>
<tr>
<th>Material</th>
<th>Young’s modulus (tension–compression) $Y$(^\text{(10}^9\text{ N/m}^2))</th>
<th>Shear modulus $S$(^\text{(10}^9\text{ N/m}^2))</th>
<th>Bulk modulus $B$(^\text{(10}^9\text{ N/m}^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>70</td>
<td>25</td>
<td>75</td>
</tr>
<tr>
<td>Bone – tension</td>
<td>16</td>
<td>80</td>
<td>8</td>
</tr>
<tr>
<td>Bone – compression</td>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brass</td>
<td>90</td>
<td>35</td>
<td>75</td>
</tr>
<tr>
<td>Brick</td>
<td>15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Concrete</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Glass</td>
<td>70</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>Granite</td>
<td>45</td>
<td>20</td>
<td>45</td>
</tr>
<tr>
<td>Hair (human)</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hardwood</td>
<td>15</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Iron, cast</td>
<td>100</td>
<td>40</td>
<td>90</td>
</tr>
<tr>
<td>Lead</td>
<td>16</td>
<td>5</td>
<td>50</td>
</tr>
<tr>
<td>Marble</td>
<td>60</td>
<td>20</td>
<td>70</td>
</tr>
<tr>
<td>Nylon</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Polystyrene</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Silk</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Spider thread</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Steel</td>
<td>210</td>
<td>80</td>
<td>130</td>
</tr>
<tr>
<td>Tendon</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Acetone</td>
<td></td>
<td>0.7</td>
<td></td>
</tr>
<tr>
<td>Ethanol</td>
<td></td>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td>Glycerin</td>
<td></td>
<td>4.5</td>
<td></td>
</tr>
<tr>
<td>Mercury</td>
<td></td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Mercury</td>
<td></td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Water</td>
<td></td>
<td>2.2</td>
<td></td>
</tr>
</tbody>
</table>

1. Approximate and average values. Young’s moduli $Y$ for tension and compression sometimes differ but are averaged here. Bone has significantly different Young’s moduli for tension and compression.
Strategy

The force is equal to the maximum tension, or $F = 3.0 \times 10^6$ N. The cross-sectional area is $\pi r^2 = 2.46 \times 10^{-3}$ m$^2$. The equation $\Delta L = \frac{FL}{YA}$ can be used to find the change in length.

Solution

All quantities are known. Thus,

$$\Delta L = \left(\frac{1}{210 \times 10^9 \text{ N/m}^2}\right)\left(3.0 \times 10^6 \text{ N/m}^2\right)\left(3020 \text{ m}\right)$$

$$= 18 \text{ m}.$$  \hspace{1cm} (5.31)

Discussion

This is quite a stretch, but only about 0.6% of the unsupported length. Effects of temperature upon length might be important in these environments.

Bones, on the whole, do not fracture due to tension or compression. Rather they generally fracture due to sideways impact or bending, resulting in the bone shearing or snapping. The behavior of bones under tension and compression is important because it determines the load the bones can carry. Bones are classified as weight-bearing structures such as columns in buildings and trees. Weight-bearing structures have special features; columns in building have steel-reinforcing rods while trees and bones are fibrous. The bones in different parts of the body serve different structural functions and are prone to different stresses. Thus the bone in the top of the femur is arranged in thin sheets separated by marrow while in other places the bones can be cylindrical and filled with marrow or just solid. Overweight people have a tendency toward bone damage due to sustained compressions in bone joints and tendons.

Another biological example of Hooke's law occurs in tendons. Functionally, the tendon (the tissue connecting muscle to bone) must stretch easily at first when a force is applied, but offer a much greater restoring force for a greater strain. Figure 5.15 shows a stress-strain relationship for a human tendon. Some tendons have a high collagen content so there is relatively little strain, or length change; others, like support tendons (as in the leg) can change length up to 10%. Note that this stress-strain curve is nonlinear, since the slope of the line changes in different regions. In the first part of the stretch called the toe region, the fibers in the tendon begin to align in the direction of the stress—this is called uncrimping. In the linear region, the fibrils will be stretched, and in the failure region individual fibers begin to break. A simple model of this relationship can be illustrated by springs in parallel: different springs are activated at different lengths of stretch. Examples of this are given in the problems at end of this chapter. Ligaments (tissue connecting bone to bone) behave in a similar way.

Unlike bones and tendons, which need to be strong as well as elastic, the arteries and lungs need to be very stretchable. The elastic properties of the arteries are essential for blood flow. The pressure in the arteries increases and arterial walls stretch when the blood is pumped out of the heart. When the aortic valve shuts, the pressure in the arteries drops and the arterial walls...
relax to maintain the blood flow. When you feel your pulse, you are feeling exactly this—the elastic behavior of the arteries as the blood gushes through with each pump of the heart. If the arteries were rigid, you would not feel a pulse. The heart is also an organ with special elastic properties. The lungs expand with muscular effort when we breathe in but relax freely and elastically when we breathe out. Our skins are particularly elastic, especially for the young. A young person can go from 100 kg to 60 kg with no visible sag in their skins. The elasticity of all organs reduces with age. Gradual physiological aging through reduction in elasticity starts in the early 20s.

**Example 5.4 Calculating Deformation: How Much Does Your Leg Shorten When You Stand on It?**

Calculate the change in length of the upper leg bone (the femur) when a 70.0 kg man supports 62.0 kg of his mass on it, assuming the bone to be equivalent to a uniform rod that is 40.0 cm long and 2.00 cm in radius.

**Strategy**
The force is equal to the weight supported, or

\[ F = mg = (62.0 \text{ kg})(9.80 \text{ m/s}^2) = 607.6 \text{ N}, \]

and the cross-sectional area is \( \pi r^2 = 1.257 \times 10^{-3} \text{ m}^2 \). The equation \( \Delta L = \frac{F}{YA}L_0 \) can be used to find the change in length.

**Solution**
All quantities except \( \Delta L \) are known. Note that the compression value for Young’s modulus for bone must be used here. Thus,

\[ \Delta L = \left( \frac{1}{9 \times 10^9 \text{ N/m}^2} \right) \left( \frac{607.6 \text{ N}}{1.257 \times 10^{-3} \text{ m}^2} \right) (0.400 \text{ m}) \]

\[ = 2 \times 10^{-5} \text{ m}. \]

**Discussion**
This small change in length seems reasonable, consistent with our experience that bones are rigid. In fact, even the rather large forces encountered during strenuous physical activity do not compress or bend bones by large amounts. Although bone is rigid compared with fat or muscle, several of the substances listed in Table 5.3 have larger values of Young’s modulus \( Y \). In other words, they are more rigid.

The equation for change in length is traditionally rearranged and written in the following form:

\[ \frac{F}{A} = \frac{Y}{L_0} \Delta L. \]

The ratio of force to area, \( \frac{F}{A} \), is defined as stress (measured in \( \text{N/m}^2 \)), and the ratio of the change in length to length, \( \frac{\Delta L}{L_0} \), is defined as strain (a unitless quantity). In other words,

\[ \text{stress} = Y \times \text{strain}. \]

In this form, the equation is analogous to Hooke’s law, with stress analogous to force and strain analogous to deformation. If we again rearrange this equation to the form

\[ F = YA \frac{\Delta L}{L_0}, \]

we see that it is the same as Hooke’s law with a proportionality constant

\[ k = \frac{YA}{L_0}. \]

This general idea—that force and the deformation it causes are proportional for small deformations—applies to changes in length, sideways bending, and changes in volume.

**Stress**
The ratio of force to area, \( \frac{F}{A} \), is defined as stress measured in \( \text{N/m}^2 \).
Strain

The ratio of the change in length to length, \( \frac{\Delta L}{L_0} \), is defined as strain (a unitless quantity). In other words,

\[
\text{stress} = Y \times \text{strain}.
\]

(5.38)

Sideways Stress: Shear Modulus

Figure 5.16 illustrates what is meant by a sideways stress or a shearing force. Here the deformation is called \( \Delta x \) and it is perpendicular to \( L_0 \), rather than parallel as with tension and compression. Shear deformation behaves similarly to tension and compression and can be described with similar equations. The expression for shear deformation is

\[
\Delta x = \frac{1}{S} \frac{F}{A} L_0,
\]

(5.39)

where \( S \) is the shear modulus (see Table 5.3) and \( F \) is the force applied perpendicular to \( L_0 \) and parallel to the cross-sectional area \( A \). Again, to keep the object from accelerating, there are actually two equal and opposite forces \( F \) applied across opposite faces, as illustrated in Figure 5.16. The equation is logical—for example, it is easier to bend a long thin pencil (small \( A \)) than a short thick one, and both are more easily bent than similar steel rods (large \( S \)).

Shear Deformation

\[
\Delta x = \frac{1}{S} \frac{F}{A} L_0,
\]

(5.40)

where \( S \) is the shear modulus and \( F \) is the force applied perpendicular to \( L_0 \) and parallel to the cross-sectional area \( A \).

Figure 5.16 Shearing forces are applied perpendicular to the length \( L_0 \) and parallel to the area \( A \), producing a deformation \( \Delta x \). Vertical forces are not shown, but it should be kept in mind that in addition to the two shearing forces, \( F \), there must be supporting forces to keep the object from rotating. The distorting effects of these supporting forces are ignored in this treatment. The weight of the object also is not shown, since it is usually negligible compared with forces large enough to cause significant deformations.

Examination of the shear moduli in Table 5.3 reveals some telling patterns. For example, shear moduli are less than Young’s moduli for most materials. Bone is a remarkable exception. Its shear modulus is not only greater than its Young’s modulus, but it is as large as that of steel. This is why bones are so rigid.

The spinal column (consisting of 26 vertebral segments separated by discs) provides the main support for the head and upper part of the body. The spinal column has normal curvature for stability, but this curvature can be increased, leading to increased shearing forces on the lower vertebrae. Discs are better at withstanding compressional forces than shear forces. Because the spine is not vertical, the weight of the upper body exerts some of both. Pregnant women and people that are overweight (with large abdomens) need to move their shoulders back to maintain balance, thereby increasing the curvature in their spine and so increasing the shear component of the stress. An increased angle due to more curvature increases the shear forces along the plane. These higher shear forces increase the risk of back injury through ruptured discs. The lumbosacral disc (the wedge shaped disc below the last vertebrae) is particularly at risk because of its location.

The shear moduli for concrete and brick are very small; they are too highly variable to be listed. Concrete used in buildings can withstand compression, as in pillars and arches, but is very poor against shear, as might be encountered in heavily loaded floors or during earthquakes. Modern structures were made possible by the use of steel and steel-reinforced concrete. Almost by definition, liquids and gases have shear moduli near zero, because they flow in response to shearing forces.
Example 5.5 Calculating Force Required to Deform: That Nail Does Not Bend Much Under a Load

Find the mass of the picture hanging from a steel nail as shown in Figure 5.17, given that the nail bends only 1.80 µm. (Assume the shear modulus is known to two significant figures.)

Figure 5.17 Side view of a nail with a picture hung from it. The nail flexes very slightly (shown much larger than actual) because of the shearing effect of the supported weight. Also shown is the upward force of the wall on the nail, illustrating that there are equal and opposite forces applied across opposite cross sections of the nail. See Example 5.5 for a calculation of the mass of the picture.

Strategy
The force $F$ on the nail (neglecting the nail’s own weight) is the weight of the picture $w$. If we can find $w$, then the mass of the picture is just $\frac{w}{g}$. The equation $\Delta x = \frac{1}{S A} F L_0$ can be solved for $F$.

Solution
Solving the equation $\Delta x = \frac{1}{S A} F L_0$ for $F$, we see that all other quantities can be found:

$$F = \frac{S A \Delta x}{L_0}.$$  \hfill (5.41)

$S$ is found in Table 5.3 and is $S = 80 \times 10^9$ N/m$^2$. The radius $r$ is 0.750 mm (as seen in the figure), so the cross-sectional area is

$$A = \pi r^2 = 1.77 \times 10^{-6} \text{ m}^2.$$  \hfill (5.42)

The value for $L_0$ is also shown in the figure. Thus,

$$F = \frac{(80 \times 10^9 \text{ N/m}^2)(1.77 \times 10^{-6} \text{ m}^2)}{(5.00 \times 10^{-3} \text{ m})}(1.80 \times 10^{-6} \text{ m}) = 51 \text{ N}.$$  \hfill (5.43)

This 51 N force is the weight $w$ of the picture, so the picture’s mass is

$$m = \frac{w}{g} = \frac{F}{g} = 5.2 \text{ kg}.$$  \hfill (5.44)

Discussion
This is a fairly massive picture, and it is impressive that the nail flexes only 1.80 µm—an amount undetectable to the unaided eye.

Changes in Volume: Bulk Modulus

An object will be compressed in all directions if inward forces are applied evenly on all its surfaces as in Figure 5.18. It is relatively easy to compress gases and extremely difficult to compress liquids and solids. For example, air in a wine bottle is compressed when it is corked. But if you try corkscrewing a brim-full bottle, you cannot compress the wine—some must be removed if the cork is to be inserted. The reason for these different compressibilities is that atoms and molecules are separated by large empty spaces in gases but packed close together in liquids and solids. To compress a gas, you must force its atoms and molecules closer together. To compress liquids and solids, you must actually compress their atoms and molecules, and very strong electromagnetic forces in them oppose this compression.
An inward force on all surfaces compresses this cube. Its change in volume is proportional to the force per unit area and its original volume, and is related to the compressibility of the substance.

We can describe the compression or volume deformation of an object with an equation. First, we note that a force “applied evenly” is defined to have the same stress, or ratio of force to area $\frac{F}{A}$ on all surfaces. The deformation produced is a change in volume $\Delta V$, which is found to behave very similarly to the shear, tension, and compression previously discussed. (This is not surprising, since a compression of the entire object is equivalent to compressing each of its three dimensions.) The relationship of the change in volume to other physical quantities is given by

$$\Delta V = \frac{1}{B} \frac{F}{A} V_0,$$

where $B$ is the bulk modulus (see Table 5.3), $V_0$ is the original volume, and $\frac{F}{A}$ is the force per unit area applied uniformly inward on all surfaces. Note that no bulk moduli are given for gases.

What are some examples of bulk compression of solids and liquids? One practical example is the manufacture of industrial-grade diamonds by compressing carbon with an extremely large force per unit area. The carbon atoms rearrange their crystalline structure into the more tightly packed pattern of diamonds. In nature, a similar process occurs deep underground, where extremely large forces result from the weight of overlying material. Another natural source of large compressive forces is the pressure created by the weight of water, especially in deep parts of the oceans. Water exerts an inward force on all surfaces of a submerged object, and even on the water itself. At great depths, water is measurably compressed, as the following example illustrates.

### Example 5.6 Calculating Change in Volume with Deformation: How Much Is Water Compressed at Great Ocean Depths?

Calculate the fractional decrease in volume ($\frac{\Delta V}{V_0}$) for seawater at 5.00 km depth, where the force per unit area is $5.00 \times 10^7$ N / m$^2$.

**Strategy**

Equation $\Delta V = \frac{1}{B} \frac{F}{A} V_0$ is the correct physical relationship. All quantities in the equation except $\frac{\Delta V}{V_0}$ are known.

**Solution**

Solving for the unknown $\frac{\Delta V}{V_0}$ gives

$$\frac{\Delta V}{V_0} = \frac{1}{B} \frac{F}{A}.$$

Substituting known values with the value for the bulk modulus $B$ from Table 5.3,

$$\frac{\Delta V}{V_0} = \frac{5.00 \times 10^7}{2.2 \times 10^9} \text{ N/m}^2 = 0.023 = 2.3\%.$$

**Discussion**

Although measurable, this is not a significant decrease in volume considering that the force per unit area is about 500
atmospheres (1 million pounds per square foot). Liquids and solids are extraordinarily difficult to compress.

Conversely, very large forces are created by liquids and solids when they try to expand but are constrained from doing so—which is equivalent to compressing them to less than their normal volume. This often occurs when a contained material warms up, since most materials expand when their temperature increases. If the materials are tightly constrained, they deform or break their container. Another very common example occurs when water freezes. Water, unlike most materials, expands when it freezes, and it can easily fracture a boulder, rupture a biological cell, or crack an engine block that gets in its way.

Other types of deformations, such as torsion or twisting, behave analogously to the tension, shear, and bulk deformations considered here.

Masses & Springs

(Glossary)

deflection: change in shape due to the application of force

drag force: $F_D$, found to be proportional to the square of the speed of the object; mathematically

$$F_D \propto v^2$$

$$F_D = \frac{1}{2} C \rho A v^2,$$

where $C$ is the drag coefficient, $A$ is the area of the object facing the fluid, and $\rho$ is the density of the fluid

friction: a force that opposes relative motion or attempts at motion between systems in contact

Hooke's law: proportional relationship between the force $F$ on a material and the deformation $\Delta L$ it causes, $F = k \Delta L$

kinetic friction: a force that opposes the motion of two systems that are in contact and moving relative to one another

magnitude of kinetic friction: $f_k = \mu_k N$, where $\mu_k$ is the coefficient of kinetic friction

magnitude of static friction: $f_s \leq \mu_s N$, where $\mu_s$ is the coefficient of static friction and $N$ is the magnitude of the normal force

shear deformation: deformation perpendicular to the original length of an object

static friction: a force that opposes the motion of two systems that are in contact and are not moving relative to one another

Stokes' law: $F_s = 6 \pi r \eta v$, where $r$ is the radius of the object, $\eta$ is the viscosity of the fluid, and $v$ is the object's velocity

strain: ratio of change in length to original length

stress: ratio of force to area

tensile strength: the breaking stress that will cause permanent deformation or fraction of a material

Section Summary

5.1 Friction

• Friction is a contact force between systems that opposes the motion or attempted motion between them. Simple friction is proportional to the normal force $N$ pushing the systems together. (A normal force is always perpendicular to the contact surface between systems.) Friction depends on both of the materials involved. The magnitude of static friction $f_s$ between systems stationary relative to one another is given by

$$f_s \leq \mu_s N,$$

where $\mu_s$ is the coefficient of static friction, which depends on both of the materials.

• The kinetic friction force $f_k$ between systems moving relative to one another is given by
\[ f_k = \mu_k N, \]

where \( \mu_k \) is the coefficient of kinetic friction, which also depends on both materials.

### 5.2 Drag Forces

- Drag forces acting on an object moving in a fluid oppose the motion. For larger objects (such as a baseball) moving at a velocity \( v \) in air, the drag force is given by
  \[ F_D = \frac{1}{2} C \rho A v^2, \]
  where \( C \) is the drag coefficient (typical values are given in Table 5.2), \( A \) is the area of the object facing the fluid, and \( \rho \) is the fluid density.
- For small objects (such as a bacterium) moving in a denser medium (such as water), the drag force is given by Stokes’ law,
  \[ F_s = 6 \pi \eta r v, \]
  where \( r \) is the radius of the object, \( \eta \) is the fluid viscosity, and \( v \) is the object’s velocity.

### 5.3 Elasticity: Stress and Strain

- Hooke’s law is given by
  \[ F = k \Delta L, \]
  where \( \Delta L \) is the amount of deformation (the change in length), \( F \) is the applied force, and \( k \) is a proportionality constant that depends on the shape and composition of the object and the direction of the force. The relationship between the deformation and the applied force can also be written as
  \[ \Delta L = \frac{1}{Y} \frac{F}{A} L_0, \]
  where \( Y \) is Young’s modulus, which depends on the substance, \( A \) is the cross-sectional area, and \( L_0 \) is the original length.
- The ratio of force to area, \( \frac{F}{A} \), is defined as stress, measured in N/m².
- The ratio of the change in length to length, \( \frac{\Delta L}{L_0} \), is defined as strain (a unitless quantity). In other words,
  \[ \text{stress} = Y \times \text{strain}. \]
- The expression for shear deformation is
  \[ \Delta x = \frac{1}{S} \frac{F}{A} L_0, \]
  where \( S \) is the shear modulus and \( F \) is the force applied perpendicular to \( L_0 \) and parallel to the cross-sectional area \( A \).
- The relationship of the change in volume to other physical quantities is given by
  \[ \Delta V = \frac{1}{B} \frac{F}{A} V_0, \]
  where \( B \) is the bulk modulus, \( V_0 \) is the original volume, and \( \frac{F}{A} \) is the force per unit area applied uniformly inward on all surfaces.

### Conceptual Questions

#### 5.1 Friction

1. Define normal force. What is its relationship to friction when friction behaves simply?
2. The glue on a piece of tape can exert forces. Can these forces be a type of simple friction? Explain, considering especially that tape can stick to vertical walls and even to ceilings.
3. When you learn to drive, you discover that you need to let up slightly on the brake pedal as you come to a stop or the car will stop with a jerk. Explain this in terms of the relationship between static and kinetic friction.
4. When you push a piece of chalk across a chalkboard, it sometimes screeches because it rapidly alternates between slipping and sticking to the board. Describe this process in more detail, in particular explaining how it is related to the fact that kinetic friction is less than static friction. (The same slip-grab process occurs when tires screech on pavement.)

#### 5.2 Drag Forces

5. Athletes such as swimmers and bicyclists wear body suits in competition. Formulate a list of pros and cons of such suits.
6. Two expressions were used for the drag force experienced by a moving object in a liquid. One depended upon the speed, while the other was proportional to the square of the speed. In which types of motion would each of these expressions be more applicable than the other one?

7. As cars travel, oil and gasoline leaks onto the road surface. If a light rain falls, what does this do to the control of the car? Does a heavy rain make any difference?

8. Why can a squirrel jump from a tree branch to the ground and run away undamaged, while a human could break a bone in such a fall?

5.3 Elasticity: Stress and Strain

9. The elastic properties of the arteries are essential for blood flow. Explain the importance of this in terms of the characteristics of the flow of blood (pulsating or continuous).

10. What are you feeling when you feel your pulse? Measure your pulse rate for 10 s and for 1 min. Is there a factor of 6 difference?

11. Examine different types of shoes, including sports shoes and thongs. In terms of physics, why are the bottom surfaces designed as they are? What differences will dry and wet conditions make for these surfaces?

12. Would you expect your height to be different depending upon the time of day? Why or why not?

13. Would you expect a large or small stress to be required to deform a spider web? Why is this elasticity an important feature for a spider web?

14. Explain why pregnant women often suffer from back strain late in their pregnancy.

15. An old carpenter’s trick to keep nails from bending when they are pounded into hard materials is to grip the center of the nail firmly with pliers. Why does this help?

16. When a glass bottle full of vinegar warms up, both the vinegar and the glass expand, but vinegar expands significantly more with temperature than glass. The bottle will break if it was filled to its tightly capped lid. Explain why, and also explain how a pocket of air above the vinegar would prevent the break. (This is the function of the air above liquids in glass containers.)
Problems & Exercises

5.1 Friction

1. A physics major is cooking breakfast when he notices that the frictional force between his steel spatula and his Teflon frying pan is only 0.200 N. Knowing the coefficient of kinetic friction between the two materials, he quickly calculates the normal force. What is it?

2. (a) When rebuilding her car’s engine, a physics major must exert 300 N of force to insert a dry steel piston into a steel cylinder. What is the magnitude of the normal force between the piston and cylinder? (b) What is the magnitude of the force she would have to exert if the steel parts were oiled?

3. (a) What is the maximum frictional force in the knee joint of a person who supports 66.0 kg of her mass on that knee? (b) During strenuous exercise it is possible to exert forces to the joints that are easily ten times greater than the weight being supported. What is the maximum force of friction under such conditions? The frictional forces in joints are relatively small in all circumstances except when the joints deteriorate, such as from injury or arthritis. Increased frictional forces can cause further damage and pain.

4. Suppose you have a 120-kg wooden crate resting on a wood floor. (a) What maximum force can you exert horizontally on the crate without moving it? (b) If you continue to exert this force once the crate starts to slip, what will the magnitude of its acceleration then be?

5. (a) If half of the weight of a small 1.00×10⁻³ kg utility truck is supported by its two drive wheels, what is the magnitude of the maximum acceleration it can achieve on dry concrete? (b) Will a metal cabinet lying on the wooden bed of the truck slip if it accelerates at this rate? (c) Solve both problems assuming the truck has four-wheel drive.

6. A team of eight dogs pulls a sled with waxed wood runners on wet snow (mush!). The dogs have average masses of 19.0 kg, and the loaded sled with its rider has a mass of 210 kg. (a) Calculate the magnitude of the acceleration starting from rest if each dog exerts an average force of 185 N backward on the snow. (b) What is the magnitude of the acceleration once the sled starts to move? (c) For both situations, calculate the magnitude of the force in the coupling between the dogs and the sled.

7. Consider the 65.0-kg ice skater being pushed by two others shown in Figure 5.19. (a) Find the direction and magnitude of \( \mathbf{F}_{\text{tot}} \), the total force exerted on her by the others, given that the magnitudes \( F_1 \) and \( F_2 \) are 26.4 N and 18.6 N, respectively. (b) What is her initial acceleration if she is initially stationary and wearing steel-bladed skates that point in the direction of \( \mathbf{F}_{\text{tot}} \)? (c) What is her acceleration assuming she is already moving in the direction of \( \mathbf{F}_{\text{tot}} \)? (Remember that friction always acts in the direction opposite that of motion or attempted motion between surfaces in contact.)

Figure 5.19

8. Show that the acceleration of any object down a frictionless incline that makes an angle \( \theta \) with the horizontal is \( a = g \sin \theta \). (Note that this acceleration is independent of mass.)

9. Show that the acceleration of any object down an incline where friction behaves simply (that is, where \( f_k = \mu_k N \)) is \( a = g(\sin \theta - \mu_k \cos \theta) \). Note that the acceleration is independent of mass and reduces to the expression found in the previous problem when friction becomes negligibly small (\( \mu_k = 0 \)).

10. Calculate the deceleration of a snow boarder going up a 5.0º slope assuming the coefficient of friction for waxed wood on wet snow. The result of Exercise 5.9 may be useful, but be careful to consider the fact that the snow boarder is going uphill. Explicitly show how you follow the steps in Problem-Solving Strategies.

11. (a) Calculate the acceleration of a skier heading down a 10.0º slope, assuming the coefficient of friction for waxed wood on wet snow. (b) Find the angle of the slope down which this skier could coast at a constant velocity. You can neglect air resistance in both parts, and you will find the result of Exercise 5.9 to be useful. Explicitly show how you follow the steps in Problem-Solving Strategies.

12. If an object is to rest on an incline without slipping, then friction must equal the component of the weight of the object parallel to the incline. This requires greater and greater friction for steeper slopes. Show that the maximum angle of an incline above the horizontal for which an object will not slide down is \( \theta = \tan^{-1} \mu_k \). You may use the result of the previous problem. Assume that \( a = 0 \) and that static friction has reached its maximum value.
13. Calculate the maximum deceleration of a car that is heading down a 6° slope (one that makes an angle of 6° with the horizontal) under the following road conditions. You may assume that the weight of the car is evenly distributed on all four tires and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the deceleration. (Ignore rolling.) Calculate for a car: (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that \( \mu_s = 0.100 \), the same as for shoes on ice.

14. Calculate the maximum acceleration of a car that is heading up a 4° slope (one that makes an angle of 4° with the horizontal) under the following road conditions. Assume that only half the weight of the car is supported by the two drive wheels and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the acceleration. (Ignore rolling.) (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that \( \mu_s = 0.100 \), the same as for shoes on ice.

15. Repeat Exercise 5.14 for a car with four-wheel drive.

16. A freight train consists of two 8.00 \times 10^5 \text{ kg} engines and 45 cars with average masses of 5.50 \times 10^5 \text{ kg}. (a) What force must each engine exert backward on the track to accelerate the train at a rate of 5.00 \times 10^{-2} \text{ m/s}^2 if the force of friction is 7.50 \times 10^5 \text{ N}, assuming the engines exert identical forces? This is not a large frictional force for such a massive system. Rolling friction for trains is small, and consequently trains are very energy-efficient transportation systems. (b) What is the magnitude of the force in the coupling between the 37th and 38th cars (this is the force each exerts on the other), assuming all cars have the same mass and that friction is evenly distributed among all of the cars and engines?

17. Consider the 52.0-kg mountain climber in Figure 5.20. (a) Find the tension in the rope and the force that the mountain climber must exert with her feet on the vertical rock face to remain stationary. Assume that the force is exerted parallel to her legs. Also, assume negligible force exerted by her arms. (b) What is the minimum coefficient of friction between her shoes and the cliff?

18. A contestant in a winter sporting event pushes a 45.0-kg block of ice across a frozen lake as shown in Figure 5.21(a). (a) Calculate the minimum force \( F \) he must exert to get the block moving. (b) What is the magnitude of its acceleration once it starts to move, if that force is maintained?
19. Repeat Exercise 5.18 with the contestant pulling the block of ice with a rope over his shoulder at the same angle above the horizontal as shown in Figure 5.21(b).

![Figure 5.21](image)

Which method of sliding a block of ice requires less force—(a) pushing or (b) pulling at the same angle above the horizontal?

5.2 Drag Forces

20. The terminal velocity of a person falling in air depends upon the weight and the area of the person facing the fluid. Find the terminal velocity (in meters per second and kilometers per hour) of an 80.0-kg skydiver falling in a headfirst position with a surface area of $0.140 \text{ m}^2$.

21. A 60-kg and a 90-kg skydiver jump from an airplane at an altitude of 6000 m, both falling in a headfirst position. Make some assumption on their frontal areas and calculate their terminal velocities. How long will it take for each skydiver to reach the ground (assuming the time to reach terminal velocity is small)? Assume all values are accurate to three significant digits.

22. A 560-g squirrel with a surface area of $144 \text{ cm}^2$ falls from a 5.0-m tree to the ground. Estimate its terminal velocity. (Use a drag coefficient for a horizontal skydiver.) What will be the velocity of a 56-kg person hitting the ground, assuming no drag contribution in such a short distance?

23. To maintain a constant speed, the force provided by a car’s engine must equal the drag force plus the force of friction of the road (the rolling resistance). (a) What are the magnitudes of drag forces at 70 km/h and 100 km/h for a Toyota Camry? (Drag area is $0.70 \text{ m}^2$) (b) What is the magnitude of drag force at 70 km/h and 100 km/h for a Hummer H2? (Drag area is $2.44 \text{ m}^2$) Assume all values are accurate to three significant digits.

24. By what factor does the drag force on a car increase as it goes from 65 to 110 km/h?

25. Calculate the speed a spherical rain drop would achieve falling from 5.00 km (a) in the absence of air drag (b) with air drag. Take the size across of the drop to be 4 mm, the density to be $1.00 \times 10^3 \text{ kg/m}^3$, and the surface area to be $\pi r^2$.

26. Using Stokes’ law, verify that the units for viscosity are kilograms per meter per second.

27. Find the terminal velocity of a spherical bacterium (diameter $2.00 \mu\text{m}$) falling in water. You will first need to note that the drag force is equal to the weight at terminal velocity. Take the density of the bacterium to be $1.10 \times 10^3 \text{ kg/m}^3$.

28. Stokes’ law describes sedimentation of particles in liquids and can be used to measure viscosity. Particles in liquids achieve terminal velocity quickly. One can measure the time it takes for a particle to fall a certain distance and then use Stokes’ law to calculate the viscosity of the liquid. Suppose a steel ball bearing (density $7.8 \times 10^3 \text{ kg/m}^3$, diameter $3.0 \text{ mm}$) is dropped in a container of motor oil. It takes 12 s to fall a distance of 0.60 m. Calculate the viscosity of the oil.

5.3 Elasticity: Stress and Strain

29. During a wrestling match, a 150 kg wrestler briefly stands on one hand during a maneuver designed to perplex his already moribund adversary. By how much does the upper arm bone shorten in length? The bone can be represented by a uniform rod 38.0 cm long and 1.80 cm in radius. Her mass is 60.0 kg.

30. During an acrobatic act, one performer swings upside down hanging from a trapeze holding another, also upside-down, performer by the legs. If the upward force on the lower performer is three times her weight, how much do the bones (the femurs) in her upper legs stretch? You may assume each is equivalent to a uniform rod 35.0 cm long and 1.80 cm in radius.

31. (a) The “lead” in pencils is a graphite composition with a Young’s modulus of about $1 \times 10^9 \text{ N/m}^2$. Calculate the change in length of the lead in an automatic pencil if you tap it straight into the pencil with a force of 4.0 N. The lead is 0.50 mm in diameter and 60 mm long. (b) Is the answer reasonable? That is, does it seem to be consistent with what you have observed when using pencils?

32. TV broadcast antennas are the tallest artificial structures on Earth. In 1987, a 72.0-kg physicist placed himself and 400 kg of equipment at the top of one 610-m high antenna to perform gravity experiments. By how much was the antenna compressed, if we consider it to be equivalent to a steel cylinder 0.150 m in radius?

33. (a) By how much does a 65.0-kg mountain climber stretch her 0.800-cm diameter nylon rope when she hangs 35.0 m below a rock outcropping? (b) Does the answer seem to be consistent with what you have observed for nylon ropes? Would it make sense if the rope were actually a bungee cord?

34. A 20.0-m tall hollow aluminum flagpole is equivalent in stiffness to a solid cylinder 4.00 cm in diameter. A strong wind bends the pole much as a horizontal force of 900 N exerted at the top would. How far to the side does the top of the pole flex?
35. As an oil well is drilled, each new section of drill pipe supports its own weight and that of the pipe and drill bit beneath it. Calculate the stretch in a new 6.00 m length of steel pipe that supports 3.00 km of pipe having a mass of 20.0 kg/m and a 100-kg drill bit. The pipe is equivalent in stiffness to a solid cylinder 5.00 cm in diameter.

36. Calculate the force a piano tuner applies to stretch a steel piano wire 8.00 mm, if the wire is originally 0.850 mm in diameter and 1.35 m long.

37. A vertebra is subjected to a shearing force of 500 N. Find the shear deformation, taking the vertebra to be a cylinder 3.00 cm high and 4.00 cm in diameter.

38. A disk between vertebrae in the spine is subjected to a shearing force of 600 N. Find its shear deformation, taking it to have the shear modulus of \(1 \times 10^9\) \(\text{N/m}^2\). The disk is equivalent to a solid cylinder 0.700 cm high and 4.00 cm in diameter.

39. When using a pencil eraser, you exert a vertical force of 6.00 N at a distance of 2.00 cm from the hardwood-eraser joint. The pencil is 6.00 mm in diameter and is held at an angle of 20.0° to the horizontal. (a) By how much does the wood flex perpendicular to its length? (b) How much is it compressed lengthwise?

40. To consider the effect of wires hung on poles, we take data from Example 4.8, in which tensions in wires supporting a traffic light were calculated. The left wire made an angle 30.0° below the horizontal with the top of its pole and carried a tension of 108 N. The 12.0 m tall hollow aluminum pole is equivalent in stiffness to a 4.50 cm diameter solid cylinder. (a) How far is it bent to the side? (b) By how much is it compressed?

41. A farmer making grape juice fills a glass bottle to the brim and caps it tightly. The juice expands more than the glass when it warms up, in such a way that the volume increases by 0.2% (that is, \(\Delta V / V_0 = 2 \times 10^{-3}\)) relative to the space available. Calculate the magnitude of the normal force exerted by the juice per square centimeter if its bulk modulus is \(1.8 \times 10^9\) \(\text{N/m}^2\), assuming the bottle does not break. In view of your answer, do you think the bottle will survive?

42. (a) When water freezes, its volume increases by 9.05% (that is, \(\Delta V / V_0 = 9.05 \times 10^{-2}\)). What force per unit area is water capable of exerting on a container when it freezes? (It is acceptable to use the bulk modulus of water in this problem.) (b) Is it surprising that such forces can fracture engine blocks, boulders, and the like?

43. This problem returns to the tightrope walker studied in Example 4.6, who created a tension of \(3.94 \times 10^3\) N in a wire making an angle 5.0° below the horizontal with each supporting pole. Calculate how much this tension stretches the steel wire if it was originally 15 m long and 0.50 cm in diameter.
Figure 6.1 This Australian Grand Prix Formula 1 race car moves in a circular path as it makes the turn. Its wheels also spin rapidly—the latter completing many revolutions, the former only part of one (a circular arc). The same physical principles are involved in each. (credit: Richard Munckton)

Chapter Outline

6.1. Rotation Angle and Angular Velocity
- Define arc length, rotation angle, radius of curvature and angular velocity.
- Calculate the angular velocity of a car wheel spin.

6.2. Centripetal Acceleration
- Establish the expression for centripetal acceleration.
- Explain the centrifuge.

6.3. Centripetal Force
- Calculate coefficient of friction on a car tire.
- Calculate ideal speed and angle of a car on a turn.

6.4. Fictitious Forces and Non-inertial Frames: The Coriolis Force
- Discuss the inertial frame of reference.
- Discuss the non-inertial frame of reference.
- Describe the effects of the Coriolis force.

6.5. Newton’s Universal Law of Gravitation
- Explain Earth’s gravitational force.
- Describe the gravitational effect of the Moon on Earth.
- Discuss weightlessness in space.
- Examine the Cavendish experiment

- State Kepler’s laws of planetary motion.
- Derive the third Kepler’s law for circular orbits.
- Discuss the Ptolemaic model of the universe.

Introduction to Uniform Circular Motion and Gravitation

Many motions, such as the arc of a bird’s flight or Earth’s path around the Sun, are curved. Recall that Newton’s first law tells us that motion is along a straight line at constant speed unless there is a net external force. We will therefore study not only motion along curves, but also the forces that cause it, including gravitational forces. In some ways, this chapter is a continuation of
Dynamics: Newton's Laws of Motion as we study more applications of Newton's laws of motion.

This chapter deals with the simplest form of curved motion, uniform circular motion, motion in a circular path at constant speed. Studying this topic illustrates most concepts associated with rotational motion and leads to the study of many new topics we group under the name rotation. Pure rotational motion occurs when points in an object move in circular paths centered on one point. Pure translational motion is motion with no rotation. Some motion combines both types, such as a rotating hockey puck moving along ice.

### 6.1 Rotation Angle and Angular Velocity

In Kinematics, we studied motion along a straight line and introduced such concepts as displacement, velocity, and acceleration. Two-Dimensional Kinematics dealt with motion in two dimensions. Projectile motion is a special case of two-dimensional kinematics in which the object is projected into the air, while being subject to the gravitational force, and lands a distance away. In this chapter, we consider situations where the object does not land but moves in a curve. We begin the study of uniform circular motion by defining two angular quantities needed to describe rotational motion.

#### Rotation Angle

When objects rotate about some axis—for example, when the CD (compact disc) in Figure 6.2 rotates about its center—each point in the object follows a circular arc. Consider a line from the center of the CD to its edge. Each pit used to record sound along this line moves through the same angle in the same amount of time. The rotation angle is the amount of rotation and is analogous to linear distance. We define the rotation angle $\Delta \theta$ to be the ratio of the arc length to the radius of curvature:

$$\Delta \theta = \frac{\Delta s}{r}.$$  \hspace{1cm} (6.1)

![Figure 6.2](image_url)

**Figure 6.2** All points on a CD travel in circular arcs. The pits along a line from the center to the edge all move through the same angle $\Delta \theta$ in a time $\Delta t$.

![Figure 6.3](image_url)

**Figure 6.3** The radius of a circle is rotated through an angle $\Delta \theta$. The arc length $\Delta s$ is described on the circumference.

The arc length $\Delta s$ is the distance traveled along a circular path as shown in Figure 6.3. Note that $r$ is the radius of curvature of the circular path.
We know that for one complete revolution, the arc length is the circumference of a circle of radius \( r \). The circumference of a circle is \( 2\pi r \). Thus for one complete revolution the rotation angle is

\[ \Delta \theta = \frac{2\pi r}{r} = 2\pi. \] (6.2)

This result is the basis for defining the units used to measure rotation angles, \( \Delta \theta \) to be radians (rad), defined so that

\[ 2\pi \text{ rad} = 1 \text{ revolution}. \] (6.3)

A comparison of some useful angles expressed in both degrees and radians is shown in Table 6.1.

### Table 6.1 Comparison of Angular Units

<table>
<thead>
<tr>
<th>Degree Measures</th>
<th>Radian Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>( \frac{\pi}{6} )</td>
</tr>
<tr>
<td>60°</td>
<td>( \frac{\pi}{3} )</td>
</tr>
<tr>
<td>90°</td>
<td>( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>120°</td>
<td>( \frac{2\pi}{3} )</td>
</tr>
<tr>
<td>135°</td>
<td>( \frac{3\pi}{4} )</td>
</tr>
<tr>
<td>180°</td>
<td>( \pi )</td>
</tr>
</tbody>
</table>

Figure 6.4 Points 1 and 2 rotate through the same angle \( \Delta \theta \), but point 2 moves through a greater arc length \( \Delta s \) because it is at a greater distance from the center of rotation \( r \).

If \( \Delta \theta = 2\pi \) rad, then the CD has made one complete revolution, and every point on the CD is back at its original position. Because there are 360° in a circle or one revolution, the relationship between radians and degrees is thus

\[ 2\pi \text{ rad} = 360° \] (6.4)

so that

\[ 1 \text{ rad} = \frac{360°}{2\pi} \approx 57.3°. \] (6.5)

### Angular Velocity

How fast is an object rotating? We define angular velocity \( \omega \) as the rate of change of an angle. In symbols, this is

\[ \omega = \frac{\Delta \theta}{\Delta t}. \] (6.6)

where an angular rotation \( \Delta \theta \) takes place in a time \( \Delta t \). The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s).
Angular velocity $\omega$ is analogous to linear velocity $v$. To get the precise relationship between angular and linear velocity, we again consider a pit on the rotating CD. This pit moves an arc length $\Delta s$ in a time $\Delta t$, and so it has a linear velocity

$$v = \frac{\Delta s}{\Delta t}. \quad (6.7)$$

From $\Delta \theta = \frac{\Delta s}{r}$ we see that $\Delta s = r \Delta \theta$. Substituting this into the expression for $v$ gives

$$v = \frac{r \Delta \theta}{\Delta t} = r \omega. \quad (6.8)$$

We write this relationship in two different ways and gain two different insights:

$$v = r \omega \text{ or } \omega = \frac{v}{r}. \quad (6.9)$$

The first relationship in $v = r \omega$ or $\omega = \frac{v}{r}$ states that the linear velocity $v$ is proportional to the distance from the center of rotation, thus, it is largest for a point on the rim (largest $r$), as you might expect. We can also call this linear speed $v$ of a point on the rim the tangential speed. The second relationship in $v = r \omega$ or $\omega = \frac{v}{r}$ can be illustrated by considering the tire of a moving car. Note that the speed of a point on the rim of the tire is the same as the speed $v$ of the car. See Figure 6.5. So the faster the car moves, the faster the tire spins—large $v$ means a large $\omega$, because $v = r \omega$. Similarly, a larger-radius tire rotating at the same angular velocity ($\omega$) will produce a greater linear speed ($v$) for the car.

![Figure 6.5](image_url)

**Figure 6.5** A car moving at a velocity $v$ to the right has a tire rotating with an angular velocity $\omega$. The speed of the tread of the tire relative to the axle is $v$, the same as if the car were jacked up. Thus the car moves forward at linear velocity $v = r \omega$, where $r$ is the tire radius. A larger angular velocity for the tire means a greater velocity for the car.

**Example 6.1 How Fast Does a Car Tire Spin?**

Calculate the angular velocity of a 0.300 m radius car tire when the car travels at 15.0 m/s (about 54 km/h). See Figure 6.5.

**Strategy**
Because the linear speed of the tire rim is the same as the speed of the car, we have $v = 15.0$ m/s. The radius of the tire is given to be $r = 0.300$ m. Knowing $v$ and $r$, we can use the second relationship in $v = r \omega$, $\omega = \frac{v}{r}$ to calculate the angular velocity.

**Solution**
To calculate the angular velocity, we will use the following relationship:

$$\omega = \frac{v}{r}. \quad (6.10)$$

Substituting the knowns,

$$\omega = \frac{15.0 \text{ m/s}}{0.300 \text{ m}} = 50.0 \text{ rad/s}. \quad (6.11)$$
Discussion
When we cancel units in the above calculation, we get 50.0/s. But the angular velocity must have units of rad/s. Because radians are actually unitless (radians are defined as a ratio of distance), we can simply insert them into the answer for the angular velocity. Also note that if an earth mover with much larger tires, say 1.20 m in radius, were moving at the same speed of 15.0 m/s, its tires would rotate more slowly. They would have an angular velocity
\[
\omega = \frac{15.0 \text{ m/s}}{1.20 \text{ m}} = 12.5 \text{ rad/s}. \quad (6.12)
\]
Both \( \omega \) and \( v \) have directions (hence they are angular and linear velocities, respectively). Angular velocity has only two directions with respect to the axis of rotation—it is either clockwise or counterclockwise. Linear velocity is tangent to the path, as illustrated in Figure 6.6.

Take-Home Experiment
Tie an object to the end of a string and swing it around in a horizontal circle above your head (swing at your wrist). Maintain uniform speed as the object swings and measure the angular velocity of the motion. What is the approximate speed of the object? Identify a point close to your hand and take appropriate measurements to calculate the linear speed at this point. Identify other circular motions and measure their angular velocities.

![Figure 6.6](http://legacy.cnx.org/content/m42083/1.23/#fs-id1167062339457)

Figure 6.6 As an object moves in a circle, here a fly on the edge of an old-fashioned vinyl record, its instantaneous velocity is always tangent to the circle. The direction of the angular velocity is clockwise in this case.

Ladybug Revolution
Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug’s x,y position, velocity, and acceleration using vectors or graphs.

6.2 Centripetal Acceleration
We know from kinematics that acceleration is a change in velocity, either in its magnitude or in its direction, or both. In uniform circular motion, the direction of the velocity changes constantly, so there is always an associated acceleration, even though the magnitude of the velocity might be constant. You experience this acceleration yourself when you turn a corner in your car. (If you hold the wheel steady during a turn and move at constant speed, you are in uniform circular motion.) What you notice is a sideways acceleration because you and the car are changing direction. The sharper the curve and the greater your speed, the more noticeable this acceleration will become. In this section we examine the direction and magnitude of that acceleration.

Figure 6.8 shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation (the center of the circular path). This pointing is shown with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the centripetal acceleration \( a_c \); centripetal means “toward the center” or “center seeking.”
The direction of centripetal acceleration is toward the center of curvature, but what is its magnitude? Note that the triangle formed by the velocity vectors and the one formed by the radii $r$ and $\Delta s$ are similar. Both the triangles ABC and PQR are isosceles triangles (two equal sides). The two equal sides of the velocity vector triangle are the speeds $v_1 = v_2 = v$. Using the properties of two similar triangles, we obtain

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}.$$  \hspace{1cm} (6.13)

Acceleration is $\Delta v / \Delta t$, and so we first solve this expression for $\Delta v$:

$$\Delta v = \frac{v}{r} \Delta s.$$  \hspace{1cm} (6.14)

Then we divide this by $\Delta t$, yielding

$$\frac{\Delta v}{\Delta t} = \frac{\frac{v}{r} \Delta s}{\Delta t}.$$  \hspace{1cm} (6.15)

Finally, noting that $\Delta v / \Delta t = a_c$ and that $\Delta s / \Delta t = v$, the linear or tangential speed, we see that the magnitude of the centripetal acceleration is

$$a_c = \frac{v^2}{r},$$  \hspace{1cm} (6.16)

which is the acceleration of an object in a circle of radius $r$ at a speed $v$. So, centripetal acceleration is greater at high speeds and in sharp curves (smaller radius), as you have noticed when driving a car. But it is a bit surprising that $a_c$ is proportional to speed squared, implying, for example, that it is four times as hard to take a curve at 100 km/h than at 50 km/h. A sharp corner has a small radius, so that $a_c$ is greater for tighter turns, as you have probably noticed.

It is also useful to express $a_c$ in terms of angular velocity. Substituting $v = r\omega$ into the above expression, we find

$$a_c = (r\omega)^2 / r = r\omega^2.$$  \hspace{1cm} (6.17)

We can express the magnitude of centripetal acceleration using either of two equations:

$$a_c = \frac{v^2}{r}; \quad a_c = r\omega^2.$$  \hspace{1cm} (6.17)

Recall that the direction of $a_c$ is toward the center. You may use whichever expression is more convenient, as illustrated in examples below.

A centrifuge (see Figure 6.9b) is a rotating device used to separate specimens of different densities. High centripetal acceleration significantly decreases the time it takes for separation to occur, and makes separation possible with small samples. Centrifuges are used in a variety of applications in science and medicine, including the separation of single cell suspensions such as bacteria, viruses, and blood cells from a liquid medium and the separation of macromolecules, such as DNA and protein, from a solution. Centrifuges are often rated in terms of their centripetal acceleration relative to acceleration due to gravity ($g$); maximum centripetal acceleration of several hundred thousand $g$ is possible in a vacuum. Human centrifuges, extremely large centrifuges, have been used to test the tolerance of astronauts to the effects of accelerations larger than that of Earth’s gravity.
Example 6.2 How Does the Centripetal Acceleration of a Car Around a Curve Compare with That Due to Gravity?

What is the magnitude of the centripetal acceleration of a car following a curve of radius 500 m at a speed of 25.0 m/s (about 90 km/h)? Compare the acceleration with that due to gravity for this fairly gentle curve taken at highway speed. See Figure 6.9(a).

Strategy

Because \( v \) and \( r \) are given, the first expression in \( a_c = \frac{v^2}{r} \); \( a_c = r\omega^2 \) is the most convenient to use.

Solution

Entering the given values of \( v = 25.0 \) m/s and \( r = 500 \) m into the first expression for \( a_c \) gives

\[
   a_c = \frac{v^2}{r} = \frac{(25.0 \text{ m/s})^2}{500 \text{ m}} = 1.25 \text{ m/s}^2.
\]

Discussion

To compare this with the acceleration due to gravity \( (g = 9.80 \text{ m/s}^2) \), we take the ratio of \( a_c / g = \frac{1.25 \text{ m/s}^2}{(9.80 \text{ m/s}^2)} = 0.128 \). Thus, \( a_c = 0.128 \text{ g} \) and is noticeable especially if you were not wearing a seat belt.

Figure 6.9 (a) The car following a circular path at constant speed is accelerated perpendicular to its velocity, as shown. The magnitude of this centripetal acceleration is found in Example 6.2. (b) A particle of mass in a centrifuge is rotating at constant angular velocity. It must be accelerated perpendicular to its velocity or it would continue in a straight line. The magnitude of the necessary acceleration is found in Example 6.3.
Example 6.3 How Big Is the Centripetal Acceleration in an Ultracentrifuge?

Calculate the centripetal acceleration of a point 7.50 cm from the axis of an ultracentrifuge spinning at $7.5 \times 10^4$ rev/min. Determine the ratio of this acceleration to that due to gravity. See Figure 6.9(b).

**Strategy**

The term rev/min stands for revolutions per minute. By converting this to radians per second, we obtain the angular velocity $\omega$. Because $r$ is given, we can use the second expression in the equation $a_c = \frac{v^2}{r}$; $a_c = r\omega^2$ to calculate the centripetal acceleration.

**Solution**

To convert $7.50 \times 10^4$ rev/min to radians per second, we use the facts that one revolution is $2\pi$ rad and one minute is 60.0 s. Thus,

$$\omega = 7.50 \times 10^4 \text{ rev/min} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \times \frac{1 \text{ min}}{60.0 \text{ s}} = 7854 \text{ rad/s}. \quad (6.19)$$

Now the centripetal acceleration is given by the second expression in $a_c = \frac{v^2}{r}$; $a_c = r\omega^2$ as

$$a_c = r\omega^2. \quad (6.20)$$

Converting 7.50 cm to meters and substituting known values gives

$$a_c = (0.0750 \text{ m})(7854 \text{ rad/s})^2 = 4.63 \times 10^6 \text{ m/s}^2. \quad (6.21)$$

Note that the unitless radians are discarded in order to get the correct units for centripetal acceleration. Taking the ratio of $a_c$ to $g$ yields

$$\frac{a_c}{g} = \frac{4.63 \times 10^6}{9.80} = 4.72 \times 10^5. \quad (6.22)$$

**Discussion**

This last result means that the centripetal acceleration is 472,000 times as strong as $g$. It is no wonder that such high $\omega$ centrifuges are called ultracentrifuges. The extremely large accelerations involved greatly decrease the time needed to cause the sedimentation of blood cells or other materials.

Of course, a net external force is needed to cause any acceleration, just as Newton proposed in his second law of motion. So a net external force is needed to cause a centripetal acceleration. In Centripetal Force, we will consider the forces involved in circular motion.

6.3 Centripetal Force

Any force or combination of forces can cause a centripetal or radial acceleration. Just a few examples are the tension in the rope on a tether ball, the force of Earth’s gravity on the Moon, friction between roller skates and a rink floor, a banked roadway’s force on a car, and forces on the tube of a spinning centrifuge.

Any net force causing uniform circular motion is called a centripetal force. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton’s second law of motion, net force is mass times acceleration: net $F = ma$. For uniform circular motion, the acceleration is the centripetal acceleration— $a = a_c$.

Thus, the magnitude of centripetal force $F_c$ is

$$F_c = ma_c. \quad (6.23)$$
By using the expressions for centripetal acceleration $a_c$ from $a_c = \frac{v^2}{r}$; $a_c = r\omega^2$, we get two expressions for the centripetal force $F_c$ in terms of mass, velocity, angular velocity, and radius of curvature:

$$F_c = m\frac{v^2}{r}; \quad F_c = mr\omega^2.$$  \hfill (6.24)

You may use whichever expression for centripetal force is more convenient. Centripetal force $F_c$ is always perpendicular to the path and pointing to the center of curvature, because $a_c$ is perpendicular to the velocity and pointing to the center of curvature. Note that if you solve the first expression for $r$, you get

$$r = \frac{mv^2}{F_c}. \hfill (6.25)$$

This implies that for a given mass and velocity, a large centripetal force causes a small radius of curvature—that is, a tight curve.

![Figure 6.11](image)

**Figure 6.11** The frictional force supplies the centripetal force and is numerically equal to it. Centripetal force is perpendicular to velocity and causes uniform circular motion. The larger the $F_c$, the smaller the radius of curvature $r$ and the sharper the curve. The second curve has the same $v$, but a larger $F_c$ produces a smaller $r'$.

### Example 6.4 What Coefficient of Friction Do Car Tires Need on a Flat Curve?

(a) Calculate the centripetal force exerted on a 900 kg car that negotiates a 500 m radius curve at 25.0 m/s.

(b) Assuming an unbanked curve, find the minimum static coefficient of friction, between the tires and the road, static friction being the reason that keeps the car from slipping (see Figure 6.12).

**Strategy and Solution for (a)**

We know that $F_c = \frac{mv^2}{r}$. Thus,

$$F_c = \frac{mv^2}{r} = \frac{(900 \text{ kg})(25.0 \text{ m/s})^2}{500 \text{ m}} = 1125 \text{ N}. \hfill (6.26)$$

**Strategy for (b)**

Figure 6.12 shows the forces acting on the car on an unbanked (level ground) curve. Friction is to the left, keeping the car from slipping, and because it is the only horizontal force acting on the car, the friction is the centripetal force in this case. We know that the maximum static friction (at which the tires roll but do not slip) is $\mu_s N$, where $\mu_s$ is the static coefficient of friction and $N$ is the normal force. The normal force equals the car’s weight on level ground, so that $N = mg$. Thus the centripetal force in this situation is
Now we have a relationship between centripetal force and the coefficient of friction. Using the first expression for $F_c$ from the equation

\[
F_c = \frac{mv^2}{r}
\]

\[
F_c = mr\omega^2
\]

\[
m\frac{v^2}{r} = \mu_s mg.
\]

We solve this for $\mu_s$, noting that mass cancels, and obtain

\[
\mu_s = \frac{v^2}{rg}
\]

**Solution for (b)**

Substituting the knowns,

\[
\mu_s = \frac{(25.0 \text{ m/s})^2}{(500 \text{ m})(9.80 \text{ m/s}^2)} = 0.13.
\]

(Because coefficients of friction are approximate, the answer is given to only two digits.)

**Discussion**

We could also solve part (a) using the first expression in $F_c = \frac{mv^2}{r}$, because $m$, $v$, and $r$ are given. The coefficient of friction found in part (b) is much smaller than is typically found between tires and roads. The car will still negotiate the curve if the coefficient is greater than 0.13, because static friction is a responsive force, being able to assume a value less than but no more than $\mu_s N$. A higher coefficient would also allow the car to negotiate the curve at a higher speed, but if the coefficient of friction is less, the safe speed would be less than 25 m/s. Note that mass cancels, implying that in this example, it does not matter how heavily loaded the car is to negotiate the turn. Mass cancels because friction is assumed proportional to the normal force, which in turn is proportional to mass. If the surface of the road were banked, the normal force would be less as will be discussed below.

---

**Figure 6.12** This car on level ground is moving away and turning to the left. The centripetal force causing the car to turn in a circular path is due to friction between the tires and the road. A minimum coefficient of friction is needed, or the car will move in a larger-radius curve and leave the roadway.

Let us now consider **banked curves**, where the slope of the road helps you negotiate the curve. See Figure 6.13. The greater the angle $\theta$, the faster you can take the curve. Race tracks for bikes as well as cars, for example, often have steeply banked curves. In an “ideally banked curve,” the angle $\theta$ is such that you can negotiate the curve at a certain speed without the aid of...
friction between the tires and the road. We will derive an expression for $\theta$ for an ideally banked curve and consider an example related to it.

For ideal banking, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force $N$ in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively. In cases in which forces are not parallel, it is most convenient to consider components along perpendicular axes—in this case, the vertical and horizontal directions.

Figure 6.13 shows a free body diagram for a car on a frictionless banked curve. If the angle $\theta$ is ideal for the speed and radius, then the net external force will equal the necessary centripetal force. The only two external forces acting on the car are its weight $w$ and the normal force of the road $N$. (A frictionless surface can only exert a force perpendicular to the surface—that is, a normal force.) These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude $mv^2/r$. Because this is the crucial force and it is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, and so this must equal the centripetal force—that is,

$$N \sin \theta = \frac{mv^2}{r}. \quad (6.32)$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From the figure, we see that the vertical component of the normal force is $N \cos \theta$, and the only other vertical force is the car’s weight. These must be equal in magnitude; thus,

$$N \cos \theta = mg. \quad (6.33)$$

Now we can combine the last two equations to eliminate $N$ and get an expression for $\theta$, as desired. Solving the second equation for $N = mg / (\cos \theta)$, and substituting this into the first yields

$$mg \sin \theta = \frac{mv^2}{r} \quad (6.34)$$

$$mg \tan(\theta) = \frac{mv^2}{r} \quad (6.35)$$

$$\tan \theta = \frac{v^2}{rg}.$$  

Taking the inverse tangent gives

$$\theta = \tan^{-1} \left( \frac{v^2}{rg} \right) \text{ (ideally banked curve, no friction).} \quad (6.36)$$

This expression can be understood by considering how $\theta$ depends on $v$ and $r$. A large $\theta$ will be obtained for a large $v$ and a small $r$. That is, roads must be steeply banked for high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve is frictionless. Note that $\theta$ does not depend on the mass of the vehicle.

**Example 6.5 What Is the Ideal Speed to Take a Steeply Banked Tight Curve?**

Curves on some test tracks and race courses, such as the Daytona International Speedway in Florida, are very steeply banked. This banking, with the aid of tire friction and very stable car configurations, allows the curves to be taken at very
high speed. To illustrate, calculate the speed at which a 100 m radius curve banked at 65.0° should be driven if the road is frictionless.

**Strategy**
We first note that all terms in the expression for the ideal angle of a banked curve except for speed are known; thus, we need only rearrange it so that speed appears on the left-hand side and then substitute known quantities.

**Solution**
Starting with

\[ \tan \theta = \frac{v^2}{rg} \quad (6.37) \]

we get

\[ v = (rg \tan \theta)^{1/2}. \quad (6.38) \]

Noting that \( \tan 65.0^\circ = 2.14 \), we obtain

\[ v = \left[ (100 \text{ m})(9.80 \text{ m/s}^2)(2.14) \right]^{1/2} \]

\[ = 45.8 \text{ m/s.} \quad (6.39) \]

**Discussion**
This is just about 165 km/h, consistent with a very steeply banked and rather sharp curve. Tire friction enables a vehicle to take the curve at significantly higher speeds.

Calculations similar to those in the preceding examples can be performed for a host of interesting situations in which centripetal force is involved—a number of these are presented in this chapter’s Problems and Exercises.

---

**Take-Home Experiment**
Ask a friend or relative to swing a golf club or a tennis racquet. Take appropriate measurements to estimate the centripetal acceleration of the end of the club or racquet. You may choose to do this in slow motion.

---

**Gravity and Orbits**
Move the sun, earth, moon and space station to see how it affects their gravitational forces and orbital paths. Visualize the sizes and distances between different heavenly bodies, and turn off gravity to see what would happen without it!

(This media type is not supported in this reader. Click to open media in browser.)

**Figure 6.14**

---

### 6.4 Fictitious Forces and Non-inertial Frames: The Coriolis Force

What do taking off in a jet airplane, turning a corner in a car, riding a merry-go-round, and the circular motion of a tropical cyclone have in common? Each exhibits fictitious forces—unreal forces that arise from motion and may seem real, because the observer’s frame of reference is accelerating or rotating.

When taking off in a jet, most people would agree it feels as if you are being pushed back into the seat as the airplane accelerates down the runway. Yet a physicist would say that you tend to remain stationary while the seat pushes forward on you, and there is no real force backward on you. An even more common experience occurs when you make a tight curve in your car—say, to the right. You feel as if you are thrown (that is, forced) toward the left relative to the car. Again, a physicist would say that you are going in a straight line but the car moves to the right, and there is no real force on you to the left. Recall Newton’s first law.
Figure 6.15 (a) The car driver feels herself forced to the left relative to the car when she makes a right turn. This is a fictitious force arising from the use of the car as a frame of reference. (b) In the Earth’s frame of reference, the driver moves in a straight line, obeying Newton’s first law, and the car moves to the right. There is no real force to the left on the driver relative to Earth. There is a real force to the right on the car to make it turn. We can reconcile these points of view by examining the frames of reference used. Let us concentrate on people in a car. Passengers instinctively use the car as a frame of reference, while a physicist uses Earth. The physicist chooses Earth because it is very nearly an inertial frame of reference—one in which all forces are real (that is, in which all forces have an identifiable physical origin). In such a frame of reference, Newton’s laws of motion take the form given in Dynamics: Newton’s Laws of Motion. The car is a non-inertial frame of reference because it is accelerated to the side. The force to the left sensed by car passengers is a fictitious force having no physical origin. There is nothing real pushing them left—the car, as well as the driver, is actually accelerating to the right.

Let us now take a mental ride on a merry-go-round—specifically, a rapidly rotating playground merry-go-round. You take the merry-go-round to be your frame of reference because you rotate together. In that non-inertial frame, you feel a fictitious force, named centrifugal force (not to be confused with centripetal force), trying to throw you off. You must hang on tightly to counteract the centrifugal force. In Earth’s frame of reference, there is no force trying to throw you off. Rather you must hang on to make yourself go in a circle because otherwise you would go in a straight line, right off the merry-go-round.

This inertial effect, carrying you away from the center of rotation if there is no centripetal force to cause circular motion, is put to good use in centrifuges (see Figure 6.17). A centrifuge spins a sample very rapidly, as mentioned earlier in this chapter. Viewed from the rotating frame of reference, the fictitious centrifugal force throws particles outward, hastening their sedimentation. The greater the angular velocity, the greater the centrifugal force. But what really happens is that the inertia of the particles carries them along a line tangent to the circle while the test tube is forced in a circular path by a centripetal force.
Centrifuges use inertia to perform their task. Particles in the fluid sediment come out because their inertia carries them away from the center of rotation. The large angular velocity of the centrifuge quickens the sedimentation. Ultimately, the particles will come into contact with the test tube walls, which will then supply the centripetal force needed to make them move in a circle of constant radius.

Let us now consider what happens if something moves in a frame of reference that rotates. For example, what if you slide a ball directly away from the center of the merry-go-round, as shown in Figure 6.18? The ball follows a straight path relative to Earth (assuming negligible friction) and a path curved to the right on the merry-go-round’s surface. A person standing next to the merry-go-round sees the ball moving straight and the merry-go-round rotating underneath it. In the merry-go-round’s frame of reference, we explain the apparent curve to the right by using a fictitious force, called the Coriolis force, that causes the ball to curve to the right. The fictitious Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths and allows us to apply Newton’s Laws in non-inertial frames of reference.

Looking down on the counterclockwise rotation of a merry-go-round, we see that a ball slid straight toward the edge follows a path curved to the right. The person slides the ball toward point B, starting at point A. Both points rotate to the shaded positions (A’ and B’) shown in the time that the ball follows the curved path in the rotating frame and a straight path in Earth’s frame.

Up until now, we have considered Earth to be an inertial frame of reference with little or no worry about effects due to its rotation. Yet such effects do exist—in the rotation of weather systems, for example. Most consequences of Earth’s rotation can be qualitatively understood by analogy with the merry-go-round. Viewed from above the North Pole, Earth rotates counterclockwise, as does the merry-go-round in Figure 6.18. As on the merry-go-round, any motion in Earth’s northern hemisphere experiences a
Coriolis force to the right. Just the opposite occurs in the southern hemisphere; there, the force is to the left. Because Earth's angular velocity is small, the Coriolis force is usually negligible, but for large-scale motions, such as wind patterns, it has substantial effects.

The Coriolis force causes hurricanes in the northern hemisphere to rotate in the counterclockwise direction, while the tropical cyclones (what hurricanes are called below the equator) in the southern hemisphere rotate in the clockwise direction. The terms hurricane, typhoon, and tropical storm are regionally-specific names for tropical cyclones, storm systems characterized by low pressure centers, strong winds, and heavy rains. Figure 6.19 helps show how these rotations take place. Air flows toward any region of low pressure, and tropical cyclones contain particularly low pressures. Thus winds flow toward the center of a tropical cyclone or a low-pressure weather system at the surface. In the northern hemisphere, these inward winds are deflected to the right, as shown in the figure, producing a counterclockwise circulation at the surface for low-pressure zones of any type. Low pressure at the surface is associated with rising air, which also produces cooling and cloud formation, making low-pressure patterns quite visible from space. Conversely, wind circulation around high-pressure zones is clockwise in the northern hemisphere but is less visible because high pressure is associated with sinking air, producing clear skies.

The rotation of tropical cyclones and the path of a ball on a merry-go-round can just as well be explained by inertia and the rotation of the system underneath. When non-inertial frames are used, fictitious forces, such as the Coriolis force, must be invented to explain the curved path. There is no identifiable physical source for these fictitious forces. In an inertial frame, inertia explains the path, and no force is found to be without an identifiable source. Either view allows us to describe nature, but a view in an inertial frame is the simplest and truest, in the sense that all forces have real origins and explanations.

![Figure 6.19](credit: NASA)

6.5 Newton's Universal Law of Gravitation

What do aching feet, a falling apple, and the orbit of the Moon have in common? Each is caused by the gravitational force. Our feet are strained by supporting our weight—the force of Earth’s gravity on us. An apple falls from a tree because of the same force acting a few meters above Earth’s surface. And the Moon orbits Earth because gravity is able to supply the necessary centripetal force at a distance of hundreds of millions of meters. In fact, the same force causes planets to orbit the Sun, stars to orbit the center of the galaxy, and galaxies to cluster together. Gravity is another example of underlying simplicity in nature. It is the weakest of the four basic forces found in nature, and in some ways the least understood. It is a force that acts at a distance, without physical contact, and is expressed by a formula that is valid everywhere in the universe, for masses and distances that vary from the tiny to the immense.

Sir Isaac Newton was the first scientist to precisely define the gravitational force, and to show that it could explain both falling bodies and astronomical motions. See Figure 6.20. But Newton was not the first to suspect that the same force caused both our weight and the motion of planets. His forerunner Galileo Galilei had contended that falling bodies and planetary motions had the same cause. Some of Newton’s contemporaries, such as Robert Hooke, Christopher Wren, and Edmund Halley, had also made some progress toward understanding gravitation. But Newton was the first to propose an exact mathematical form and to use that form to show that the motion of heavenly bodies should be conic sections—circles, ellipses, parabolas, and hyperbolas. This theoretical prediction was a major triumph—it had been known for some time that moons, planets, and comets follow such paths, but no one had been able to propose a mechanism that caused them to follow these paths and not others.
According to early accounts, Newton was inspired to make the connection between falling bodies and astronomical motions when he saw an apple fall from a tree and realized that if the gravitational force could extend above the ground to a tree, it might also reach the Sun. The inspiration of Newton’s apple is a part of worldwide folklore and may even be based in fact. Great importance is attached to it because Newton’s universal law of gravitation and his laws of motion answered very old questions about nature and gave tremendous support to the notion of underlying simplicity and unity in nature. Scientists still expect underlying simplicity to emerge from their ongoing inquiries into nature.

The gravitational force is relatively simple. It is always attractive, and it depends only on the masses involved and the distance between them. Stated in modern language, Newton’s universal law of gravitation states that every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Gravitational attraction is along a line joining the centers of mass of these two bodies. The magnitude of the force is the same on each, consistent with Newton’s third law.

Misconception Alert
The magnitude of the force on each object (one has larger mass than the other) is the same, consistent with Newton’s third law.

The bodies we are dealing with tend to be large. To simplify the situation we assume that the body acts as if its entire mass is concentrated at one specific point called the center of mass (CM), which will be further explored in Linear Momentum and...
Collisions. For two bodies having masses $m$ and $M$ with a distance $r$ between their centers of mass, the equation for Newton’s universal law of gravitation is

$$ F = \frac{G m M}{r^2}, \quad (6.40) $$

where $F$ is the magnitude of the gravitational force and $G$ is a proportionality factor called the gravitational constant. $G$ is a universal gravitational constant—that is, it is thought to be the same everywhere in the universe. It has been measured experimentally to be

$$ G = 6.674 \times 10^{-11} \text{N} \cdot \text{m}^2 \text{kg}^{-2} \quad (6.41) $$

in SI units. Note that the units of $G$ are such that a force in newtons is obtained from $F = \frac{G m M}{r^2}$, when considering masses in kilograms and distance in meters. For example, two 1.000 kg masses separated by 1.000 m will experience a gravitational attraction of $6.674 \times 10^{-11}$ N. This is an extraordinarily small force. The small magnitude of the gravitational force is consistent with everyday experience. We are unaware that even large objects like mountains exert gravitational forces on us. In fact, our body weight is the force of attraction of the entire Earth on us with a mass of $6 \times 10^{24}$ kg.

Recall that the acceleration due to gravity $g$ is about $9.80 \text{ m/s}^2$ on Earth. We can now determine why this is so. The weight of an object $mg$ is the gravitational force between it and Earth. Substituting $mg$ for $F$ in Newton’s universal law of gravitation gives

$$ mg = \frac{G M m}{r^2}, \quad (6.42) $$

where $m$ is the mass of the object, $M$ is the mass of Earth, and $r$ is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See Figure 6.22. The mass $m$ of the object cancels, leaving an equation for $g$:

$$ g = \frac{G M}{r^2}, \quad (6.43) $$

Substituting known values for Earth’s mass and radius (to three significant figures),

$$ g = \left(6.67 \times 10^{-11} \text{N} \cdot \text{m}^2 \text{kg}^{-2}\right) \times \frac{5.98 \times 10^{24} \text{kg}}{(6.38 \times 10^6 \text{m})^2}, \quad (6.44) $$

and we obtain a value for the acceleration of a falling body:

$$ g = 9.80 \text{ m/s}^2. \quad (6.45) $$

Figure 6.22 The distance between the centers of mass of Earth and an object on its surface is very nearly the same as the radius of Earth, because Earth is so much larger than the object.

This is the expected value and is independent of the body’s mass. Newton’s law of gravitation takes Galileo’s observation that all masses fall with the same acceleration a step further, explaining the observation in terms of a force that causes objects to fall—in fact, in terms of a universally existing force of attraction between masses.

Take-Home Experiment

Take a marble, a ball, and a spoon and drop them from the same height. Do they hit the floor at the same time? If you drop a piece of paper as well, does it behave like the other objects? Explain your observations.
Making Connections

Attempts are still being made to understand the gravitational force. As we shall see in Particle Physics (https://legacy.cnx.org/content/m42667/latest/), modern physics is exploring the connections of gravity to other forces, space, and time. General relativity alters our view of gravitation, leading us to think of gravitation as bending space and time.

In the following example, we make a comparison similar to one made by Newton himself. He noted that if the gravitational force caused the Moon to orbit Earth, then the acceleration due to gravity should equal the centripetal acceleration of the Moon in its orbit. Newton found that the two accelerations agreed "pretty nearly."

Example 6.6 Earth’s Gravitational Force Is the Centripetal Force Making the Moon Move in a Curved Path

(a) Find the acceleration due to Earth’s gravity at the distance of the Moon.

(b) Calculate the centripetal acceleration needed to keep the Moon in its orbit (assuming a circular orbit about a fixed Earth), and compare it with the value of the acceleration due to Earth’s gravity that you have just found.

Strategy for (a)

This calculation is the same as the one finding the acceleration due to gravity at Earth’s surface, except that $r$ is the distance from the center of Earth to the center of the Moon. The radius of the Moon’s nearly circular orbit is $3.84 \times 10^8$ m.

Solution for (a)

Substituting known values into the expression for $g$ found above, remembering that $M$ is the mass of Earth not the Moon, yields

$$g = \frac{GM}{r^2} = \left(6.67 \times 10^{-11} \text{N} \cdot \text{m}^2/\text{kg}^2\right) \times \frac{5.98 \times 10^{24} \text{kg}}{(3.84 \times 10^8 \text{m})^2}$$

$$= 2.70 \times 10^{-3} \text{m/s}^2$$

Strategy for (b)

Centripetal acceleration can be calculated using either form of

$$a_c = \frac{v^2}{r}$$

$$a_c = r\omega^2$$

We choose to use the second form:

$$a_c = r\omega^2$$

where $\omega$ is the angular velocity of the Moon about Earth.

Solution for (b)

Given that the period (the time it takes to make one complete rotation) of the Moon’s orbit is 27.3 days, (d) and using

$$1 \text{d} \times 24 \text{hr/d} \times 60 \text{min/hr} \times 60 \frac{\text{s}}{\text{min}} = 86,400 \text{ s}$$

we see that

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi \text{ rad}}{(27.3 \text{ d})(86,400 \text{ s/d})} = 2.66 \times 10^{-6} \text{rad/s}.$$  

The centripetal acceleration is

$$a_c = r\omega^2 = (3.84 \times 10^8 \text{ m})(2.66 \times 10^{-6} \text{ rad/s})^2$$

$$= 2.72 \times 10^{-3} \text{ m/s}^2$$

The direction of the acceleration is toward the center of the Earth.

Discussion

The centripetal acceleration of the Moon found in (b) differs by less than 1% from the acceleration due to Earth’s gravity found in (a). This agreement is approximate because the Moon’s orbit is slightly elliptical, and Earth is not stationary (rather the Earth-Moon system rotates about its center of mass, which is located some 1700 km below Earth’s surface). The clear implication is that Earth’s gravitational force causes the Moon to orbit Earth.
Why does Earth not remain stationary as the Moon orbits it? This is because, as expected from Newton’s third law, if Earth exerts a force on the Moon, then the Moon should exert an equal and opposite force on Earth (see Figure 6.23). We do not sense the Moon’s effect on Earth’s motion, because the Moon’s gravity moves our bodies right along with Earth but there are other signs on Earth that clearly show the effect of the Moon’s gravitational force as discussed in Satellites and Kepler’s Laws: An Argument for Simplicity.

![Figure 6.23](image)

Figure 6.23 (a) Earth and the Moon rotate approximately once a month around their common center of mass. (b) Their center of mass orbits the Sun in an elliptical orbit, but Earth’s path around the Sun has “wiggles” in it. Similar wiggles in the paths of stars have been observed and are considered direct evidence of planets orbiting those stars. This is important because the planets’ reflected light is often too dim to be observed.

**Tides**

Ocean tides are one very observable result of the Moon’s gravity acting on Earth. Figure 6.24 is a simplified drawing of the Moon’s position relative to the tides. Because water easily flows on Earth’s surface, a high tide is created on the side of Earth nearest to the Moon, where the Moon’s gravitational pull is strongest. Why is there also a high tide on the opposite side of Earth? The answer is that Earth is pulled toward the Moon more than the water on the far side, because Earth is closer to the Moon. So the water on the side of Earth closest to the Moon is pulled away from Earth, and Earth is pulled away from water on the far side. As Earth rotates, the tidal bulge (an effect of the tidal forces between an orbiting natural satellite and the primary planet that it orbits) keeps its orientation with the Moon. Thus there are two tides per day (the actual tidal period is about 12 hours and 25.2 minutes), because the Moon moves in its orbit each day as well.

![Figure 6.24](image)

Figure 6.24 The Moon causes ocean tides by attracting the water on the near side more than Earth, and by attracting Earth more than the water on the far side. The distances and sizes are not to scale. For this simplified representation of the Earth-Moon system, there are two high and two low tides per day at any location, because Earth rotates under the tidal bulge.

The Sun also affects tides, although it has about half the effect of the Moon. However, the largest tides, called spring tides, occur when Earth, the Moon, and the Sun are aligned. The smallest tides, called neap tides, occur when the Sun is at a 90° angle to the Earth-Moon alignment.
Tides are not unique to Earth but occur in many astronomical systems. The most extreme tides occur where the gravitational force is the strongest and varies most rapidly, such as near black holes (see Figure 6.26). A few likely candidates for black holes have been observed in our galaxy. These have masses greater than the Sun but have diameters only a few kilometers across. The tidal forces near them are so great that they can actually tear matter from a companion star.

"Weightlessness" and Microgravity

In contrast to the tremendous gravitational force near black holes is the apparent gravitational field experienced by astronauts orbiting Earth. What is the effect of “weightlessness” upon an astronaut who is in orbit for months? Or what about the effect of weightlessness upon plant growth? Weightlessness doesn’t mean that an astronaut is not being acted upon by the gravitational force. There is no “zero gravity” in an astronaut’s orbit. The term just means that the astronaut is in free-fall, accelerating with the acceleration due to gravity. If an elevator cable breaks, the passengers inside will be in free fall and will experience weightlessness. You can experience short periods of weightlessness in some rides in amusement parks.
Microgravity refers to an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface. Many interesting biology and physics topics have been studied over the past three decades in the presence of microgravity. Of immediate concern is the effect on astronauts of extended times in outer space, such as at the International Space Station. Researchers have observed that muscles will atrophy (waste away) in this environment. There is also a corresponding loss of bone mass. Study continues on cardiovascular adaptation to space flight. On Earth, blood pressure is usually higher in the feet than in the head, because the higher column of blood exerts a downward force on it, due to gravity. When standing, 70% of your blood is below the level of the heart, while in a horizontal position, just the opposite occurs. What difference does the absence of this pressure differential have upon the heart?

Some findings in human physiology in space can be clinically important to the management of diseases back on Earth. On a somewhat negative note, spaceflight is known to affect the human immune system, possibly making the crew members more vulnerable to infectious diseases. Experiments flown in space also have shown that some bacteria grow faster in microgravity than they do on Earth. However, on a positive note, studies indicate that microbial antibiotic production can increase by a factor of two in space-grown cultures. One hopes to be able to understand these mechanisms so that similar successes can be achieved on the ground. In another area of physics space research, inorganic crystals and protein crystals have been grown in outer space that have much higher quality than any grown on Earth, so crystallography studies on their structure can yield much better results.

Plants have evolved with the stimulus of gravity and with gravity sensors. Roots grow downward and shoots grow upward. Plants might be able to provide a life support system for long duration space missions by regenerating the atmosphere, purifying water, and producing food. Some studies have indicated that plant growth and development are not affected by gravity, but there is still uncertainty about structural changes in plants grown in a microgravity environment.

The Cavendish Experiment: Then and Now

As previously noted, the universal gravitational constant $G$ is determined experimentally. This definition was first done accurately by Henry Cavendish (1731–1810), an English scientist, in 1798, more than 100 years after Newton published his universal law of gravitation. The measurement of $G$ is very basic and important because it determines the strength of one of the four forces in nature. Cavendish's experiment was very difficult because he measured the tiny gravitational attraction between two ordinary-sized masses (tens of kilograms at most), using apparatus like that in Figure 6.28. Remarkably, his value for $G$ differs by less than 1% from the best modern value.

One important consequence of knowing $G$ was that an accurate value for Earth's mass could finally be obtained. This was done by measuring the acceleration due to gravity as accurately as possible and then calculating the mass of Earth $M$ from the relationship Newton's universal law of gravitation gives

$$mg = \frac{GMm}{r^2},$$  \hspace{1cm} (6.52)

where $m$ is the mass of the object, $M$ is the mass of Earth, and $r$ is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See Figure 6.21. The mass $m$ of the object cancels, leaving an equation for $g$:

$$g = \frac{GM}{r^2},$$  \hspace{1cm} (6.53)

Rearranging to solve for $M$ yields

$$M = \frac{gr^2}{G}.$$  \hspace{1cm} (6.54)
So \( M \) can be calculated because all quantities on the right, including the radius of Earth \( r \), are known from direct measurements. We shall see in Satellites and Kepler’s Laws: An Argument for Simplicity that knowing \( G \) also allows for the determination of astronomical masses. Interestingly, of all the fundamental constants in physics, \( G \) is by far the least well determined.

The Cavendish experiment is also used to explore other aspects of gravity. One of the most interesting questions is whether the gravitational force depends on substance as well as mass—for example, whether one kilogram of lead exerts the same gravitational pull as one kilogram of water. A Hungarian scientist named Roland von Eötvös pioneered this inquiry early in the 20th century. He found, with an accuracy of five parts per billion, that the gravitational force does not depend on the substance. Such experiments continue today, and have improved upon Eötvös’ measurements. Cavendish-type experiments such as those of Eric Adelberger and others at the University of Washington, have also put severe limits on the possibility of a fifth force and have verified a major prediction of general relativity—that gravitational energy contributes to rest mass. Ongoing measurements there use a torsion balance and a parallel plate (not spheres, as Cavendish used) to examine how Newton’s law of gravitation works over sub-millimeter distances. On this small-scale, do gravitational effects depart from the inverse square law? So far, no deviation has been observed.

![Figure 6.28](https://i.imgur.com/3Q5Q5Q.png) Cavendish used an apparatus like this to measure the gravitational attraction between the two suspended spheres (\( m \)) and the two on the stand (\( M \)) by observing the amount of torsion (twisting) created in the fiber. Distance between the masses can be varied to check the dependence of the force on distance. Modern experiments of this type continue to explore gravity.

### 6.6 Satellites and Kepler’s Laws: An Argument for Simplicity

Examples of gravitational orbits abound. Hundreds of artificial satellites orbit Earth together with thousands of pieces of debris. The Moon’s orbit about Earth has intrigued humans from time immemorial. The orbits of planets, asteroids, meteors, and comets about the Sun are no less interesting. If we look further, we see almost unimaginable numbers of stars, galaxies, and other celestial objects orbiting one another and interacting through gravity.

All these motions are governed by gravitational force, and it is possible to describe them to various degrees of precision. Precise descriptions of complex systems must be made with large computers. However, we can describe an important class of orbits without the use of computers, and we shall find it instructive to study them. These orbits have the following characteristics:

1. A small mass \( m \) orbits a much larger mass \( M \). This allows us to view the motion as if \( M \) were stationary—in fact, as if from an inertial frame of reference placed on \( M \)—without significant error. Mass \( m \) is the satellite of \( M \), if the orbit is gravitationally bound.

2. The system is isolated from other masses. This allows us to neglect any small effects due to outside masses.

The conditions are satisfied, to good approximation, by Earth’s satellites (including the Moon), by objects orbiting the Sun, and by the satellites of other planets. Historically, planets were studied first, and there is a classical set of three laws, called Kepler’s laws of planetary motion, that describe the orbits of all bodies satisfying the two previous conditions (not just planets in our solar system). These descriptive laws are named for the German astronomer Johannes Kepler (1571–1630), who devised them after careful study (over some 20 years) of a large amount of meticulously recorded observations of planetary motion done by Tycho Brahe (1546–1601). Such careful collection and detailed recording of methods and data are hallmarks of good science. Data constitute the evidence from which new interpretations and meanings can be constructed.

**Kepler’s Laws of Planetary Motion**

**Kepler’s First Law**

The orbit of each planet about the Sun is an ellipse with the Sun at one focus.
Figure 6.29 (a) An ellipse is a closed curve such that the sum of the distances from a point on the curve to the two foci ($f_1$ and $f_2$) is a constant. You can draw an ellipse as shown by putting a pin at each focus, and then placing a string around a pencil and the pins and tracing a line on paper. A circle is a special case of an ellipse in which the two foci coincide (thus any point on the circle is the same distance from the center). (b) For any closed gravitational orbit, $m$ follows an elliptical path with $M$ at one focus. Kepler’s first law states this fact for planets orbiting the Sun.

Kepler’s Second Law

Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times (see Figure 6.30).

Kepler’s Third Law

The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun. In equation form, this is

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}, \quad (6.55)$$

where $T$ is the period (time for one orbit) and $r$ is the average radius. This equation is valid only for comparing two small masses orbiting the same large one. Most importantly, this is a descriptive equation only, giving no information as to the cause of the equality.
Figure 6.30 The shaded regions have equal areas. It takes equal times for \( M \) to go from A to B, from C to D, and from E to F. The mass \( m \) moves fastest when it is closest to \( M \). Kepler’s second law was originally devised for planets orbiting the Sun, but it has broader validity.

Note again that while, for historical reasons, Kepler’s laws are stated for planets orbiting the Sun, they are actually valid for all bodies satisfying the two previously stated conditions.

**Example 6.7 Find the Time for One Orbit of an Earth Satellite**

Given that the Moon orbits Earth each 27.3 d and that it is an average distance of \( 3.84 \times 10^8 \) m from the center of Earth, calculate the period of an artificial satellite orbiting at an average altitude of 1500 km above Earth’s surface.

**Strategy**

The period, or time for one orbit, is related to the radius of the orbit by Kepler’s third law, given in mathematical form in

\[
\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}. \tag{6.56}
\]

Let us use the subscript 1 for the Moon and the subscript 2 for the satellite. We are asked to find \( T_2 \). The given information tells us that the orbital radius of the Moon is \( r_1 = 3.84 \times 10^8 \) m, and that the period of the Moon is \( T_1 = 27.3 \) d. The height of the artificial satellite above Earth’s surface is given, and so we must add the radius of Earth (6380 km) to get \( r_2 = (1500 + 6380) \) km = 7880 km. Now all quantities are known, and so \( T_2 \) can be found.

**Solution**

Kepler’s third law is

\[
\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}. \tag{6.57}
\]

To solve for \( T_2 \), we cross-multiply and take the square root, yielding

\[
T_2^2 = T_1^2 \left( \frac{r_2}{r_1} \right)^3 \tag{6.58}
\]

Substituting known values yields

\[
T_2 = T_1 \left( \frac{r_2}{r_1} \right)^{3/2} \tag{6.59}
\]

\[
T_2 = 27.3 \text{ d} \times \frac{24.0 \text{ h}}{1 \text{ d}} \times \left( \frac{7880 \text{ km}}{3.84 \times 10^5 \text{ km}} \right)^{3/2} = 1.93 \text{ h.}
\]

**Discussion**

This is a reasonable period for a satellite in a fairly low orbit. It is interesting that any satellite at this altitude will orbit in the
same amount of time. This fact is related to the condition that the satellite’s mass is small compared with that of Earth.

People immediately search for deeper meaning when broadly applicable laws, like Kepler’s, are discovered. It was Newton who took the next giant step when he proposed the law of universal gravitation. While Kepler was able to discover what was happening, Newton discovered that gravitational force was the cause.

**Derivation of Kepler’s Third Law for Circular Orbits**

We shall derive Kepler’s third law, starting with Newton’s laws of motion and his universal law of gravitation. The point is to demonstrate that the force of gravity is the cause for Kepler’s laws (although we will only derive the third one).

Let us consider a circular orbit of a small mass \( m \) around a large mass \( M \), satisfying the two conditions stated at the beginning of this section. Gravity supplies the centripetal force to mass \( m \). Starting with Newton’s second law applied to circular motion,

\[
 F_{\text{net}} = ma = m\frac{v^2}{r}. \tag{6.60}
\]

The net external force on mass \( m \) is gravity, and so we substitute the force of gravity for \( F_{\text{net}} \):

\[
 \frac{GmM}{r^2} = m\frac{v^2}{r}. \tag{6.61}
\]

The mass \( m \) cancels, yielding

\[
 \frac{GM}{r} = \frac{v^2}{r}. \tag{6.62}
\]

The fact that \( m \) cancels out is another aspect of the oft-noted fact that at a given location all masses fall with the same acceleration. Here we see that at a given orbital radius \( r \), all masses orbit at the same speed. (This was implied by the result of the preceding worked example.) Now, to get at Kepler’s third law, we must get the period \( T \) into the equation. By definition, period \( T \) is the time for one complete orbit. Now the average speed \( v \) is the circumference divided by the period—that is,

\[
 v = \frac{2\pi r}{T}. \tag{6.63}
\]

Substituting this into the previous equation gives

\[
 GM = \frac{4\pi^2 r^2}{T^2}. \tag{6.64}
\]

Solving for \( T^2 \) yields

\[
 T^2 = \frac{4\pi^2}{GM} r^3. \tag{6.65}
\]

Using subscripts 1 and 2 to denote two different satellites, and taking the ratio of the last equation for satellite 1 to satellite 2 yields

\[
 \frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}. \tag{6.66}
\]

This is Kepler’s third law. Note that Kepler’s third law is valid only for comparing satellites of the same parent body, because only then does the mass of the parent body \( M \) cancel.

Now consider what we get if we solve \( T^2 = \frac{4\pi^2}{GM} r^3 \) for the ratio \( r^3 / T^2 \). We obtain a relationship that can be used to determine the mass \( M \) of a parent body from the orbits of its satellites:

\[
 \frac{r^3}{T^2} = \frac{G}{4\pi^2}M. \tag{6.67}
\]

If \( r \) and \( T \) are known for a satellite, then the mass \( M \) of the parent can be calculated. This principle has been used extensively to find the masses of heavenly bodies that have satellites. Furthermore, the ratio \( r^3 / T^2 \) should be a constant for all satellites of the same parent body (because \( r^3 / T^2 = GM / 4\pi^2 \)). (See Table 6.2).

It is clear from Table 6.2 that the ratio of \( r^3 / T^2 \) is constant, at least to the third digit, for all listed satellites of the Sun, and for those of Jupiter. Small variations in that ratio have two causes—uncertainties in the \( r \) and \( T \) data, and perturbations of the
orbits due to other bodies. Interestingly, those perturbations can be—and have been—used to predict the location of new planets and moons. This is another verification of Newton’s universal law of gravitation.

### Making Connections

Newton’s universal law of gravitation is modified by Einstein’s general theory of relativity, as we shall see in Particle Physics [https://legacy.cnx.org/content/m42667/latest]. Newton’s gravity is not seriously in error—it was and still is an extremely good approximation for most situations. Einstein’s modification is most noticeable in extremely large gravitational fields, such as near black holes. However, general relativity also explains such phenomena as small but long-known deviations of the orbit of the planet Mercury from classical predictions.

### The Case for Simplicity

The development of the universal law of gravitation by Newton played a pivotal role in the history of ideas. While it is beyond the scope of this text to cover that history in any detail, we note some important points. The definition of planet set in 2006 by the International Astronomical Union (IAU) states that in the solar system, a planet is a celestial body that:

1. is in orbit around the Sun,
2. has sufficient mass to assume hydrostatic equilibrium and
3. has cleared the neighborhood around its orbit.

A non-satellite body fulfilling only the first two of the above criteria is classified as “dwarf planet.”

In 2006, Pluto was demoted to a ‘dwarf planet’ after scientists revised their definition of what constitutes a “true” planet.

### Table 6.2 Orbital Data and Kepler’s Third Law

<table>
<thead>
<tr>
<th>Parent</th>
<th>Satellite</th>
<th>Average orbital radius r(km)</th>
<th>Period T(y)</th>
<th>( r^3 / T^2 ) (km(^3) / y(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth</td>
<td>Moon</td>
<td>3.84×10(^5)</td>
<td>0.07481</td>
<td>1.01×10(^{19})</td>
</tr>
<tr>
<td>Sun</td>
<td>Mercury</td>
<td>5.79×10(^7)</td>
<td>0.2409</td>
<td>3.34×10(^{24})</td>
</tr>
<tr>
<td>Venus</td>
<td></td>
<td>1.082×10(^8)</td>
<td>0.6150</td>
<td>3.35×10(^{24})</td>
</tr>
<tr>
<td>Earth</td>
<td></td>
<td>1.496×10(^8)</td>
<td>1.000</td>
<td>3.35×10(^{24})</td>
</tr>
<tr>
<td>Mars</td>
<td></td>
<td>2.279×10(^8)</td>
<td>1.881</td>
<td>3.35×10(^{24})</td>
</tr>
<tr>
<td>Jupiter</td>
<td></td>
<td>7.783×10(^8)</td>
<td>11.86</td>
<td>3.35×10(^{24})</td>
</tr>
<tr>
<td>Saturn</td>
<td></td>
<td>1.427×10(^9)</td>
<td>29.46</td>
<td>3.35×10(^{24})</td>
</tr>
<tr>
<td>Neptune</td>
<td></td>
<td>4.497×10(^9)</td>
<td>164.8</td>
<td>3.35×10(^{24})</td>
</tr>
<tr>
<td>Pluto</td>
<td></td>
<td>5.90×10(^9)</td>
<td>248.3</td>
<td>3.33×10(^{24})</td>
</tr>
<tr>
<td>Jupiter</td>
<td>Io</td>
<td>4.22×10(^5)</td>
<td>0.00485 (1.77 d)</td>
<td>3.19×10(^{21})</td>
</tr>
<tr>
<td></td>
<td>Europa</td>
<td>6.71×10(^5)</td>
<td>0.00972 (3.55 d)</td>
<td>3.20×10(^{21})</td>
</tr>
<tr>
<td></td>
<td>Ganymede</td>
<td>1.07×10(^6)</td>
<td>0.0196 (7.16 d)</td>
<td>3.19×10(^{21})</td>
</tr>
<tr>
<td></td>
<td>Callisto</td>
<td>1.88×10(^6)</td>
<td>0.0457 (16.19 d)</td>
<td>3.20×10(^{21})</td>
</tr>
</tbody>
</table>

The universal law of gravitation is a good example of a physical principle that is very broadly applicable. That single equation for the gravitational force describes all situations in which gravity acts. It gives a cause for a vast number of effects, such as the orbits of the planets and moons in the solar system. It epitomizes the underlying unity and simplicity of physics.

Before the discoveries of Kepler, Copernicus, Galileo, Newton, and others, the solar system was thought to revolve around Earth as shown in Figure 6.31(a). This is called the Ptolemaic view, for the Greek philosopher who lived in the second century AD. This model is characterized by a list of facts for the motions of planets with no cause and effect explanation. There tended to be a different rule for each heavenly body and a general lack of simplicity.

Figure 6.31(b) represents the modern or Copernican model. In this model, a small set of rules and a single underlying force explain not only all motions in the solar system, but all other situations involving gravity. The breadth and simplicity of the laws of physics are compelling. As our knowledge of nature has grown, the basic simplicity of its laws has become ever more evident.
Figure 6.31 (a) The Ptolemaic model of the universe has Earth at the center with the Moon, the planets, the Sun, and the stars revolving about it in complex superpositions of circular paths. This geocentric model, which can be made progressively more accurate by adding more circles, is purely descriptive, containing no hints as to what are the causes of these motions. (b) The Copernican model has the Sun at the center of the solar system. It is fully explained by a small number of laws of physics, including Newton's universal law of gravitation.

Glossary

angular velocity: $\omega$, the rate of change of the angle with which an object moves on a circular path

arc length: $\Delta s$, the distance traveled by an object along a circular path

banked curve: the curve in a road that is sloping in a manner that helps a vehicle negotiate the curve

center of mass: the point where the entire mass of an object can be thought to be concentrated

centrifugal force: a fictitious force that tends to throw an object off when the object is rotating in a non-inertial frame of reference

centripetal acceleration: the acceleration of an object moving in a circle, directed toward the center

centripetal force: any net force causing uniform circular motion

Coriolis force: the fictitious force causing the apparent deflection of moving objects when viewed in a rotating frame of reference

fictitious force: a force having no physical origin

gravitational constant, $G$: a proportionality factor used in the equation for Newton’s universal law of gravitation; it is a universal constant—that is, it is thought to be the same everywhere in the universe

ideal angle: the angle at which a car can turn safely on a steep curve, which is in proportion to the ideal speed

ideal banking: the sloping of a curve in a road, where the angle of the slope allows the vehicle to negotiate the curve at a certain speed without the aid of friction between the tires and the road; the net external force on the vehicle equals the horizontal centripetal force in the absence of friction

ideal speed: the maximum safe speed at which a vehicle can turn on a curve without the aid of friction between the tire and the road

microgravity: an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface

Newton’s universal law of gravitation: every particle in the universe attracts every other particle with a force along a line joining them; the force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them

non-inertial frame of reference: an accelerated frame of reference

pit: a tiny indentation on the spiral track moulded into the top of the polycarbonate layer of CD

radians: a unit of angle measurement
radius of curvature: radius of a circular path

rotation angle: the ratio of the arc length to the radius of curvature on a circular path: $\Delta \theta = \frac{\Delta s}{r}$

ultracentrifuge: a centrifuge optimized for spinning a rotor at very high speeds

uniform circular motion: the motion of an object in a circular path at constant speed

### Section Summary

#### 6.1 Rotation Angle and Angular Velocity
- Uniform circular motion is motion in a circle at constant speed. The rotation angle $\Delta \theta$ is defined as the ratio of the arc length to the radius of curvature:

  $$\Delta \theta = \frac{\Delta s}{r},$$

  where arc length $\Delta s$ is distance traveled along a circular path and $r$ is the radius of curvature of the circular path. The quantity $\Delta \theta$ is measured in units of radians (rad), for which $2\pi \text{ rad} = 360^o = 1$ revolution.

- The conversion between radians and degrees is $1 \text{ rad} = 57.3^0$.

- Angular velocity $\omega$ is the rate of change of an angle,

  $$\omega = \frac{\Delta \theta}{\Delta t},$$

  where a rotation $\Delta \theta$ takes place in a time $\Delta t$. The units of angular velocity are radians per second (rad/s). Linear velocity $v$ and angular velocity $\omega$ are related by

  $$v = r \omega \text{ or } \omega = \frac{v}{r}.$$

#### 6.2 Centripetal Acceleration
- Centripetal acceleration $a_c$ is the acceleration experienced while in uniform circular motion. It always points toward the center of rotation. It is perpendicular to the linear velocity $v$ and has the magnitude

  $$a_c = \frac{v^2}{r}; \quad a_c = r \omega^2.$$

- The unit of centripetal acceleration is $\text{m/s}^2$.

#### 6.3 Centripetal Force
- Centripetal force $F_c$ is any force causing uniform circular motion. It is a "center-seeking" force that always points toward the center of rotation. It is perpendicular to linear velocity $v$ and has magnitude

  $$F_c = ma_c,$$

  which can also be expressed as

  $$F_c = m\frac{v^2}{r} \quad \text{or} \quad F_c = mr\omega^2.$$

#### 6.4 Fictitious Forces and Non-inertial Frames: The Coriolis Force
- Rotating and accelerated frames of reference are non-inertial.

- Fictitious forces, such as the Coriolis force, are needed to explain motion in such frames.

#### 6.5 Newton's Universal Law of Gravitation
- Newton’s universal law of gravitation: Every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. In equation form, this is

  $$F = \frac{GmM}{r^2},$$

  where $F$ is the magnitude of the gravitational force. $G$ is the gravitational constant, given by $G = 6.674 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$. 

---

This OpenStax book is available for free at https://legacy.cnx.org/content/col11951/1.1
• Newton’s law of gravitation applies universally.

6.6 Satellites and Kepler’s Laws: An Argument for Simplicity
• Kepler’s laws are stated for a small mass \( m \) orbiting a larger mass \( M \) in near-isolation. Kepler’s laws of planetary motion are then as follows:
  
  Kepler’s first law
  The orbit of each planet about the Sun is an ellipse with the Sun at one focus.

  Kepler’s second law
  Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times.

  Kepler’s third law
  The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun:

  \[
  \frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3},
  \]

  where \( T \) is the period (time for one orbit) and \( r \) is the average radius of the orbit.

• The period and radius of a satellite’s orbit about a larger body \( M \) are related by

  \[
  T^2 = \frac{4\pi^2}{GM}r^3
  \]

  or

  \[
  \frac{r^3}{T^2} = \frac{G}{4\pi^2}M.
  \]

Conceptual Questions

6.1 Rotation Angle and Angular Velocity
1. There is an analogy between rotational and linear physical quantities. What rotational quantities are analogous to distance and velocity?

6.2 Centripetal Acceleration
2. Can centripetal acceleration change the speed of circular motion? Explain.

6.3 Centripetal Force
3. If you wish to reduce the stress (which is related to centripetal force) on high-speed tires, would you use large- or small-diameter tires? Explain.

4. Define centripetal force. Can any type of force (for example, tension, gravitational force, friction, and so on) be a centripetal force? Can any combination of forces be a centripetal force?

5. If centripetal force is directed toward the center, why do you feel that you are ‘thrown’ away from the center as a car goes around a curve? Explain.
6. Race car drivers routinely cut corners as shown in Figure 6.32. Explain how this allows the curve to be taken at the greatest speed.

![Figure 6.32](image)

**Figure 6.32** Two paths around a race track curve are shown. Race car drivers will take the inside path (called cutting the corner) whenever possible because it allows them to take the curve at the highest speed.

7. A number of amusement parks have rides that make vertical loops like the one shown in Figure 6.33. For safety, the cars are attached to the rails in such a way that they cannot fall off. If the car goes over the top at just the right speed, gravity alone will supply the centripetal force. What other force acts and what is its direction if:
   (a) The car goes over the top at faster than this speed?
   (b) The car goes over the top at slower than this speed?

![Figure 6.33](image)

**Figure 6.33** Amusement rides with a vertical loop are an example of a form of curved motion.

8. What is the direction of the force exerted by the car on the passenger as the car goes over the top of the amusement ride pictured in Figure 6.33 under the following circumstances:
   (a) The car goes over the top at such a speed that the gravitational force is the only force acting?
   (b) The car goes over the top faster than this speed?
   (c) The car goes over the top slower than this speed?

9. As a skater forms a circle, what force is responsible for making her turn? Use a free body diagram in your answer.
10. Suppose a child is riding on a merry-go-round at a distance about halfway between its center and edge. She has a lunch box resting on wax paper, so that there is very little friction between it and the merry-go-round. Which path shown in Figure 6.34 will the lunch box take when she lets go? The lunch box leaves a trail in the dust on the merry-go-round. Is that trail straight, curved to the left, or curved to the right? Explain your answer.

Figure 6.34 A child riding on a merry-go-round releases her lunch box at point P. This is a view from above the clockwise rotation. Assuming it slides with negligible friction, will it follow path A, B, or C, as viewed from Earth’s frame of reference? What will be the shape of the path it leaves in the dust on the merry-go-round?

11. Do you feel yourself thrown to either side when you negotiate a curve that is ideally banked for your car’s speed? What is the direction of the force exerted on you by the car seat?

12. Suppose a mass is moving in a circular path on a frictionless table as shown in figure. In the Earth’s frame of reference, there is no centrifugal force pulling the mass away from the centre of rotation, yet there is a very real force stretching the string attaching the mass to the nail. Using concepts related to centripetal force and Newton’s third law, explain what force stretches the string, identifying its physical origin.

Figure 6.35 A mass attached to a nail on a frictionless table moves in a circular path. The force stretching the string is real and not fictional. What is the physical origin of the force on the string?

6.4 Fictitious Forces and Non-inertial Frames: The Coriolis Force

13. When a toilet is flushed or a sink is drained, the water (and other material) begins to rotate about the drain on the way down. Assuming no initial rotation and a flow initially directly straight toward the drain, explain what causes the rotation and which direction it has in the northern hemisphere. (Note that this is a small effect and in most toilets the rotation is caused by directional water jets.) Would the direction of rotation reverse if water were forced up the drain?

14. Is there a real force that throws water from clothes during the spin cycle of a washing machine? Explain how the water is removed.
15. In one amusement park ride, riders enter a large vertical barrel and stand against the wall on its horizontal floor. The barrel is spun up and the floor drops away. Riders feel as if they are pinned to the wall by a force something like the gravitational force. This is a fictitious force sensed and used by the riders to explain events in the rotating frame of reference of the barrel. Explain in an inertial frame of reference (Earth is nearly one) what pins the riders to the wall, and identify all of the real forces acting on them.

16. Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

17. Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not $9.80 \text{ m/s}^2$. Who do you agree with and why?

18. A non-rotating frame of reference placed at the center of the Sun is very nearly an inertial one. Why is it not exactly an inertial frame?

6.5 Newton's Universal Law of Gravitation

19. Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

20. Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not $9.80 \text{ m/s}^2$. Who do you agree with and why?

21. Draw a free body diagram for a satellite in an elliptical orbit showing why its speed increases as it approaches its parent body and decreases as it moves away.

22. Newton's laws of motion and gravity were among the first to convincingly demonstrate the underlying simplicity and unity in nature. Many other examples have since been discovered, and we now expect to find such underlying order in complex situations. Is there proof that such order will always be found in new explorations?

6.6 Satellites and Kepler's Laws: An Argument for Simplicity

23. In what frame(s) of reference are Kepler's laws valid? Are Kepler's laws purely descriptive, or do they contain causal information?
6.1 Rotation Angle and Angular Velocity

1. Semi-trailer trucks have an odometer on one hub of a trailer wheel. The hub is weighted so that it does not rotate, but it contains gears to count the number of wheel revolutions—it then calculates the distance traveled. If the wheel has a 1.15 m diameter and goes through 200,000 rotations, how many kilometers should the odometer read?

2. Microwave ovens rotate at a rate of about 6 rev/min. What is this in revolutions per second? What is the angular velocity in radians per second?

3. An automobile with 0.260 m radius tires travels 80,000 km before wearing them out. How many revolutions do the tires make, neglecting any backing up and any change in radius due to wear?

4. (a) What is the period of rotation of Earth in seconds? (b) What is the angular velocity of Earth? (c) Given that Earth has a radius of $6.4\times10^6$ m at its equator, what is the linear velocity at Earth’s surface?

5. A baseball pitcher brings his arm forward during a pitch, rotating the forearm about the elbow. If the velocity of the ball in the pitcher’s hand is 35.0 m/s and the ball is 0.300 m from the elbow joint, what is the angular velocity of the forearm?

6. In lacrosse, a ball is thrown from a net on the end of a stick by rotating the stick and forearm about the elbow. If the angular velocity of the ball about the elbow joint is 30.0 rad/s and the ball is 1.30 m from the elbow joint, what is the velocity of the ball?

7. A truck with 0.420-m-radius tires travels at 32.0 m/s. What is the angular velocity of the rotating tires in radians per second? What is this in rev/min?

8. Integrated Concepts

When kicking a football, the kicker rotates his leg about the hip joint.

(a) If the velocity of the tip of the kicker’s shoe is 35.0 m/s and the hip joint is 1.05 m from the tip of the shoe, what is the shoe tip’s angular velocity?

(b) The shoe is in contact with the initially stationary 0.500 kg football for 20.0 ms. What average force is exerted on the football to give it a velocity of 20.0 m/s?

(c) Find the maximum range of the football, neglecting air resistance.

9. Construct Your Own Problem

Consider an amusement park ride in which participants are rotated about a vertical axis in a cylinder with vertical walls. Once the angular velocity reaches its full value, the floor drops away and friction between the walls and the riders prevents them from sliding down. Construct a problem in which you calculate the necessary angular velocity that assures the riders will not slide down the wall. Include a free body diagram of a single rider. Among the variables to consider are the radius of the cylinder and the coefficients of friction between the riders’ clothing and the wall.

6.2 Centripetal Acceleration

10. A fairground ride spins its occupants inside a flying saucer-shaped container. If the horizontal circular path the riders follow has an 8.00 m radius, at how many revolutions per minute will the riders be subjected to a centripetal acceleration whose magnitude is 1.50 times that due to gravity?

11. A runner taking part in the 200 m dash must run around the end of a track that has a circular arc with a radius of curvature of 30.0 m. If he completes the 200 m dash in 23.2 s and runs at constant speed throughout the race, what is the magnitude of his centripetal acceleration as he runs the curved portion of the track?

12. Taking the age of Earth to be about $4\times10^9$ years and assuming its orbital radius of $1.5\times10^{11}$ m has not changed and is circular, calculate the approximate total distance Earth has traveled since its birth (in a frame of reference stationary with respect to the Sun).

13. The propeller of a World War II fighter plane is 2.30 m in diameter.

(a) What is its angular velocity in radians per second if it spins at 1200 rev/min?

(b) What is the linear speed of its tip at this angular velocity if the plane is stationary on the tarmac?

(c) What is the centripetal acceleration of the propeller tip under these conditions? Calculate it in meters per second squared and convert to multiples of $g$.

14. An ordinary workshop grindstone has a radius of 7.50 cm and rotates at 6500 rev/min.

(a) Calculate the magnitude of the centripetal acceleration at its edge in meters per second squared and convert it to multiples of $g$.

(b) What is the linear speed of a point on its edge?

15. Helicopter blades withstand tremendous stresses. In addition to supporting the weight of a helicopter, they are spun at rapid rates and experience large centripetal accelerations, especially at the tip.

(a) Calculate the magnitude of the centripetal acceleration at the tip of a 4.00 m long helicopter blade that rotates at 300 rev/min.

(b) Compare the linear speed of the tip with the speed of sound (taken to be 340 m/s).

16. Olympic ice skaters are able to spin at about 5 rev/s.

(a) What is their angular velocity in radians per second?

(b) What is the centripetal acceleration of the skater’s nose if it is 0.120 m from the axis of rotation?

(c) An exceptional skater named Dick Button was able to spin much faster in the 1950s than anyone since—at about 9 rev/s. What was the centripetal acceleration of the tip of his nose, assuming it is at 0.120 m radius?

(d) Comment on the magnitudes of the accelerations found. It is reputed that Button ruptured small blood vessels during his spins.

17. What percentage of the acceleration at Earth’s surface is the acceleration due to gravity at the position of a satellite located 300 km above Earth?
18. Verify that the linear speed of an ultracentrifuge is about 0.50 km/s, and Earth in its orbit is about 30 km/s by calculating:
   (a) The linear speed of a point on an ultracentrifuge 0.100 m from its center, rotating at 50,000 rev/min.
   (b) The linear speed of Earth in its orbit about the Sun (use data from the text on the radius of Earth’s orbit and approximate it as being circular).
19. A rotating space station is said to create “artificial gravity”—a loosely-defined term used for an acceleration that would be crudely similar to gravity. The outer wall of the rotating space station would become a floor for the astronauts, and centripetal acceleration supplied by the floor would allow astronauts to exercise and maintain muscle and bone strength more naturally than in non-rotating space environments. If the space station is 200 m in diameter, what angular velocity would produce an “artificial gravity” of 9.80 m/s² at the rim?
20. At takeoff, a commercial jet has a 60.0 m/s speed. Its tires have a diameter of 0.850 m.
   (a) At how many rev/min are the tires rotating?
   (b) What is the centripetal acceleration at the edge of the tire?
   (c) With what force must a determined 1.00×10⁻¹⁵ kg bacterium cling to the rim?
   (d) Take the ratio of this force to the bacterium’s weight.
21. Integrated Concepts
   Riders in an amusement park ride shaped like a Viking ship hung from a large pivot are rotated back and forth like a rigid pendulum. Sometime near the middle of the ride, the ship is momentarily motionless at the top of its circular arc. The ship then swings down under the influence of gravity.
   (a) Assuming negligible friction, find the speed of the riders at the bottom of its arc, given the system’s center of mass travels in an arc having a radius of 14.0 m and the riders are near the center of mass.
   (b) What is the centripetal acceleration at the bottom of the arc?
   (c) Draw a free body diagram of the forces acting on a rider at the bottom of the arc.
   (d) Find the force exerted by the ride on a 60.0 kg rider and compare it to her weight.
   (e) Discuss whether the answer seems reasonable.
22. Unreasonable Results
   A mother pushes her child on a swing so that his speed is 9.00 m/s at the lowest point of his path. The swing is suspended 2.00 m above the child’s center of mass.
   (a) What is the magnitude of the centripetal acceleration of the child at the low point?
   (b) What is the magnitude of the force the child exerts on the seat if his mass is 18.0 kg?
   (c) What is unreasonable about these results?
   (d) Which premises are unreasonable or inconsistent?

6.3 Centripetal Force
23. (a) A 22.0 kg child is riding a playground merry-go-round that is rotating at 40.0 rev/min. What centripetal force must she exert to stay on if she is 1.25 m from its center?
   (b) What centripetal force does she need to stay on an amusement park merry-go-round that rotates at 3.00 rev/min if she is 8.00 m from its center?
   (c) Compare each force with her weight.
24. Calculate the centripetal force on the end of a 100 m (radius) wind turbine blade that is rotating at 0.5 rev/s. Assume the mass is 4 kg.
25. What is the ideal banking angle for a gentle turn of 1.20 km radius on a highway with a 105 km/h speed limit (about 65 mi/h), assuming everyone travels at the limit?
26. What is the ideal speed to take a 100 m radius curve banked at a 20.0° angle?
27. (a) What is the radius of a bobsled turn banked at 75.0° and taken at 30.0 m/s, assuming it is ideally banked?
   (b) Calculate the centripetal acceleration.
   (c) Does this acceleration seem large to you?
28. Part of riding a bicycle involves leaning at the correct angle when making a turn, as seen in Figure 6.36. To be stable, the force exerted by the ground must be on a line going through the center of gravity. The force on the bicycle wheel can be resolved into two perpendicular components—friction parallel to the road (this must supply the centripetal force), and the vertical normal force (which must equal the system’s weight).

(a) Show that \( \theta \) (as defined in the figure) is related to the speed \( v \) and radius of curvature \( r \) of the turn in the same way as for an ideally banked roadway—that is,

\[
\theta = \tan^{-1} \frac{v^2}{rg}
\]

(b) Calculate \( \theta \) for a 12.0 m/s turn of radius 30.0 m (as in a race).

29. A large centrifuge, like the one shown in Figure 6.37(a), is used to expose aspiring astronauts to accelerations similar to those experienced in rocket launches and atmospheric reentries.

(a) At what angular velocity is the centripetal acceleration \( 10g \) if the rider is 15.0 m from the center of rotation?

(b) The rider’s cage hangs on a pivot at the end of the arm, allowing it to swing outward during rotation as shown in Figure 6.37(b). At what angle \( \theta \) below the horizontal will the cage hang when the centripetal acceleration is \( 10g \)? (Hint: The arm supplies centripetal force and supports the weight of the cage. Draw a free body diagram of the forces to see what the angle \( \theta \) should be.)

30. Integrated Concepts

If a car takes a banked curve at less than the ideal speed, friction is needed to keep it from sliding toward the inside of the curve (a real problem on icy mountain roads). (a) Calculate the ideal speed to take a 100 m radius curve banked at 15.0\(^\circ\), (b) What is the minimum coefficient of friction needed for a frightened driver to take the same curve at 20.0 km/h?
31. Modern roller coasters have vertical loops like the one shown in Figure 6.38. The radius of curvature is smaller at the top than on the sides so that the downward centripetal acceleration at the top will be greater than the acceleration due to gravity, keeping the passengers pressed firmly into their seats. What is the speed of the roller coaster at the top of the loop if the radius of curvature there is 15.0 m and the downward acceleration of the car is 1.50 g?

![Figure 6.38 Teardrop-shaped loops are used in the latest roller coasters so that the radius of curvature gradually decreases to a minimum at the top. This means that the centripetal acceleration builds from zero to a maximum at the top and gradually decreases again. A circular loop would cause a jolting change in acceleration at entry, a disadvantage discovered long ago in railroad curve design. With a small radius of curvature at the top, the centripetal acceleration can more easily be kept greater than g so that the passengers do not lose contact with their seats nor do they need seat belts to keep them in place.](image)

32. Unreasonable Results

(a) Calculate the minimum coefficient of friction needed for a car to negotiate an unbanked 50.0 m radius curve at 30.0 m/s.
(b) What is unreasonable about the result?
(c) Which premises are unreasonable or inconsistent?

6.5 Newton’s Universal Law of Gravitation

33. (a) Calculate Earth’s mass given the acceleration due to gravity at the North Pole is 9.830 m/s² and the radius of the Earth is 6371 km from center to pole.
(b) Compare this with the accepted value of $\frac{5.979 \times 10^{24}}{5} \text{ kg}$.

34. (a) Calculate the magnitude of the acceleration due to gravity on the surface of the Moon.
(b) Calculate the magnitude of the acceleration due to gravity at Earth due to the Sun.
(c) Take the ratio of the Moon’s acceleration to the Sun’s and comment on why the tides are predominantly due to the Moon in spite of this number.

35. (a) What is the acceleration due to gravity on the surface of the Moon?
(b) On the surface of Mars? The mass of Mars is $6.418 \times 10^{23}$ kg and its radius is $3.38 \times 10^6$ m.

36. (a) Calculate the acceleration due to gravity on the surface of the Sun.
(b) By what factor would your weight increase if you could stand on the Sun? (Never mind that you cannot.)

37. The Moon and Earth rotate about their common center of mass, which is located about 4700 km from the center of Earth. (This is 1690 km below the surface.)
(a) Calculate the magnitude of the acceleration due to the Moon’s gravity at that point.
(b) Calculate the magnitude of the centripetal acceleration of the center of Earth as it rotates about that point once each lunar month (about 27.3 d) and compare it with the acceleration found in part (a). Comment on whether or not they are equal and why they should or should not be.

38. Solve part (b) of Example 6.6 using $a_c = v^2/r$.

39. Astrology, that unlikely and vague pseudoscience, makes much of the position of the planets at the moment of one’s birth. The only known force a planet exerts on Earth is gravitational.

(a) Calculate the magnitude of the gravitational force exerted on a 4.20 kg baby by a 100 kg father 0.200 m away at birth (he is assisting, so he is close to the child).
(b) Calculate the magnitude of the force on the baby due to Jupiter if it is at its closest distance to Earth, some $6.29 \times 10^{11}$ m away. How does the force of Jupiter on the baby compare to the force of the father on the baby? Other objects in the room and the hospital building also exert similar gravitational forces. (Of course, there could be an unknown force acting, but scientists first need to be convinced that there is even an effect, much less that an unknown force causes it.)

40. The existence of the dwarf planet Pluto was proposed based on irregularities in Neptune’s orbit. Pluto was subsequently discovered near its predicted position. But it now appears that the discovery was fortuitous, because Pluto is small and the irregularities in Neptune’s orbit were not well known. To illustrate that Pluto has a minor effect on the orbit of Neptune compared with the closest planet to Neptune:
(a) Calculate the acceleration due to gravity at Neptune due to Pluto when they are $4.50 \times 10^{12}$ m apart, as they are at present. The mass of Pluto is $1.4 \times 10^{22}$ kg.
(b) Calculate the acceleration due to gravity at Neptune due to Uranus, presently about $2.50 \times 10^{12}$ m apart, and compare it with that due to Pluto. The mass of Uranus is $8.62 \times 10^{25}$ kg.
41. (a) The Sun orbits the Milky Way galaxy once each $2.60 \times 10^8$ y, with a roughly circular orbit averaging $3.00 \times 10^4$ light years in radius. (A light year is the distance traveled by light in 1 y.) Calculate the centripetal acceleration of the Sun in its galactic orbit. Does your result support the contention that a nearly inertial frame of reference can be located at the Sun? 
(b) Calculate the average speed of the Sun in its galactic orbit. Does the answer surprise you?

42. Unreasonable Result
A mountain 10.0 km from a person exerts a gravitational force on him equal to 2.00% of his weight.
(a) Calculate the mass of the mountain.
(b) Compare the mountain’s mass with that of Earth.
(c) What is unreasonable about these results?
(d) Which premises are unreasonable or inconsistent? (Note that accurate gravitational measurements can easily detect the effect of nearby mountains and variations in local geology.)

6.6 Satellites and Kepler’s Laws: An Argument for Simplicity
43. A geosynchronous Earth satellite is one that has an orbital period of precisely 1 day. Such orbits are useful for communication and weather observation because the satellite remains above the same point on Earth (provided it orbits in the equatorial plane in the same direction as Earth’s rotation). Calculate the radius of such an orbit based on the data for the moon in Table 6.2.

44. Calculate the mass of the Sun based on data for Earth’s orbit and compare the value obtained with the Sun’s actual mass.

45. Find the mass of Jupiter based on data for the orbit of one of its moons, and compare your result with its actual mass.

46. Find the ratio of the mass of Jupiter to that of Earth based on data in Table 6.2.

47. Astronomical observations of our Milky Way galaxy indicate that it has a mass of about $8.0 \times 10^{11}$ solar masses. A star orbiting on the galaxy’s periphery is about $6.0 \times 10^4$ light years from its center. (a) What should the orbital period of that star be? (b) If its period is $6.0 \times 10^7$ years instead, what is the mass of the galaxy? Such calculations are used to imply the existence of “dark matter” in the universe and have indicated, for example, the existence of very massive black holes at the centers of some galaxies.

48. Integrated Concepts
Space debris left from old satellites and their launchers is becoming a hazard to other satellites. (a) Calculate the speed of a satellite in an orbit 900 km above Earth’s surface. (b) Suppose a loose rivet is in an orbit of the same radius that intersects the satellite’s orbit at an angle of $90^\circ$ relative to Earth. What is the velocity of the rivet relative to the satellite just before striking it? (c) Given the rivet is 3.00 mm in size, how long will its collision with the satellite last? (d) If its mass is 0.500 g, what is the average force it exerts on the satellite? (e) How much energy in joules is generated by the collision? (The satellite’s velocity does not change appreciably, because its mass is much greater than the rivet’s.)

49. Unreasonable Results
(a) Based on Kepler’s laws and information on the orbital characteristics of the Moon, calculate the orbital radius for an Earth satellite having a period of 1.00 h. (b) What is unreasonable about this result? (c) What is unreasonable or inconsistent about the premise of a 1.00 h orbit?

50. Construct Your Own Problem
On February 14, 2000, the NEAR spacecraft was successfully inserted into orbit around Eros, becoming the first artificial satellite of an asteroid. Construct a problem in which you determine the orbital speed for a satellite near Eros. You will need to find the mass of the asteroid and consider such things as a safe distance for the orbit. Although Eros is not spherical, calculate the acceleration due to gravity on its surface at a point an average distance from its center of mass. Your instructor may also wish to have you calculate the escape velocity from this point on Eros.
Figure 7.1 How many forms of energy can you identify in this photograph of a wind farm in Iowa? (credit: Jürgen from Sandesneben, Germany, Wikimedia Commons)

Chapter Outline

7.1. Work: The Scientific Definition
- Explain how an object must be displaced for a force on it to do work.
- Explain how relative directions of force and displacement determine whether the work done is positive, negative, or zero.

7.2. Kinetic Energy and the Work-Energy Theorem
- Explain work as a transfer of energy and net work as the work done by the net force.
- Explain and apply the work-energy theorem.

7.3. Gravitational Potential Energy
- Explain gravitational potential energy in terms of work done against gravity.
- Show that the gravitational potential energy of an object of mass \( m \) at height \( h \) on Earth is given by \( PE_g = mgh \).
- Show how knowledge of the potential energy as a function of position can be used to simplify calculations and explain physical phenomena.

7.4. Conservative Forces and Potential Energy
- Define conservative force, potential energy, and mechanical energy.
- Explain the potential energy of a spring in terms of its compression when Hooke's law applies.
- Use the work-energy theorem to show how having only conservative forces implies conservation of mechanical energy.

7.5. Nonconservative Forces
- Define nonconservative forces and explain how they affect mechanical energy.
- Show how the principle of conservation of energy can be applied by treating the conservative forces in terms of their potential energies and any nonconservative forces in terms of the work they do.

7.6. Conservation of Energy
- Explain the law of the conservation of energy.
- Describe some of the many forms of energy.
- Define efficiency of an energy conversion process as the fraction left as useful energy or work, rather than being transformed, for example, into thermal energy.

7.7. Power
- Calculate power by calculating changes in energy over time.
- Examine power consumption and calculations of the cost of energy consumed.

7.8. Work, Energy, and Power in Humans
- Explain the human body's consumption of energy when at rest vs. when engaged in activities that do useful work.
- Calculate the conversion of chemical energy in food into useful work.

7.9. World Energy Use
- Describe the distinction between renewable and nonrenewable energy sources.
- Explain why the inevitable conversion of energy to less useful forms makes it necessary to conserve energy resources.
Introduction to Work, Energy, and Energy Resources

Energy plays an essential role both in everyday events and in scientific phenomena. You can no doubt name many forms of energy, from that provided by our foods, to the energy we use to run our cars, to the sunlight that warms us on the beach. You can also cite examples of what people call energy that may not be scientific, such as someone having an energetic personality. Not only does energy have many interesting forms, it is involved in almost all phenomena, and is one of the most important concepts of physics. What makes it even more important is that the total amount of energy in the universe is constant. Energy can change forms, but it cannot appear from nothing or disappear without a trace. Energy is thus one of a handful of physical quantities that we say is conserved.

Conservation of energy (as physicists like to call the principle that energy can neither be created nor destroyed) is based on experiment. Even as scientists discovered new forms of energy, conservation of energy has always been found to apply. Perhaps the most dramatic example of this was supplied by Einstein when he suggested that mass is equivalent to energy (his famous equation \( E = mc^2 \)).

From a societal viewpoint, energy is one of the major building blocks of modern civilization. Energy resources are key limiting factors to economic growth. The world use of energy resources, especially oil, continues to grow, with ominous consequences economically, socially, politically, and environmentally. We will briefly examine the world’s energy use patterns at the end of this chapter.

There is no simple, yet accurate, scientific definition for energy. Energy is characterized by its many forms and the fact that it is conserved. We can loosely define energy as the ability to do work, admitting that in some circumstances not all energy is available to do work. Because of the association of energy with work, we begin the chapter with a discussion of work. Work is intimately related to energy and how energy moves from one system to another or changes form.

7.1 Work: The Scientific Definition

What It Means to Do Work

The scientific definition of work differs in some ways from its everyday meaning. Certain things we think of as hard work, such as writing an exam or carrying a heavy load on level ground, are not work as defined by a scientist. The scientific definition of work reveals its relationship to energy—whenever work is done, energy is transferred.

For work, in the scientific sense, to be done, a force must be exerted and there must be displacement in the direction of the force.

Formally, the work done on a system by a constant force is defined to be the product of the component of the force in the direction of motion times the distance through which the force acts. For one-way motion in one dimension, this is expressed in equation form as

\[
W = |F| \cos \theta |d|,
\]

where \( W \) is work, \( d \) is the displacement of the system, and \( \theta \) is the angle between the force vector \( F \) and the displacement vector \( d \), as in Figure 7.2. We can also write this as

\[
W = Fd \cos \theta.
\]

To find the work done on a system that undergoes motion that is not one-way or that is in two or three dimensions, we divide the motion into one-way one-dimensional segments and add up the work done over each segment.
Figure 7.2 Examples of work. (a) The work done by the force $\mathbf{F}$ on this lawn mower is $Fd \cos \theta$. Note that $F \cos \theta$ is the component of the force in the direction of motion. (b) A person holding a briefcase does no work on it, because there is no displacement. No energy is transferred to or from the briefcase. (c) The person moving the briefcase horizontally at a constant speed does no work on it, and transfers no energy to it. (d) Work is done on the briefcase by carrying it up stairs at constant speed, because there is necessarily a component of force $\mathbf{F}$ in the direction of the motion. Energy is transferred to the briefcase and could in turn be used to do work. (e) When the briefcase is lowered, energy is transferred out of the briefcase and into an electric generator. Here the work done on the briefcase by the generator is negative, removing energy from the briefcase, because $\mathbf{F}$ and $\mathbf{d}$ are in opposite directions.

To examine what the definition of work means, let us consider the other situations shown in Figure 7.2. The person holding the briefcase in Figure 7.2(b) does no work, for example. Here $d = 0$, so $W = 0$. Why is it you get tired just holding a load? The answer is that your muscles are doing work against one another, but they are doing no work on the system of interest (the "briefcase-Earth system"—see Gravitational Potential Energy for more details). There must be displacement for work to be done, and there must be a component of the force in the direction of the motion. For example, the person carrying the briefcase...
on level ground in Figure 7.2(c) does no work on it, because the force is perpendicular to the motion. That is, \( \cos 90^\circ = 0 \), and so \( W = 0 \).

In contrast, when a force exerted on the system has a component in the direction of motion, such as in Figure 7.2(d), work is done—energy is transferred to the briefcase. Finally, in Figure 7.2(e), energy is transferred from the briefcase to a generator. There are two good ways to interpret this energy transfer. One interpretation is that the briefcase’s weight does work on the generator, giving it energy. The other interpretation is that the generator does negative work on the briefcase, thus removing energy from it. The drawing shows the latter, with the force from the generator upward on the briefcase, and the displacement downward. This makes \( \theta = 180^\circ \), and \( \cos 180^\circ = -1 \); therefore, \( W \) is negative.

Calculating Work

Work and energy have the same units. From the definition of work, we see that those units are force times distance. Thus, in SI units, work and energy are measured in newton-meters. A newton-meter is given the special name joule (J), and 1 J = 1 N \( \cdot \) m = 1 kg \( \cdot \) m\(^2\)/s\(^2\). One joule is not a large amount of energy; it would lift a small 100-gram apple a distance of about 1 meter.

### Example 7.1 Calculating the Work You Do to Push a Lawn Mower Across a Large Lawn

How much work is done on the lawn mower by the person in Figure 7.2(a) if he exerts a constant force of 75.0 N at an angle 35º below the horizontal and pushes the mower 25.0 m on level ground? Convert the amount of work from joules to kilocalories and compare it with this person’s average daily intake of 10,000 kJ (about 2400 kcal) of food energy. One calorie (1 cal) of heat is the amount required to warm 1 g of water by 1°C, and is equivalent to 4.186 J, while one food calorie (1 kcal) is equivalent to 4186 J.

#### Strategy

We can solve this problem by substituting the given values into the definition of work done on a system, stated in the equation \( W = Fd \cos \theta \). The force, angle, and displacement are given, so that only the work \( W \) is unknown.

#### Solution

The equation for the work is

\[
W = Fd \cos \theta. \tag{7.4}
\]

Substituting the known values gives

\[
W = (75.0 \text{ N})(25.0 \text{ m}) \cos (35.0^\circ) \tag{7.5}
\]

\[
= 1536 \text{ J} = 1.54 \times 10^3 \text{ J}. \]

Converting the work in joules to kilocalories yields

\[
\frac{W}{2400 \text{ kcal}} = 1.53 \times 10^{-4}. \tag{7.6}
\]

#### Discussion

This ratio is a tiny fraction of what the person consumes, but it is typical. Very little of the energy released in the consumption of food is used to do work. Even when we “work” all day long, less than 10% of our food energy intake is used to do work and more than 90% is converted to thermal energy or stored as chemical energy in fat.

### 7.2 Kinetic Energy and the Work-Energy Theorem

#### Work Transfers Energy

What happens to the work done on a system? Energy is transferred into the system, but in what form? Does it remain in the system or move on? The answers depend on the situation. For example, if the lawn mower in Figure 7.2(a) is pushed just hard enough to keep it going at a constant speed, then energy put into the mower by the person is removed continuously by friction, and eventually leaves the system in the form of heat transfer. In contrast, work done on the briefcase by the person carrying it up stairs in Figure 7.2(d) is stored in the briefcase-Earth system and can be recovered at any time, as shown in Figure 7.2(e). In fact, the building of the pyramids in ancient Egypt is an example of storing energy in a system by doing work on the system. Some of the energy imparted to the stone blocks in lifting them during construction of the pyramids remains in the stone-Earth system and has the potential to do work.

In this section we begin the study of various types of work and forms of energy. We will find that some types of work leave the energy of a system constant, for example, whereas others change the system in some way, such as making it move. We will also develop definitions of important forms of energy, such as the energy of motion.
Net Work and the Work-Energy Theorem

We know from the study of Newton’s laws in *Dynamics: Force and Newton’s Laws of Motion* that net force causes acceleration. We will see in this section that work done by the net force gives a system energy of motion, and in the process we will also find an expression for the energy of motion.

Let us start by considering the total, or net, work done on a system. Net work is defined to be the sum of work on an object. The net work can be written in terms of the net force on an object, \( F_{\text{net}} \). In equation form, this is \( W_{\text{net}} = F_{\text{net}}d \cos \theta \) where \( \theta \) is the angle between the force vector and the displacement vector.

Figure 7.3(a) shows a graph of force versus displacement for the component of the force in the direction of the displacement—that is, an \( F \cos \theta \) vs. \( d \) graph. In this case, \( F \cos \theta \) is constant. You can see that the area under the graph is \( Fd \cos \theta \), or the work done. Figure 7.3(b) shows a more general process where the force varies. The area under the curve is divided into strips, each having an average force \( (F \cos \theta)_{(\text{ave})} \). The work done is \( (F \cos \theta)_{(\text{ave})}d_i \) for each strip, and the total work done is the sum of the \( W_i \). Thus the total work done is the total area under the curve, a useful property to which we shall refer later.

Figure 7.3 (a) A graph of \( F \cos \theta \) vs. \( d \), when \( F \cos \theta \) is constant. The area under the curve represents the work done by the force. (b) A graph of \( F \cos \theta \) vs. \( d \) in which the force varies. The work done for each interval is the area of each strip; thus, the total area under the curve equals the total work done.

Net work will be simpler to examine if we consider a one-dimensional situation where a force is used to accelerate an object in a direction parallel to its initial velocity. Such a situation occurs for the package on the roller belt conveyor system shown in Figure 7.4.

Figure 7.4 A package on a roller belt is pushed horizontally through a distance \( d \).

The force of gravity and the normal force acting on the package are perpendicular to the displacement and do no work. Moreover, they are also equal in magnitude and opposite in direction so they cancel in calculating the net force. The net force
arises solely from the horizontal applied force $F_{\text{app}}$ and the horizontal friction force $f$. Thus, as expected, the net force is parallel to the displacement, so that $\theta = 0^\circ$ and $\cos \theta = 1$, and the net work is given by

$$W_{\text{net}} = F_{\text{net}}d.$$  \hfill (7.7)

The effect of the net force $F_{\text{net}}$ is to accelerate the package from $v_0$ to $v$. The kinetic energy of the package increases, indicating that the net work done on the system is positive. (See Example 7.2.) By using Newton’s second law, and doing some algebra, we can reach an interesting conclusion. Substituting $F_{\text{net}} = ma$ from Newton’s second law gives

$$W_{\text{net}} = mad.$$  \hfill (7.8)

To get a relationship between net work and the speed given to a system by the net force acting on it, we take $d = x - x_0$ and use the equation studied in Motion Equations for Constant Acceleration in One Dimension for the change in speed over a distance $d$ if the acceleration has the constant value $a$; namely, $v^2 = v_0^2 + 2ad$ (note that $a$ appears in the expression for the net work). Solving for acceleration gives $a = \frac{v^2 - v_0^2}{2d}$. When $a$ is substituted into the preceding expression for $W_{\text{net}}$, we obtain

$$W_{\text{net}} = m\left(\frac{v^2 - v_0^2}{2d}\right)d.$$  \hfill (7.9)

The $d$ cancels, and we rearrange this to obtain

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$  \hfill (7.10)

This expression is called the work-energy theorem, and it actually applies in general (even for forces that vary in direction and magnitude), although we have derived it for the special case of a constant force parallel to the displacement. The theorem implies that the net work on a system equals the change in the quantity $\frac{1}{2}mv^2$. This quantity is our first example of a form of energy.

The Work-Energy Theorem

The net work on a system equals the change in the quantity $\frac{1}{2}mv^2$.

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$  \hfill (7.11)

The quantity $\frac{1}{2}mv^2$ in the work-energy theorem is defined to be the translational kinetic energy (KE) of a mass $m$ moving at a speed $v$. (Translational kinetic energy is distinct from rotational kinetic energy, which is considered later.) In equation form, the translational kinetic energy,

$$KE = \frac{1}{2}mv^2,$$  \hfill (7.12)

is the energy associated with translational motion. Kinetic energy is a form of energy associated with the motion of a particle, single body, or system of objects moving together.

We are aware that it takes energy to get an object, like a car or the package in Figure 7.4, up to speed, but it may be a bit surprising that kinetic energy is proportional to speed squared. This proportionality means, for example, that a car traveling at 100 km/h has four times the kinetic energy it has at 50 km/h, helping to explain why high-speed collisions are so devastating. We will now consider a series of examples to illustrate various aspects of work and energy.

Example 7.2 Calculating the Kinetic Energy of a Package

Suppose a 30.0-kg package on the roller belt conveyor system in Figure 7.4 is moving at 0.500 m/s. What is its kinetic energy?

Strategy

Because the mass $m$ and speed $v$ are given, the kinetic energy can be calculated from its definition as given in the equation $KE = \frac{1}{2}mv^2$.  

---

This OpenStax book is available for free at https://legacy.cnx.org/content/col11951/1.1
Solution
The kinetic energy is given by

\[ KE = \frac{1}{2}mv^2. \]  

(7.13)

Entering known values gives

\[ KE = 0.5(30.0 \text{ kg})(0.500 \text{ m/s})^2, \]  

(7.14)

which yields

\[ KE = 3.75 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 3.75 \text{ J}. \]  

(7.15)

Discussion
Note that the unit of kinetic energy is the joule, the same as the unit of work, as mentioned when work was first defined. It is also interesting that, although this is a fairly massive package, its kinetic energy is not large at this relatively low speed. This fact is consistent with the observation that people can move packages like this without exhausting themselves.

Example 7.3 Determining the Work to Accelerate a Package

Suppose that you push on the 30.0-kg package in Figure 7.4 with a constant force of 120 N through a distance of 0.800 m, and that the opposing friction force averages 5.00 N.

(a) Calculate the net work done on the package. (b) Solve the same problem as in part (a), this time by finding the work done by each force that contributes to the net force.

Strategy and Concept for (a)
This is a motion in one dimension problem, because the downward force (from the weight of the package) and the normal force have equal magnitude and opposite direction, so that they cancel in calculating the net force, while the applied force, friction, and the displacement are all horizontal. (See Figure 7.4.) As expected, the net work equals the net force times distance.

Solution for (a)
The net force is the push force minus friction, or 

\[ F_{\text{net}} = 120 \text{ N} - 5.00 \text{ N} = 115 \text{ N}. \]

Thus the net work is

\[ W_{\text{net}} = F_{\text{net}}d = (115 \text{ N})(0.800 \text{ m}) \]

\[ = 92.0 \text{ N} \cdot \text{m} = 92.0 \text{ J}. \]  

(7.16)

Discussion for (a)
This value is the net work done on the package. The person actually does more work than this, because friction opposes the motion. Friction does negative work and removes some of the energy the person expends and converts it to thermal energy. The net work equals the sum of the work done by each individual force.

Strategy and Concept for (b)
The forces acting on the package are gravity, the normal force, the force of friction, and the applied force. The normal force and force of gravity are each perpendicular to the displacement, and therefore do no work.

Solution for (b)
The applied force does work:

\[ W_{\text{app}} = F_{\text{app}}d \cos(0^\circ) = F_{\text{app}}d \]

\[ = (120 \text{ N})(0.800 \text{ m}) \]

\[ = 96.0 \text{ J}. \]  

(7.17)

The friction force and displacement are in opposite directions, so that \( \theta = 180^\circ \), and the work done by friction is

\[ W_{\text{fr}} = F_{\text{fr}}d \cos(180^\circ) = -F_{\text{fr}}d \]

\[ = -(5.00 \text{ N})(0.800 \text{ m}) \]

\[ = -4.00 \text{ J}. \]  

(7.18)

So the amounts of work done by gravity, by the normal force, by the applied force, and by friction are, respectively,

\[ W_{\text{gr}} = 0, \]

\[ W_{\text{N}} = 0, \]

\[ W_{\text{app}} = 96.0 \text{ J}, \]

\[ W_{\text{fr}} = -4.00 \text{ J}. \]  

(7.19)
The total work done as the sum of the work done by each force is then seen to be
\[ W_{\text{total}} = W_{\text{gr}} + W_{N} + W_{\text{app}} + W_{\text{fr}} = 92.0 \text{ J}. \] (7.20)

**Discussion for (b)**
The calculated total work \( W_{\text{total}} \) as the sum of the work by each force agrees, as expected, with the work \( W_{\text{net}} \) done by the net force. The work done by a collection of forces acting on an object can be calculated by either approach.

---

**Example 7.4 Determining Speed from Work and Energy**

Find the speed of the package in Figure 7.4 at the end of the push, using work and energy concepts.

**Strategy**
Here the work-energy theorem can be used, because we have just calculated the net work, \( W_{\text{net}} \), and the initial kinetic energy, \( \frac{1}{2} m v_0^2 \). These calculations allow us to find the final kinetic energy, \( \frac{1}{2} m v^2 \), and thus the final speed \( v \).

**Solution**
The work-energy theorem in equation form is
\[ W_{\text{net}} = \frac{1}{2} m v^2 - \frac{1}{2} m v_0^2. \] (7.21)

Solving for \( \frac{1}{2} m v^2 \) gives
\[ \frac{1}{2} m v^2 = W_{\text{net}} + \frac{1}{2} m v_0^2. \] (7.22)

Thus,
\[ \frac{1}{2} m v^2 = 92.0 \text{ J} + 3.75 \text{ J} = 95.75 \text{ J}. \] (7.23)

Solving for the final speed as requested and entering known values gives
\[ v = \sqrt{\frac{2(95.75 \text{ J})}{m}} = \sqrt{\frac{191.5 \text{ kg \cdot m}^2/\text{s}^2}{30.0 \text{ kg}}} = 2.53 \text{ m/s}. \] (7.24)

**Discussion**
Using work and energy, we not only arrive at an answer, we see that the final kinetic energy is the sum of the initial kinetic energy and the net work done on the package. This means that the work indeed adds to the energy of the package.

---

**Example 7.5 Work and Energy Can Reveal Distance, Too**

How far does the package in Figure 7.4 coast after the push, assuming friction remains constant? Use work and energy considerations.

**Strategy**
We know that once the person stops pushing, friction will bring the package to rest. In terms of energy, friction does negative work until it has removed all of the package’s kinetic energy. The work done by friction is the force of friction times the distance traveled times the cosine of the angle between the friction force and displacement; hence, this gives us a way of finding the distance traveled after the person stops pushing.

**Solution**
The normal force and force of gravity cancel in calculating the net force. The horizontal friction force is then the net force, and it acts opposite to the displacement, so \( \theta = 180^\circ \). To reduce the kinetic energy of the package to zero, the work \( W_{\text{fr}} \) by friction must be minus the kinetic energy that the package started with plus what the package accumulated due to the pushing. Thus \( W_{\text{fr}} = -95.75 \text{ J} \). Furthermore, \( W_{\text{fr}} = f d' \cos \theta = -f d' \), where \( d' \) is the distance it takes to stop. Thus,
\[ d' = \frac{-W_{\text{fr}}}{f} = \frac{-95.75 \text{ J}}{5.00 \text{ N}}, \] (7.25)

and so
This is a reasonable distance for a package to coast on a relatively friction-free conveyor system. Note that the work done by friction is negative (the force is in the opposite direction of motion), so it removes the kinetic energy.

Some of the examples in this section can be solved without considering energy, but at the expense of missing out on gaining insights about what work and energy are doing in this situation. On the whole, solutions involving energy are generally shorter and easier than those using kinematics and dynamics alone.

### 7.3 Gravitational Potential Energy

#### Work Done Against Gravity

Climbing stairs and lifting objects is work in both the scientific and everyday sense—it is work done against the gravitational force. When there is work, there is a transformation of energy. The work done against the gravitational force goes into an important form of stored energy that we will explore in this section.

Let us calculate the work done in lifting an object of mass \( m \) through a height \( h \), such as in Figure 7.5. If the object is lifted straight up at constant speed, then the force needed to lift it is equal to its weight \( mg \). The work done on the mass is then \( W = Fd = mgh \). We define this to be the **gravitational potential energy** \( (PE_g) \) put into (or gained by) the object-Earth system. This energy is associated with the state of separation between two objects that attract each other by the gravitational force. For convenience, we refer to this as the \( PE_g \) gained by the object, recognizing that this is energy stored in the gravitational field of Earth. Why do we use the word “system”? Potential energy is a property of a system rather than of a single object—due to its physical position. An object’s gravitational potential is due to its position relative to the surroundings within the Earth-object system. The force applied to the object is an external force, from outside the system. When it does positive work it increases the gravitational potential energy of the system. Because gravitational potential energy depends on relative position, we need a reference level at which to set the potential energy equal to 0. We usually choose this point to be Earth’s surface, but this point is arbitrary; what is important is the difference in gravitational potential energy, because this difference is what relates to the work done. The difference in gravitational potential energy of an object (in the Earth-object system) between two rungs of a ladder will be the same for the first two rungs as for the last two rungs.

#### Converting Between Potential Energy and Kinetic Energy

Gravitational potential energy may be converted to other forms of energy, such as kinetic energy. If we release the mass, gravitational force will do an amount of work equal to \( mgh \) on it, thereby increasing its kinetic energy by that same amount (by the work-energy theorem). We will find it more useful to consider just the conversion of \( PE_g \) to \( KE \) without explicitly considering the intermediate step of work. (See Example 7.7.) This shortcut makes it easier to solve problems using energy (if possible) rather than explicitly using forces.
More precisely, we define the change in gravitational potential energy $\Delta E_{\text{g}}$ to be

$$\Delta E_{\text{g}} = mgh,$$

where, for simplicity, we denote the change in height by $h$ rather than the usual $\Delta h$. Note that $h$ is positive when the final height is greater than the initial height, and vice versa. For example, if a 0.500-kg mass hung from a cuckoo clock is raised 1.00 m, then its change in gravitational potential energy is

$$mgh = \left(0.500 \text{ kg}\right)\left(9.80 \text{ m/s}^2\right)(1.00 \text{ m})$$

$$= 4.90 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 4.90 \text{ J}.$$  

Note that the units of gravitational potential energy turn out to be joules, the same as for work and other forms of energy. As the clock runs, the mass is lowered. We can think of the mass as gradually giving up its 4.90 J of gravitational potential energy, **without directly considering the force of gravity that does the work**.

**Using Potential Energy to Simplify Calculations**

The equation $\Delta E_{\text{g}} = mgh$ applies for any path that has a change in height of $h$, not just when the mass is lifted straight up. (See Figure 7.6.) It is much easier to calculate $mgh$ (a simple multiplication) than it is to calculate the work done along a complicated path. The idea of gravitational potential energy has the double advantage that it is very broadly applicable and it makes calculations easier. From now on, we will consider that any change in vertical position $h$ of a mass $m$ is accompanied by a change in gravitational potential energy $mgh$, and we will avoid the equivalent but more difficult task of calculating work done by or against the gravitational force.
The change in gravitational potential energy \( \Delta PE_g \) between points A and B is independent of the path. \( \Delta PE_g = mgh \) for any path between the two points. Gravity is one of a small class of forces where the work done by or against the force depends only on the starting and ending points, not on the path between them.

**Example 7.6 The Force to Stop Falling**

A 60.0-kg person jumps onto the floor from a height of 3.00 m. If he lands stiffly (with his knee joints compressing by 0.500 cm), calculate the force on the knee joints.

**Strategy**

This person’s energy is brought to zero in this situation by the work done on him by the floor as he stops. The initial \( PE_g \) is transformed into \( KE \) as he falls. The work done by the floor reduces this kinetic energy to zero.

**Solution**

The work done on the person by the floor as he stops is given by

\[
W = Fd \cos \theta = -Fd,
\]

with a minus sign because the displacement while stopping and the force from floor are in opposite directions \( \cos \theta = \cos 180^\circ = -1 \). The floor removes energy from the system, so it does negative work.

The kinetic energy the person has upon reaching the floor is the amount of potential energy lost by falling through height \( h \):

\[
KE = -\Delta PE_g = -mgh.
\]

The distance \( d \) that the person’s knees bend is much smaller than the height \( h \) of the fall, so the additional change in gravitational potential energy during the knee bend is ignored.

The work \( W \) done by the floor on the person stops the person and brings the person’s kinetic energy to zero:

\[
W = -KE = mgh.
\]

Combining this equation with the expression for \( W \) gives

\[
-Fd = mgh.
\]

Recalling that \( h \) is negative because the person fell down, the force on the knee joints is given by

\[
F = -\frac{mgh}{d} = -\frac{(60.0 \text{ kg})(9.80 \text{ m/s}^2)(-3.00 \text{ m})}{5.00 \times 10^{-3} \text{ m}} = 3.53 \times 10^5 \text{ N}.
\]
Discussion

Such a large force (500 times more than the person’s weight) over the short impact time is enough to break bones. A much better way to cushion the shock is by bending the legs or rolling on the ground, increasing the time over which the force acts. A bending motion of 0.5 m this way yields a force 100 times smaller than in the example. A kangaroo’s hopping shows this method in action. The kangaroo is the only large animal to use hopping for locomotion, but the shock in hopping is cushioned by the bending of its hind legs in each jump. (See Figure 7.7.)

![Figure 7.7](https://example.com/image.png)

**Figure 7.7** The work done by the ground upon the kangaroo reduces its kinetic energy to zero as it lands. However, by applying the force of the ground on the hind legs over a longer distance, the impact on the bones is reduced. (credit: Chris Samuel, Flickr)

**Example 7.7 Finding the Speed of a Roller Coaster from its Height**

(a) What is the final speed of the roller coaster shown in Figure 7.8 if it starts from rest at the top of the 20.0 m hill and work done by frictional forces is negligible? (b) What is its final speed (again assuming negligible friction) if its initial speed is 5.00 m/s?

![Figure 7.8](https://example.com/image.png)

**Figure 7.8** The speed of a roller coaster increases as gravity pulls it downhill and is greatest at its lowest point. Viewed in terms of energy, the roller-coaster-Earth system’s gravitational potential energy is converted to kinetic energy. If work done by friction is negligible, all $\Delta PE_g$ is converted to $KE$.

**Strategy**

The roller coaster loses potential energy as it goes downhill. We neglect friction, so that the remaining force exerted by the track is the normal force, which is perpendicular to the direction of motion and does no work. The net work on the roller coaster is then done by gravity alone. The loss of gravitational potential energy from moving downward through a distance $h$ equals the gain in kinetic energy. This can be written in equation form as $-\Delta PE_g = \Delta KE$. Using the equations for $PE_g$ and $KE$, we can solve for the final speed $v$, which is the desired quantity.
Solution for (a)

Here the initial kinetic energy is zero, so that $\Delta KE = \frac{1}{2}mv^2$. The equation for change in potential energy states that $\Delta PE_g = mgh$. Since $h$ is negative in this case, we will rewrite this as $\Delta PE_g = -mg |h|$ to show the minus sign clearly. Thus,

$$-\Delta PE_g = \Delta KE$$  \hspace{1cm} (7.34)

becomes

$$mg |h| = \frac{1}{2}mv^2.$$  \hspace{1cm} (7.35)

Solving for $v$, we find that mass cancels and that

$$v = \sqrt{2g |h|}.$$  \hspace{1cm} (7.36)

Substituting known values,

$$v = \sqrt{2(9.80 \text{ m/s}^2)(20.0 \text{ m})} = 19.8 \text{ m/s.}$$  \hspace{1cm} (7.37)

Solution for (b)

Again $-\Delta PE_g = \Delta KE$. In this case there is initial kinetic energy, so $\Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$. Thus,

$$mg |h| = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$  \hspace{1cm} (7.38)

Rearranging gives

$$\frac{1}{2}mv^2 = mg |h| + \frac{1}{2}mv_0^2.$$  \hspace{1cm} (7.39)

This means that the final kinetic energy is the sum of the initial kinetic energy and the gravitational potential energy. Mass again cancels, and

$$v = \sqrt{2g |h| + v_0^2}.$$  \hspace{1cm} (7.40)

This equation is very similar to the kinematics equation $v = \sqrt{v_0^2 + 2ad}$, but it is more general—the kinematics equation is valid only for constant acceleration, whereas our equation above is valid for any path regardless of whether the object moves with a constant acceleration. Now, substituting known values gives

$$v = \sqrt{2(9.80 \text{ m/s}^2)(20.0 \text{ m}) + (5.00 \text{ m/s})^2} = 20.4 \text{ m/s.}$$  \hspace{1cm} (7.41)

Discussion and Implications

First, note that mass cancels. This is quite consistent with observations made in Falling Objects that all objects fall at the same rate if friction is negligible. Second, only the speed of the roller coaster is considered; there is no information about its direction at any point. This reveals another general truth. When friction is negligible, the speed of a falling body depends only on its initial speed and height, and not on its mass or the path taken. For example, the roller coaster will have the same final speed whether it falls 20.0 m straight down or takes a more complicated path like the one in the figure. Third, and perhaps unexpectedly, the final speed in part (b) is greater than in part (a), but by far less than 5.00 m/s. Finally, note that speed can be found at any height along the way by simply using the appropriate value of $h$ at the point of interest.

We have seen that work done by or against the gravitational force depends only on the starting and ending points, and not on the path between, allowing us to define the simplifying concept of gravitational potential energy. We can do the same thing for a few other forces, and we will see that this leads to a formal definition of the law of conservation of energy.

Making Connections: Take-Home Investigation—Converting Potential to Kinetic Energy

One can study the conversion of gravitational potential energy into kinetic energy in this experiment. On a smooth, level surface, use a ruler of the kind that has a groove running along its length and a book to make an incline (see Figure 7.9). Place a marble at the 10-cm position on the ruler and let it roll down the ruler. When it hits the level surface, measure the time it takes to roll one meter. Now place the marble at the 20-cm and the 30-cm positions and again measure the times it takes to roll 1 m on the level surface. Find the velocity of the marble on the level surface for all three positions. Plot velocity...
squared versus the distance traveled by the marble. What is the shape of each plot? If the shape is a straight line, the plot shows that the marble’s kinetic energy at the bottom is proportional to its potential energy at the release point.

Figure 7.9 A marble rolls down a ruler, and its speed on the level surface is measured.

### 7.4 Conservative Forces and Potential Energy

#### Potential Energy and Conservative Forces

Work is done by a force, and some forces, such as weight, have special characteristics. A **conservative force** is one, like the gravitational force, for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken. We can define a **potential energy** \(PE\) for any conservative force, just as we did for the gravitational force. For example, when you wind up a toy, an egg timer, or an old-fashioned watch, you do work against its spring and store energy in it. (We treat these springs as ideal, in that we assume there is no friction and no production of thermal energy.) This stored energy is recoverable as work, and it is useful to think of it as potential energy contained in the spring. Indeed, the reason that the spring has this characteristic is that its force is **conservative**. That is, a conservative force results in stored or potential energy. Gravitational potential energy is one example, as is the energy stored in a spring. We will also see how conservative forces are related to the conservation of energy.

#### Potential Energy of a Spring

First, let us obtain an expression for the potential energy stored in a spring \(PE_s\). We calculate the work done to stretch or compress a spring that obeys Hooke’s law. (Hooke’s law was examined in Elasticity: Stress and Strain, and states that the magnitude of force \(F\) on the spring and the resulting deformation \(\Delta L\) are proportional, \(F = k\Delta L\).) (See Figure 7.10.) For our spring, we will replace \(\Delta L\) (the amount of deformation produced by a force \(F\)) by the distance \(x\) that the spring is stretched or compressed along its length. So the force needed to stretch the spring has magnitude \(F = kx\), where \(k\) is the spring’s force constant. The force increases linearly from 0 at the start to \(kx\) in the fully stretched position. The average force is \(kx/2\). Thus the work done in stretching or compressing the spring is \(W_s = Fed = \left(\frac{kx}{2}\right)x = \frac{1}{2}kx^2\). Alternatively, we noted in Kinetic Energy and the Work-Energy Theorem that the area under a graph of \(F\) vs. \(x\) is the work done by the force. In Figure 7.10(c) we see that this area is also \(\frac{1}{2}kx^2\). We therefore define the potential energy of a spring, \(PE_s\), to be

\[
PE_s = \frac{1}{2}kx^2, \tag{7.42}
\]

where \(k\) is the spring’s force constant and \(x\) is the displacement from its undeformed position. The potential energy represents the work done on the spring and the energy stored in it as a result of stretching or compressing it a distance \(x\). The potential energy of the spring \(PE_s\) does not depend on the path taken; it depends only on the stretch or squeeze \(x\) in the final configuration.
Figure 7.10 (a) An undeformed spring has no PE\_s stored in it. (b) The force needed to stretch (or compress) the spring a distance \( x \) has a magnitude \( F = kx \), and the work done to stretch (or compress) it is \( \frac{1}{2}kx^2 \). Because the force is conservative, this work is stored as potential energy (PE\_s) in the spring, and it can be fully recovered. (c) A graph of \( F \) vs. \( x \) has a slope of \( k \), and the area under the graph is \( \frac{1}{2}kx^2 \). Thus the work done or potential energy stored is \( \frac{1}{2}kx^2 \).

The equation \( \text{PE}_s = \frac{1}{2}kx^2 \) has general validity beyond the special case for which it was derived. Potential energy can be stored in any elastic medium by deforming it. Indeed, the general definition of potential energy is energy due to position, shape, or configuration. For shape or position deformations, stored energy is \( \text{PE}_s = \frac{1}{2}kx^2 \), where \( k \) is the force constant of the particular system and \( x \) is its deformation. Another example is seen in Figure 7.11 for a guitar string.

Figure 7.11 Work is done to deform the guitar string, giving it potential energy. When released, the potential energy is converted to kinetic energy and back to potential as the string oscillates back and forth. A very small fraction is dissipated as sound energy, slowly removing energy from the string.

**Conservation of Mechanical Energy**

Let us now consider what form the work-energy theorem takes when only conservative forces are involved. This will lead us to the conservation of energy principle. The work-energy theorem states that the net work done by all forces acting on a system equals its change in kinetic energy. In equation form, this is

\[
W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \Delta KE.
\]  

(7.43)

If only conservative forces act, then

\[
W_{\text{net}} = W_c.
\]  

(7.44)
where $W_c$ is the total work done by all conservative forces. Thus,

$$W_c = \Delta KE. \quad (7.45)$$

Now, if the conservative force, such as the gravitational force or a spring force, does work, the system loses potential energy. That is, $W_c = -\Delta PE$. Therefore,

$$-\Delta PE = \Delta KE \quad (7.46)$$

or

$$\Delta KE + \Delta PE = 0. \quad (7.47)$$

This equation means that the total kinetic and potential energy is constant for any process involving only conservative forces. That is,

$$\text{KE} + \text{PE} = \text{constant} \quad \text{(conservative forces only),} \quad (7.48)$$

where $i$ and $f$ denote initial and final values. This equation is a form of the work-energy theorem for conservative forces; it is known as the conservation of mechanical energy principle. Remember that this applies to the extent that all the forces are conservative, so that friction is negligible. The total kinetic plus potential energy of a system is defined to be its mechanical energy, $(\text{KE} + \text{PE})$. In a system that experiences only conservative forces, there is a potential energy associated with each force, and the energy only changes form between KE and the various types of PE, with the total energy remaining constant.

**Example 7.8 Using Conservation of Mechanical Energy to Calculate the Speed of a Toy Car**

A 0.100-kg toy car is propelled by a compressed spring, as shown in Figure 7.12. The car follows a track that rises 0.180 m above the starting point. The spring is compressed 4.00 cm and has a force constant of 250.0 N/m. Assuming work done by friction to be negligible, find (a) how fast the car is going before it starts up the slope and (b) how fast it is going at the top of the slope.

Figure 7.12 A toy car is pushed by a compressed spring and coasts up a slope. Assuming negligible friction, the potential energy in the spring is first completely converted to kinetic energy, and then to a combination of kinetic and gravitational potential energy as the car rises. The details of the path are unimportant because all forces are conservative—the car would have the same final speed if it took the alternate path shown.

**Strategy**

The spring force and the gravitational force are conservative forces, so conservation of mechanical energy can be used. Thus,

$$\text{KE}_i + \text{PE}_i = \text{KE}_f + \text{PE}_f \quad (7.49)$$

or

$$\frac{1}{2}mv_i^2 + mgh_i + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + mgh_f + \frac{1}{2}kx_f^2, \quad (7.50)$$

where $h$ is the height (vertical position) and $x$ is the compression of the spring. This general statement looks complex but becomes much simpler when we start considering specific situations. First, we must identify the initial and final conditions in a problem; then, we enter them into the last equation to solve for an unknown.

**Solution for (a)**

This part of the problem is limited to conditions just before the car is released and just after it leaves the spring. Take the initial height to be zero, so that both $h_i$ and $h_f$ are zero. Furthermore, the initial speed $v_i$ is zero and the final compression of the spring $x_f$ is zero, and so several terms in the conservation of mechanical energy equation are zero and it simplifies to

$$\frac{1}{2}kx_i^2 = \frac{1}{2}mv_i^2. \quad (7.51)$$
In other words, the initial potential energy in the spring is converted completely to kinetic energy in the absence of friction. Solving for the final speed and entering known values yields

\[
v_f = \frac{\sqrt{kx_i^2}}{m}
\]

\[= \frac{\sqrt{250.0 \text{ N/m} \cdot 0.0400 \text{ m}}}{0.100 \text{ kg}}\]

\[= 2.00 \text{ m/s}.
\]

**Solution for (b)**

One method of finding the speed at the top of the slope is to consider conditions just before the car is released and just after it reaches the top of the slope, completely ignoring everything in between. Doing the same type of analysis to find which terms are zero, the conservation of mechanical energy becomes

\[
\frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + mgh_f.
\]

This form of the equation means that the spring’s initial potential energy is converted partly to gravitational potential energy and partly to kinetic energy. The final speed at the top of the slope will be less than at the bottom. Solving for \(v_f\) and substituting known values gives

\[
v_f = \sqrt{\frac{kx_i^2}{m} - 2gh_f}
\]

\[= \sqrt{\left(\frac{250.0 \text{ N/m}}{0.100 \text{ kg}}\right)(0.0400 \text{ m})^2 - 2(9.80 \text{ m/s}^2)(0.180 \text{ m})}\]

\[= 0.687 \text{ m/s}.
\]

**Discussion**

Another way to solve this problem is to realize that the car’s kinetic energy before it goes up the slope is converted partly to potential energy—that is, to take the final conditions in part (a) to be the initial conditions in part (b).

Note that, for conservative forces, we do not directly calculate the work they do; rather, we consider their effects through their corresponding potential energies, just as we did in Example 7.8. Note also that we do not consider details of the path taken—only the starting and ending points are important (as long as the path is not impossible). This assumption is usually a tremendous simplification, because the path may be complicated and forces may vary along the way.

---

**Energy Skate Park**

Learn about conservation of energy with a skater dude! Build tracks, ramps and jumps for the skater and view the kinetic energy, potential energy and friction as he moves. You can also take the skater to different planets or even space!

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42149/1.14/#fs-id1167066938367)

---

### 7.5 Nonconservative Forces

**Nonconservative Forces and Friction**

Forces are either conservative or nonconservative. Conservative forces were discussed in Conservative Forces and Potential Energy. A nonconservative force is one for which work depends on the path taken. Friction is a good example of a nonconservative force. As illustrated in Figure 7.14, work done against friction depends on the length of the path between the starting and ending points. Because of this dependence on path, there is no potential energy associated with nonconservative forces. An important characteristic is that the work done by a nonconservative force adds or removes mechanical energy from a system. Friction, for example, creates thermal energy that dissipates, removing energy from the system. Furthermore, even if the thermal energy is retained or captured, it cannot be fully converted back to work, so it is lost or not recoverable in that sense as well.
Figure 7.14 The amount of the happy face erased depends on the path taken by the eraser between points A and B, as does the work done against friction. Less work is done and less of the face is erased for the path in (a) than for the path in (b). The force here is friction, and most of the work goes into thermal energy that subsequently leaves the system (the happy face plus the eraser). The energy expended cannot be fully recovered.

How Nonconservative Forces Affect Mechanical Energy

Mechanical energy may not be conserved when nonconservative forces act. For example, when a car is brought to a stop by friction on level ground, it loses kinetic energy, which is dissipated as thermal energy, reducing its mechanical energy. Figure 7.15 compares the effects of conservative and nonconservative forces. We often choose to understand simpler systems such as that described in Figure 7.15(a) first before studying more complicated systems as in Figure 7.15(b).

Figure 7.15 Comparison of the effects of conservative and nonconservative forces on the mechanical energy of a system. (a) A system with only conservative forces. When a rock is dropped onto a spring, its mechanical energy remains constant (neglecting air resistance) because the force in the spring is conservative. The spring can propel the rock back to its original height, where it once again has only potential energy due to gravity. (b) A system with nonconservative forces. When the same rock is dropped onto the ground, it is stopped by nonconservative forces that dissipate its mechanical energy as thermal energy, sound, and surface distortion. The rock has lost mechanical energy.

How the Work-Energy Theorem Applies

Now let us consider what form the work-energy theorem takes when both conservative and nonconservative forces act. We will see that the work done by nonconservative forces equals the change in the mechanical energy of a system. As noted in Kinetic Energy and the Work-Energy Theorem, the work-energy theorem states that the net work on a system equals the change in its kinetic energy, or $W_{\text{net}} = \Delta KE$. The net work is the sum of the work by nonconservative forces plus the work by conservative forces. That is,

$$W_{\text{net}} = W_{\text{nc}} + W_{\text{c}},$$

(7.55)

so that

$$W_{\text{nc}} + W_{\text{c}} = \Delta KE,$$

(7.56)

where $W_{\text{nc}}$ is the total work done by all nonconservative forces and $W_{\text{c}}$ is the total work done by all conservative forces.
Consider Figure 7.16, in which a person pushes a crate up a ramp and is opposed by friction. As in the previous section, we note that work done by a conservative force comes from a loss of gravitational potential energy, so that $W_c = -\Delta PE$.

Substituting this equation into the previous one and solving for $W_{nc}$ gives

$$W_{nc} = \Delta KE + \Delta PE.$$  \hfill (7.57)

This equation means that the total mechanical energy ($KE + PE$) changes by exactly the amount of work done by nonconservative forces. In Figure 7.16, this is the work done by the person minus the work done by friction. So even if energy is not conserved for the system of interest (such as the crate), we know that an equal amount of work was done to cause the change in total mechanical energy.

We rearrange $W_{nc} = \Delta KE + \Delta PE$ to obtain

$$KE_i + PE_i + W_{nc} = KE_f + PE_f.$$  \hfill (7.58)

This means that the amount of work done by nonconservative forces adds to the mechanical energy of a system. If $W_{nc}$ is positive, then mechanical energy is increased, such as when the person pushes the crate up the ramp in Figure 7.16. If $W_{nc}$ is negative, then mechanical energy is decreased, such as when the rock hits the ground in Figure 7.15(b). If $W_{nc}$ is zero, then mechanical energy is conserved, and nonconservative forces are balanced. For example, when you push a lawn mower at constant speed on level ground, your work done is removed by the work of friction, and the mower has a constant energy.

**Applying Energy Conservation with Nonconservative Forces**

When no change in potential energy occurs, applying $KE_i + PE_i + W_{nc} = KE_f + PE_f$ amounts to applying the work-energy theorem by setting the change in kinetic energy to be equal to the net work done on the system, which in the most general case includes both conservative and nonconservative forces. But when seeking instead to find a change in total mechanical energy in situations that involve changes in both potential and kinetic energy, the previous equation $KE_i + PE_i + W_{nc} = KE_f + PE_f$ says that you can start by finding the change in mechanical energy that would have resulted from just the conservative forces, including the potential energy changes, and add to it the work done, with the proper sign, by any nonconservative forces involved.

**Example 7.9 Calculating Distance Traveled: How Far a Baseball Player Slides**

Consider the situation shown in Figure 7.17, where a baseball player slides to a stop on level ground. Using energy considerations, calculate the distance the 65.0-kg baseball player slides, given that his initial speed is 6.00 m/s and the force of friction against him is a constant 450 N.
The baseball player slides to a stop in a distance $d$. In the process, friction removes the player’s kinetic energy by doing an amount of work $fd$ equal to the initial kinetic energy.

**Strategy**

Friction stops the player by converting his kinetic energy into other forms, including thermal energy. In terms of the work-energy theorem, the work done by friction, which is negative, is added to the initial kinetic energy to reduce it to zero. The work done by friction is negative, because $f$ is in the opposite direction of the motion (that is, $\theta = 180^\circ$, and so $\cos \theta = -1$). Thus $W_{nc} = -fd$. The equation simplifies to

$$\frac{1}{2}mv_i^2 - fd = 0$$

or

$$fd = \frac{1}{2}mv_i^2.$$  \hspace{1cm} (7.60)

This equation can now be solved for the distance $d$.

**Solution**

Solving the previous equation for $d$ and substituting known values yields

$$d = \frac{mv_i^2}{2f}$$

$$= \frac{(65.0 \text{ kg})(6.00 \text{ m/s})^2}{(2)(450 \text{ N})}$$

$$= 2.60 \text{ m}.$$  \hspace{1cm} (7.61)

**Discussion**

The most important point of this example is that the amount of nonconservative work equals the change in mechanical energy. For example, you must work harder to stop a truck, with its large mechanical energy, than to stop a mosquito.

---

**Example 7.10 Calculating Distance Traveled: Sliding Up an Incline**

Suppose that the player from Example 7.9 is running up a hill having a $5.00^\circ$ incline upward with a surface similar to that in the baseball stadium. The player slides with the same initial speed, and the frictional force is still 450 N. Determine how far he slides.
Figure 7.18 The same baseball player slides to a stop on a 5.00° slope.

Strategy
In this case, the work done by the nonconservative friction force on the player reduces the mechanical energy he has from his kinetic energy at zero height, to the final mechanical energy he has by moving through distance \(d\) to reach height \(h\) along the hill, with \(h = d \sin 5.00°\). This is expressed by the equation

\[ KE + PE_i + W_{nc} = KE_f + PE_f. \]

Solution
The work done by friction is again \(W_{nc} = -fd\); initially the potential energy is \(PE_i = mg \cdot 0 = 0\) and the kinetic energy is \(KE_i = \frac{1}{2}mv_i^2\); the final energy contributions are \(KE_f = 0\) for the kinetic energy and \(PE_f = mgh = mgd \sin \theta\) for the potential energy. Substituting these values gives

\[ \frac{1}{2}mv_i^2 + 0 + (-fd) = 0 + mgd \sin \theta. \]

Solve this for \(d\) to obtain

\[ d = \frac{\frac{1}{2}mv_i^2}{f + mg \sin \theta} \]

\[ = \frac{(0.5)(65.0 \text{ kg})(6.00 \text{ m/s})^2}{450 \text{ N} + (65.0 \text{ kg})(9.80 \text{ m/s}^2) \sin (5.00°)} \]

\[ = 2.31 \text{ m}. \]

Discussion
As might have been expected, the player slides a shorter distance by sliding uphill. Note that the problem could also have been solved in terms of the forces directly and the work energy theorem, instead of using the potential energy. This method would have required combining the normal force and force of gravity vectors, which no longer cancel each other because they point in different directions, and friction, to find the net force. You could then use the net force and the net work to find the distance \(d\) that reduces the kinetic energy to zero. By applying conservation of energy and using the potential energy instead, we need only consider the gravitational potential energy \(mgh\), without combining and resolving force vectors. This simplifies the solution considerably.

Making Connections: Take-Home Investigation—Determining Friction from the Stopping Distance

This experiment involves the conversion of gravitational potential energy into thermal energy. Use the ruler, book, and marble from Take-Home Investigation—Converting Potential to Kinetic Energy. In addition, you will need a foam cup with a small hole in the side, as shown in Figure 7.19. From the 10-cm position on the ruler, let the marble roll into the cup positioned at the bottom of the ruler. Measure the distance \(d\) the cup moves before stopping. What forces caused it to stop? What happened to the kinetic energy of the marble at the bottom of the ruler? Next, place the marble at the 20-cm and the 30-cm positions and again measure the distance the cup moves after the marble enters it. Plot the distance the cup moves versus the initial marble position on the ruler. Is this relationship linear?

With some simple assumptions, you can use these data to find the coefficient of kinetic friction \(\mu_k\) of the cup on the table. The force of friction \(f\) on the cup is \(\mu_k N\), where the normal force \(N\) is just the weight of the cup plus the marble. The normal force and force of gravity do no work because they are perpendicular to the displacement of the cup, which moves...
The work done by friction is $fd$. You will need the mass of the marble as well to calculate its initial kinetic energy.

It is interesting to do the above experiment also with a steel marble (or ball bearing). Releasing it from the same positions on the ruler as you did with the glass marble, is the velocity of this steel marble the same as the velocity of the marble at the bottom of the ruler? Is the distance the cup moves proportional to the mass of the steel and glass marbles?

![Figure 7.19 Rolling a marble down a ruler into a foam cup.](http://legacy.cnx.org/content/m42150/1.16/#TheRamp)

### 7.6 Conservation of Energy

#### Law of Conservation of Energy

Energy, as we have noted, is conserved, making it one of the most important physical quantities in nature. The law of conservation of energy can be stated as follows:

Total energy is constant in any process. It may change in form or be transferred from one system to another, but the total remains the same.

We have explored some forms of energy and some ways it can be transferred from one system to another. This exploration led to the definition of two major types of energy—mechanical energy ($KE + PE$) and energy transferred via work done by nonconservative forces ($W_{nc}$). But energy takes many other forms, manifesting itself in many different ways, and we need to be able to deal with all of these before we can write an equation for the above general statement of the conservation of energy.

#### Other Forms of Energy than Mechanical Energy

At this point, we deal with all other forms of energy by lumping them into a single group called other energy ($OE$). Then we can state the conservation of energy in equation form as

$$KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f.$$  \hspace{1cm} (7.65)

All types of energy and work can be included in this very general statement of conservation of energy. Kinetic energy is $KE$, work done by a conservative force is represented by $PE$, work done by nonconservative forces is $W_{nc}$, and all other energies are included as $OE$. This equation applies to all previous examples; in those situations $OE$ was constant, and so it subtracted out and was not directly considered.

#### Making Connections: Usefulness of the Energy Conservation Principle

The fact that energy is conserved and has many forms makes it very important. You will find that energy is discussed in many contexts, because it is involved in all processes. It will also become apparent that many situations are best understood in terms of energy and that problems are often most easily conceptualized and solved by considering energy.

When does $OE$ play a role? One example occurs when a person eats. Food is oxidized with the release of carbon dioxide, water, and energy. Some of this chemical energy is converted to kinetic energy when the person moves, to potential energy when the person changes altitude, and to thermal energy (another form of $OE$).
Some of the Many Forms of Energy

What are some other forms of energy? You can probably name a number of forms of energy not yet discussed. Many of these will be covered in later chapters, but let us detail a few here. Electrical energy is a common form that is converted to many other forms and does work in a wide range of practical situations. Fuels, such as gasoline and food, carry chemical energy that can be transferred to a system through oxidation. Chemical fuel can also produce electrical energy, such as in batteries. Batteries can in turn produce light, which is a very pure form of energy. Most energy sources on Earth are in fact stored energy from the energy we receive from the Sun. We sometimes refer to this as radiant energy, or electromagnetic radiation, which includes visible light, infrared, and ultraviolet radiation. Nuclear energy comes from processes that convert measurable amounts of mass into energy. Nuclear energy is transformed into the energy of sunlight, into electrical energy in power plants, and into the energy of the heat transfer and blast in weapons. Atoms and molecules inside all objects are in random motion. This internal mechanical energy from the random motions is called thermal energy, because it is related to the temperature of the object. These and all other forms of energy can be converted into one another and can do work.

Table 7.1 gives the amount of energy stored, used, or released from various objects and in various phenomena. The range of energies and the variety of types and situations is impressive.

### Problem-Solving Strategies for Energy

You will find the following problem-solving strategies useful whenever you deal with energy. The strategies help in organizing and reinforcing energy concepts. In fact, they are used in the examples presented in this chapter. The familiar general problem-solving strategies presented earlier—involving identifying physical principles, knowns, and unknowns, checking units, and so on—continue to be relevant here.

**Step 1.** Determine the system of interest and identify what information is given and what quantity is to be calculated. A sketch will help.

**Step 2.** Examine all the forces involved and determine whether you know or are given the potential energy from the work done by the forces. Then use step 3 or step 4.

**Step 3.** If you know the potential energies for the forces that enter into the problem, then forces are all conservative, and you can apply conservation of mechanical energy simply in terms of potential and kinetic energy. The equation expressing conservation of energy is

\[ KE_i + PE_i = KE_f + PE_f. \]  

**Step 4.** If you know the potential energy for only some of the forces, possibly because some of them are nonconservative and do not have a potential energy, or if there are other energies that are not easily treated in terms of force and work, then the conservation of energy law in its most general form must be used.

\[ KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f. \]  

In most problems, one or more of the terms is zero, simplifying its solution. Do not calculate \( W_{nc} \), the work done by conservative forces; it is already incorporated in the \( PE \) terms.

**Step 5.** You have already identified the types of work and energy involved (in step 2). Before solving for the unknown, eliminate terms wherever possible to simplify the algebra. For example, choose \( h = 0 \) at either the initial or final point, so that \( PE_g \) is zero there. Then solve for the unknown in the customary manner.

**Step 6.** Check the answer to see if it is reasonable. Once you have solved a problem, reexamine the forms of work and energy to see if you have set up the conservation of energy equation correctly. For example, work done against friction should be negative, potential energy at the bottom of a hill should be less than that at the top, and so on. Also check to see that the numerical value obtained is reasonable. For example, the final speed of a skateboarder who coasts down a 3-m-high ramp could reasonably be 20 km/h, but not 80 km/h.

### Transformation of Energy

The transformation of energy from one form into others is happening all the time. The chemical energy in food is converted into thermal energy through metabolism; light energy is converted into chemical energy through photosynthesis. In a larger example, the chemical energy contained in coal is converted into thermal energy as it burns to turn water into steam in a boiler. This thermal energy in the steam in turn is converted to mechanical energy as it spins a turbine, which is connected to a generator to produce electrical energy. (In all of these examples, not all of the initial energy is converted into the forms mentioned. This important point is discussed later in this section.)

Another example of energy conversion occurs in a solar cell. Sunlight impinging on a solar cell (see Figure 7.20) produces electricity, which in turn can be used to run an electric motor. Energy is converted from the primary source of solar energy into electrical energy and then into mechanical energy.
Figure 7.20 Solar energy is converted into electrical energy by solar cells, which is used to run a motor in this solar-power aircraft. (credit: NASA)
Table 7.1 Energy of Various Objects and Phenomena

<table>
<thead>
<tr>
<th>Object/phenomenon</th>
<th>Energy in joules</th>
</tr>
</thead>
<tbody>
<tr>
<td>Big Bang</td>
<td>$10^{68}$</td>
</tr>
<tr>
<td>Energy released in a supernova</td>
<td>$10^{44}$</td>
</tr>
<tr>
<td>Fusion of all the hydrogen in Earth’s oceans</td>
<td>$10^{34}$</td>
</tr>
<tr>
<td>Annual world energy use</td>
<td>$4 \times 10^{20}$</td>
</tr>
<tr>
<td>Large fusion bomb (9 megaton)</td>
<td>$3.8 \times 10^{16}$</td>
</tr>
<tr>
<td>1 kg hydrogen (fusion to helium)</td>
<td>$6.4 \times 10^{14}$</td>
</tr>
<tr>
<td>1 kg uranium (nuclear fission)</td>
<td>$8.0 \times 10^{13}$</td>
</tr>
<tr>
<td>Hiroshima-size fission bomb (10 kiloton)</td>
<td>$4.2 \times 10^{13}$</td>
</tr>
<tr>
<td>90,000-metric ton aircraft carrier at 30 knots</td>
<td>$1.1 \times 10^{10}$</td>
</tr>
<tr>
<td>1 barrel crude oil</td>
<td>$5.9 \times 10^9$</td>
</tr>
<tr>
<td>1 ton TNT</td>
<td>$4.2 \times 10^9$</td>
</tr>
<tr>
<td>1 gallon of gasoline</td>
<td>$1.2 \times 10^8$</td>
</tr>
<tr>
<td>Daily home electricity use (developed countries)</td>
<td>$7 \times 10^7$</td>
</tr>
<tr>
<td>Daily adult food intake (recommended)</td>
<td>$1.2 \times 10^7$</td>
</tr>
<tr>
<td>1000-kg car at 90 km/h</td>
<td>$3.1 \times 10^5$</td>
</tr>
<tr>
<td>1 g fat (9.3 kcal)</td>
<td>$3.9 \times 10^4$</td>
</tr>
<tr>
<td>ATP hydrolysis reaction</td>
<td>$3.2 \times 10^4$</td>
</tr>
<tr>
<td>1 g carbohydrate (4.1 kcal)</td>
<td>$1.7 \times 10^4$</td>
</tr>
<tr>
<td>1 g protein (4.1 kcal)</td>
<td>$1.7 \times 10^4$</td>
</tr>
<tr>
<td>Tennis ball at 100 km/h</td>
<td>22</td>
</tr>
<tr>
<td>Mosquito ($10^{-2}$ g at 0.5 m/s)</td>
<td>$1.3 \times 10^{-6}$</td>
</tr>
<tr>
<td>Single electron in a TV tube beam</td>
<td>$4.0 \times 10^{-15}$</td>
</tr>
<tr>
<td>Energy to break one DNA strand</td>
<td>$10^{-19}$</td>
</tr>
</tbody>
</table>

**Efficiency**

Even though energy is conserved in an energy conversion process, the output of useful energy or work will be less than the energy input. The efficiency $\textit{Eff}$ of an energy conversion process is defined as

$$\text{Efficiency (Eff)} = \frac{\text{useful energy or work output}}{\text{total energy input}} = \frac{W_{\text{out}}}{E_{\text{in}}},$$

(7.68)

Table 7.2 lists some efficiencies of mechanical devices and human activities. In a coal-fired power plant, for example, about 40% of the chemical energy in the coal becomes useful electrical energy. The other 60% transforms into other (perhaps less useful) energy forms, such as thermal energy, which is then released to the environment through combustion gases and cooling towers.
### Table 7.2 Efficiency of the Human Body and Mechanical Devices

<table>
<thead>
<tr>
<th>Activity/device</th>
<th>Efficiency (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cycling and climbing</td>
<td>20</td>
</tr>
<tr>
<td>Swimming, surface</td>
<td>2</td>
</tr>
<tr>
<td>Swimming, submerged</td>
<td>4</td>
</tr>
<tr>
<td>Shoveling</td>
<td>3</td>
</tr>
<tr>
<td>Weightlifting</td>
<td>9</td>
</tr>
<tr>
<td>Steam engine</td>
<td>17</td>
</tr>
<tr>
<td>Gasoline engine</td>
<td>30</td>
</tr>
<tr>
<td>Diesel engine</td>
<td>35</td>
</tr>
<tr>
<td>Nuclear power plant</td>
<td>35</td>
</tr>
<tr>
<td>Coal power plant</td>
<td>42</td>
</tr>
<tr>
<td>Electric motor</td>
<td>98</td>
</tr>
<tr>
<td>Compact fluorescent light</td>
<td>20</td>
</tr>
<tr>
<td>Gas heater (residential)</td>
<td>90</td>
</tr>
<tr>
<td>Solar cell</td>
<td>10</td>
</tr>
</tbody>
</table>

---

#### Masses and Springs

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energies for each spring.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42151/1.16/#masses_springs)

---

#### 7.7 Power

**What is Power?**

*Power*—the word conjures up many images: a professional football player muscling aside his opponent, a dragster roaring away from the starting line, a volcano blowing its lava into the atmosphere, or a rocket blasting off, as in Figure 7.22.

![Figure 7.22](https://legacy.cnx.org/content/m42151/1.16/#masses_springs)

*Figure 7.22* This powerful rocket on the Space Shuttle *Endeavor* did work and consumed energy at a very high rate. (credit: NASA)

These images of power have in common the rapid performance of work, consistent with the scientific definition of **power** ($P$) as the rate at which work is done.

---

1. Representative values
Power

Power is the rate at which work is done.

\[ P = \frac{W}{t} \]  \hspace{1cm} (7.69)

The SI unit for power is the **watt** (\( W \)), where 1 watt equals 1 joule/second (\( 1 \text{ W} = 1 \text{ J/s} \)).

Because work is energy transfer, power is also the rate at which energy is expended. A 60-W light bulb, for example, expends 60 J of energy per second. Great power means a large amount of work or energy developed in a short time. For example, when a powerful car accelerates rapidly, it does a large amount of work and consumes a large amount of fuel in a short time.

**Calculating Power from Energy**

**Example 7.11 Calculating the Power to Climb Stairs**

What is the power output for a 60.0-kg woman who runs up a 3.00 m high flight of stairs in 3.50 s, starting from rest but having a final speed of 2.00 m/s? (See Figure 7.23.)

![Figure 7.23](image)

**Strategy and Concept**

The work going into mechanical energy is \( W = KE + PE_g \). At the bottom of the stairs, we take both \( KE \) and \( PE_g \) as initially zero; thus, \( W = KE_f + PE_g = \frac{1}{2}mv_f^2 + mgh \), where \( h \) is the vertical height of the stairs. Because all terms are given, we can calculate \( W \) and then divide it by time to get power.

**Solution**

Substituting the expression for \( W \) into the definition of power given in the previous equation, \( P = \frac{W}{t} \) yields

\[ P = \frac{W}{t} = \frac{\frac{1}{2}mv_f^2 + mgh}{t} \]  \hspace{1cm} (7.70)

Entering known values yields

\[ P = \frac{0.5(60.0 \text{ kg})(2.00 \text{ m/s})^2 + (60.0 \text{ kg})(9.80 \text{ m/s}^2)(3.00 \text{ m})}{3.50 \text{ s}} \]  \hspace{1cm} (7.71)

\[ = \frac{120 \text{ J} + 1764 \text{ J}}{3.50 \text{ s}} \]

\[ = 538 \text{ W}. \]

**Discussion**

The woman does 1764 J of work to move up the stairs compared with only 120 J to increase her kinetic energy; thus, most of her power output is required for climbing rather than accelerating.
It is impressive that this woman’s useful power output is slightly less than 1 horsepower (1 hp = 746 W)! People can generate more than a horsepower with their leg muscles for short periods of time by rapidly converting available blood sugar and oxygen into work output. (A horse can put out 1 hp for hours on end.) Once oxygen is depleted, power output decreases and the person begins to breathe rapidly to obtain oxygen to metabolize more food—this is known as the aerobic stage of exercise. If the woman climbed the stairs slowly, then her power output would be much less, although the amount of work done would be the same.

**Making Connections: Take-Home Investigation—Measure Your Power Rating**

Determine your own power rating by measuring the time it takes you to climb a flight of stairs. We will ignore the gain in kinetic energy, as the above example showed that it was a small portion of the energy gain. Don’t expect that your output will be more than about 0.5 hp.

**Examples of Power**

Examples of power are limited only by the imagination, because there are as many types as there are forms of work and energy. (See Table 7.3 for some examples.) Sunlight reaching Earth’s surface carries a maximum power of about 1.3 kilowatts per square meter (kW/m$^2$). A tiny fraction of this is retained by Earth over the long term. Our consumption rate of fossil fuels is far greater than the rate at which they are stored, so it is inevitable that they will be depleted. Power implies that energy is transferred, perhaps changing form. It is never possible to change one form completely into another without losing some of it as thermal energy. For example, a 60-W incandescent bulb converts only 5 W of electrical power to light, with 55 W dissipating into thermal energy. Furthermore, the typical electric power plant converts only 35 to 40% of its fuel into electricity. The remainder becomes a huge amount of thermal energy that must be dispersed as heat transfer, as rapidly as it is created. A coal-fired power plant may produce 1000 megawatts; 1 megawatt (MW) is $10^6$ W of electric power. But the power plant consumes chemical energy at a rate of about 2500 MW, creating heat transfer to the surroundings at a rate of 1500 MW. (See Figure 7.24.)

**Figure 7.24** Tremendous amounts of electric power are generated by coal-fired power plants such as this one in China, but an even larger amount of power goes into heat transfer to the surroundings. The large cooling towers here are needed to transfer heat as rapidly as it is produced. The transfer of heat is not unique to coal plants but is an unavoidable consequence of generating electric power from any fuel—nuclear, coal, oil, natural gas, or the like. (credit: Kleinolive, Wikimedia Commons)
Table 7.3 Power Output or Consumption

<table>
<thead>
<tr>
<th>Object or Phenomenon</th>
<th>Power in Watts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supernova (at peak)</td>
<td>$5 \times 10^{37}$</td>
</tr>
<tr>
<td>Milky Way galaxy</td>
<td>$10^{37}$</td>
</tr>
<tr>
<td>Crab Nebula pulsar</td>
<td>$10^{28}$</td>
</tr>
<tr>
<td>The Sun</td>
<td>$4 \times 10^{26}$</td>
</tr>
<tr>
<td>Volcanic eruption (maximum)</td>
<td>$4 \times 10^{15}$</td>
</tr>
<tr>
<td>Lightning bolt</td>
<td>$2 \times 10^{12}$</td>
</tr>
<tr>
<td>Nuclear power plant (total electric and heat transfer)</td>
<td>$3 \times 10^{9}$</td>
</tr>
<tr>
<td>Aircraft carrier (total useful and heat transfer)</td>
<td>$10^{8}$</td>
</tr>
<tr>
<td>Dragster (total useful and heat transfer)</td>
<td>$2 \times 10^{6}$</td>
</tr>
<tr>
<td>Car (total useful and heat transfer)</td>
<td>$8 \times 10^{4}$</td>
</tr>
<tr>
<td>Football player (total useful and heat transfer)</td>
<td>$5 \times 10^{3}$</td>
</tr>
<tr>
<td>Clothes dryer</td>
<td>$4 \times 10^{3}$</td>
</tr>
<tr>
<td>Person at rest (all heat transfer)</td>
<td>100</td>
</tr>
<tr>
<td>Typical incandescent light bulb (total useful and heat transfer)</td>
<td>60</td>
</tr>
<tr>
<td>Heart, person at rest (total useful and heat transfer)</td>
<td>8</td>
</tr>
<tr>
<td>Electric clock</td>
<td>3</td>
</tr>
<tr>
<td>Pocket calculator</td>
<td>$10^{-3}$</td>
</tr>
</tbody>
</table>

Power and Energy Consumption

We usually have to pay for the energy we use. It is interesting and easy to estimate the cost of energy for an electrical appliance if its power consumption rate and time used are known. The higher the power consumption rate and the longer the appliance is used, the greater the cost of that appliance. The power consumption rate is $P = \frac{W}{t} = \frac{E}{t}$, where $E$ is the energy supplied by the electricity company. So the energy consumed over a time $t$ is

$$E = Pt.$$  \hfill (7.72)

Electricity bills state the energy used in units of kilowatt-hours (kW·h), which is the product of power in kilowatts and time in hours. This unit is convenient because electrical power consumption at the kilowatt level for hours at a time is typical.

Example 7.12 Calculating Energy Costs

What is the cost of running a 0.200-kW computer 6.00 h per day for 30.0 d if the cost of electricity is $0.120 per kW·h?

**Strategy**

Cost is based on energy consumed; thus, we must find $E$ from $E = Pt$ and then calculate the cost. Because electrical energy is expressed in kW·h, at the start of a problem such as this it is convenient to convert the units into kW and hours.

**Solution**

The energy consumed in kW·h is

$$E = Pt = (0.200 \text{ kW})(6.00 \text{ h/d})(30.0 \text{ d})$$

$$= 36.0 \text{ kW·h},$$  \hfill (7.73)

and the cost is simply given by
The cost of using the computer in this example is neither exorbitant nor negligible. It is clear that the cost is a combination of power and time. When both are high, such as for an air conditioner in the summer, the cost is high.

\[
\text{cost} = (36.0 \text{ kW} \cdot \text{h})(\$0.120 \text{ per kW} \cdot \text{h}) = \$4.32 \text{ per month.}
\]

**Discussion**

The motivation to save energy has become more compelling with its ever-increasing price. Armed with the knowledge that energy consumed is the product of power and time, you can estimate costs for yourself and make the necessary value judgments about where to save energy. Either power or time must be reduced. It is most cost-effective to limit the use of high-power devices that normally operate for long periods of time, such as water heaters and air conditioners. This would not include relatively high power devices like toasters, because they are on only a few minutes per day. It would also not include electric clocks, in spite of their 24-hour-per-day usage, because they are very low power devices. It is sometimes possible to use devices that have greater efficiencies—that is, devices that consume less power to accomplish the same task. One example is the compact fluorescent light bulb, which produces over four times more light per watt of power consumed than its incandescent cousin.

Modern civilization depends on energy, but current levels of energy consumption and production are not sustainable. The likelihood of a link between global warming and fossil fuel use (with its concomitant production of carbon dioxide), has made reduction in energy use as well as a shift to non-fossil fuels of the utmost importance. Even though energy in an isolated system is a conserved quantity, the final result of most energy transformations is waste heat transfer to the environment, which is no longer useful for doing work. As we will discuss in more detail in Thermochemistry (https://legacy.cnx.org/content/m42231/latest), the potential for energy to produce useful work has been “degraded” in the energy transformation.

### 7.8 Work, Energy, and Power in Humans

**Energy Conversion in Humans**

Our own bodies, like all living organisms, are energy conversion machines. Conservation of energy implies that the chemical energy stored in food is converted into work, thermal energy, and/or stored as chemical energy in fatty tissue. (See Figure 7.25.) The fraction going into each form depends both on how much we eat and on our level of physical activity. If we eat more than is needed to do work and stay warm, the remainder goes into body fat.

![Energy conversion diagram](https://example.com/energy_conversion_diagram)

**Figure 7.25** Energy consumed by humans is converted to work, thermal energy, and stored fat. By far the largest fraction goes to thermal energy, although the fraction varies depending on the type of physical activity.

**Power Consumed at Rest**

The rate at which the body uses food energy to sustain life and to do different activities is called the metabolic rate. The total energy conversion rate of a person at rest is called the basal metabolic rate (BMR) and is divided among various systems in the body, as shown in Table 7.4. The largest fraction goes to the liver and spleen, with the brain coming next. Of course, during vigorous exercise, the energy consumption of the skeletal muscles and heart increase markedly. About 75% of the calories burned in a day go into these basic functions. The BMR is a function of age, gender, total body weight, and amount of muscle mass (which burns more calories than body fat). Athletes have a greater BMR due to this last factor.
Table 7.4 Basal Metabolic Rates (BMR)

<table>
<thead>
<tr>
<th>Organ</th>
<th>Power consumed at rest (W)</th>
<th>Oxygen consumption (mL/min)</th>
<th>Percent of BMR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liver &amp; spleen</td>
<td>23</td>
<td>67</td>
<td>27</td>
</tr>
<tr>
<td>Brain</td>
<td>16</td>
<td>47</td>
<td>19</td>
</tr>
<tr>
<td>Skeletal muscle</td>
<td>15</td>
<td>45</td>
<td>18</td>
</tr>
<tr>
<td>Kidney</td>
<td>9</td>
<td>26</td>
<td>10</td>
</tr>
<tr>
<td>Heart</td>
<td>6</td>
<td>17</td>
<td>7</td>
</tr>
<tr>
<td>Other</td>
<td>16</td>
<td>48</td>
<td>19</td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td><strong>85 W</strong></td>
<td><strong>250 mL/min</strong></td>
<td><strong>100%</strong></td>
</tr>
</tbody>
</table>

Energy consumption is directly proportional to oxygen consumption because the digestive process is basically one of oxidizing food. We can measure the energy people use during various activities by measuring their oxygen use. (See Figure 7.26.) Approximately 20 kJ of energy are produced for each liter of oxygen consumed, independent of the type of food. Table 7.5 shows energy and oxygen consumption rates (power expended) for a variety of activities.

Power of Doing Useful Work

Work done by a person is sometimes called **useful work**, which is work done on the outside world, such as lifting weights. Useful work requires a force exerted through a distance on the outside world, and so it excludes internal work, such as that done by the heart when pumping blood. Useful work does include that done in climbing stairs or accelerating to a full run, because these are accomplished by exerting forces on the outside world. Forces exerted by the body are nonconservative, so that they can change the mechanical energy (KE + PE) of the system worked upon, and this is often the goal. A baseball player throwing a ball, for example, increases both the ball’s kinetic and potential energy.

If a person needs more energy than they consume, such as when doing vigorous work, the body must draw upon the chemical energy stored in fat. So exercise can be helpful in losing fat. However, the amount of exercise needed to produce a loss in fat, or to burn off extra calories consumed that day, can be large, as Example 7.13 illustrates.

Example 7.13 Calculating Weight Loss from Exercising

If a person who normally requires an average of 12,000 kJ (3000 kcal) of food energy per day consumes 13,000 kJ per day, he will steadily gain weight. How much bicycling per day is required to work off this extra 1000 kJ?

**Solution**

Table 7.5 states that 400 W are used when cycling at a moderate speed. The time required to work off 1000 kJ at this rate is then

\[
\text{Time} = \frac{\text{energy}}{\text{energy/time}} = \frac{1000 \text{ kJ}}{400 \text{ W}} = 2500 \text{ s} = 42 \text{ min.}
\]

**Discussion**

If this person uses more energy than he or she consumes, the person’s body will obtain the needed energy by metabolizing body fat. If the person uses 13,000 kJ but consumes only 12,000 kJ, then the amount of fat loss will be

\[
\text{Fat loss} = (1000 \text{ kJ}) \left( \frac{1.0 \text{ g fat}}{39 \text{ kJ}} \right) = 26 \text{ g},
\]

assuming the energy content of fat to be 39 kJ/g.

Figure 7.26 A pulse oxymeter is an apparatus that measures the amount of oxygen in blood. A knowledge of oxygen and carbon dioxide levels indicates a person’s metabolic rate, which is the rate at which food energy is converted to another form. Such measurements can indicate the level of athletic conditioning as well as certain medical problems. (credit: UusiAjaja, Wikimedia Commons)
Table 7.5 Energy and Oxygen Consumption Rates\(^{(2)}\) (Power)

<table>
<thead>
<tr>
<th>Activity</th>
<th>Energy consumption in watts</th>
<th>Oxygen consumption in liters O(_2)/min</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sleeping</td>
<td>83</td>
<td>0.24</td>
</tr>
<tr>
<td>Sitting at rest</td>
<td>120</td>
<td>0.34</td>
</tr>
<tr>
<td>Standing relaxed</td>
<td>125</td>
<td>0.36</td>
</tr>
<tr>
<td>Sitting in class</td>
<td>210</td>
<td>0.60</td>
</tr>
<tr>
<td>Walking (5 km/h)</td>
<td>280</td>
<td>0.80</td>
</tr>
<tr>
<td>Cycling (13–18 km/h)</td>
<td>400</td>
<td>1.14</td>
</tr>
<tr>
<td>Shivering</td>
<td>425</td>
<td>1.21</td>
</tr>
<tr>
<td>Playing tennis</td>
<td>440</td>
<td>1.26</td>
</tr>
<tr>
<td>Swimming breaststroke</td>
<td>475</td>
<td>1.36</td>
</tr>
<tr>
<td>Ice skating (14.5 km/h)</td>
<td>545</td>
<td>1.56</td>
</tr>
<tr>
<td>Climbing stairs (116/min)</td>
<td>685</td>
<td>1.96</td>
</tr>
<tr>
<td>Cycling (21 km/h)</td>
<td>700</td>
<td>2.00</td>
</tr>
<tr>
<td>Running cross-country</td>
<td>740</td>
<td>2.12</td>
</tr>
<tr>
<td>Playing basketball</td>
<td>800</td>
<td>2.28</td>
</tr>
<tr>
<td>Cycling, professional racer</td>
<td>1855</td>
<td>5.30</td>
</tr>
<tr>
<td>Sprinting</td>
<td>2415</td>
<td>6.90</td>
</tr>
</tbody>
</table>

All bodily functions, from thinking to lifting weights, require energy. (See Figure 7.27.) The many small muscle actions accompanying all quiet activity, from sleeping to head scratching, ultimately become thermal energy, as do less visible muscle actions by the heart, lungs, and digestive tract. Shivering, in fact, is an involuntary response to low body temperature that pits muscles against one another to produce thermal energy in the body (and do no work). The kidneys and liver consume a surprising amount of energy, but the biggest surprise of all is that a full 25% of all energy consumed by the body is used to maintain electrical potentials in all living cells. (Nerve cells use this electrical potential in nerve impulses.) This bioelectrical energy ultimately becomes mostly thermal energy, but some is utilized to power chemical processes such as in the kidneys and liver, and in fat production.

Figure 7.27 This fMRI scan shows an increased level of energy consumption in the vision center of the brain. Here, the patient was being asked to recognize faces. (credit: NIH via Wikimedia Commons)

7.9 World Energy Use

Energy is an important ingredient in all phases of society. We live in a very interdependent world, and access to adequate and reliable energy resources is crucial for economic growth and for maintaining the quality of our lives. But current levels of energy consumption and production are not sustainable. About 40% of the world’s energy comes from oil, and much of that goes to transportation uses. Oil prices are dependent as much upon new (or foreseen) discoveries as they are upon political events and situations around the world. The U.S., with 4.5% of the world’s population, consumes 24% of the world’s oil production per year;

2. for an average 76-kg male
66% of that oil is imported!

Renewable and Nonrenewable Energy Sources

The principal energy resources used in the world are shown in Figure 7.28. The fuel mix has changed over the years but now is dominated by oil, although natural gas and solar contributions are increasing. Renewable forms of energy are those sources that cannot be used up, such as water, wind, solar, and biomass. About 85% of our energy comes from nonrenewable fossil fuels—oil, natural gas, coal. The likelihood of a link between global warming and fossil fuel use, with its production of carbon dioxide through combustion, has made, in the eyes of many scientists, a shift to non-fossil fuels of utmost importance—but it will not be easy.

![Figure 7.28 World energy consumption by source, in billions of kilowatt-hours: 2006. (credit: KVDP)](image)

The World’s Growing Energy Needs

World energy consumption continues to rise, especially in the developing countries. (See Figure 7.29.) Global demand for energy has tripled in the past 50 years and might triple again in the next 30 years. While much of this growth will come from the rapidly booming economies of China and India, many of the developed countries, especially those in Europe, are hoping to meet their energy needs by expanding the use of renewable sources. Although presently only a small percentage, renewable energy is growing very fast, especially wind energy. For example, Germany plans to meet 20% of its electricity and 10% of its overall energy needs with renewable resources by the year 2020. (See Figure 7.30.) Energy is a key constraint in the rapid economic growth of China and India. In 2003, China surpassed Japan as the world’s second largest consumer of oil. However, over 1/3 of this is imported. Unlike most Western countries, coal dominates the commercial energy resources of China, accounting for 2/3 of its energy consumption. In 2009 China surpassed the United States as the largest generator of \( \text{CO}_2 \). In India, the main energy resources are biomass (wood and dung) and coal. Half of India’s oil is imported. About 70% of India’s electricity is generated by highly polluting coal. Yet there are sizeable strides being made in renewable energy. India has a rapidly growing wind energy base, and it has the largest solar cooking program in the world.

![Figure 7.29 Past and projected world energy use (source: Based on data from U.S. Energy Information Administration, 2011)](image)
Table 7.6 displays the 2006 commercial energy mix by country for some of the prime energy users in the world. While non-renewable sources dominate, some countries get a sizeable percentage of their electricity from renewable resources. For example, about 67% of New Zealand’s electricity demand is met by hydroelectric. Only 10% of the U.S. electricity is generated by renewable resources, primarily hydroelectric. It is difficult to determine total contributions of renewable energy in some countries with a large rural population, so these percentages in this table are left blank.

<table>
<thead>
<tr>
<th>Country</th>
<th>Consumption, in EJ ($10^{18}$ J)</th>
<th>Oil</th>
<th>Natural Gas</th>
<th>Coal</th>
<th>Nuclear</th>
<th>Hydro</th>
<th>Other Renewables</th>
<th>Electricity Use per capita (kWh/yr)</th>
<th>Energy Use per capita (GJ/yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>5.4</td>
<td>34%</td>
<td>17%</td>
<td>44%</td>
<td>0%</td>
<td>3%</td>
<td>1%</td>
<td>10000</td>
<td>260</td>
</tr>
<tr>
<td>Brazil</td>
<td>9.6</td>
<td>48%</td>
<td>7%</td>
<td>5%</td>
<td>1%</td>
<td>35%</td>
<td>2%</td>
<td>2000</td>
<td>50</td>
</tr>
<tr>
<td>China</td>
<td>63</td>
<td>22%</td>
<td>3%</td>
<td>69%</td>
<td>1%</td>
<td>6%</td>
<td></td>
<td>1500</td>
<td>35</td>
</tr>
<tr>
<td>Egypt</td>
<td>2.4</td>
<td>50%</td>
<td>41%</td>
<td>1%</td>
<td>0%</td>
<td>6%</td>
<td></td>
<td>990</td>
<td>32</td>
</tr>
<tr>
<td>Germany</td>
<td>16</td>
<td>37%</td>
<td>24%</td>
<td>24%</td>
<td>11%</td>
<td>1%</td>
<td>3%</td>
<td>6400</td>
<td>173</td>
</tr>
<tr>
<td>India</td>
<td>15</td>
<td>34%</td>
<td>7%</td>
<td>52%</td>
<td>1%</td>
<td>5%</td>
<td></td>
<td>470</td>
<td>13</td>
</tr>
<tr>
<td>Indonesia</td>
<td>4.9</td>
<td>51%</td>
<td>26%</td>
<td>16%</td>
<td>0%</td>
<td>2%</td>
<td>3%</td>
<td>420</td>
<td>22</td>
</tr>
<tr>
<td>Japan</td>
<td>24</td>
<td>48%</td>
<td>14%</td>
<td>21%</td>
<td>12%</td>
<td>4%</td>
<td>1%</td>
<td>7100</td>
<td>176</td>
</tr>
<tr>
<td>New Zealand</td>
<td>0.44</td>
<td>32%</td>
<td>26%</td>
<td>6%</td>
<td>0%</td>
<td>11%</td>
<td>19%</td>
<td>8500</td>
<td>102</td>
</tr>
<tr>
<td>Russia</td>
<td>31</td>
<td>19%</td>
<td>53%</td>
<td>16%</td>
<td>5%</td>
<td>6%</td>
<td></td>
<td>5700</td>
<td>202</td>
</tr>
<tr>
<td>U.S.</td>
<td>105</td>
<td>40%</td>
<td>23%</td>
<td>22%</td>
<td>8%</td>
<td>3%</td>
<td>1%</td>
<td>12500</td>
<td>340</td>
</tr>
<tr>
<td>World</td>
<td>432</td>
<td>39%</td>
<td>23%</td>
<td>24%</td>
<td>6%</td>
<td>6%</td>
<td>2%</td>
<td>2600</td>
<td>71</td>
</tr>
</tbody>
</table>

Energy and Economic Well-being

The last two columns in this table examine the energy and electricity use per capita. Economic well-being is dependent upon energy use, and in most countries higher standards of living, as measured by GDP (gross domestic product) per capita, are matched by higher levels of energy consumption per capita. This is borne out in Figure 7.31. Increased efficiency of energy use will change this dependency. A global problem is balancing energy resource development against the harmful effects upon the environment in its extraction and use.
Conserving Energy

As we finish this chapter on energy and work, it is relevant to draw some distinctions between two sometimes misunderstood terms in the area of energy use. As has been mentioned elsewhere, the “law of the conservation of energy” is a very useful principle in analyzing physical processes. It is a statement that cannot be proven from basic principles, but is a very good bookkeeping device, and no exceptions have ever been found. It states that the total amount of energy in an isolated system will always remain constant. Related to this principle, but remarkably different from it, is the important philosophy of energy conservation. This concept has to do with seeking to decrease the amount of energy used by an individual or group through (1) reduced activities (e.g., turning down thermostats, driving fewer kilometers) and/or (2) increasing conversion efficiencies in the performance of a particular task—such as developing and using more efficient room heaters, cars that have greater miles-per-gallon ratings, energy-efficient compact fluorescent lights, etc.

Since energy in an isolated system is not destroyed or created or generated, one might wonder why we need to be concerned about our energy resources, since energy is a conserved quantity. The problem is that the final result of most energy transformations is waste heat transfer to the environment and conversion to energy forms no longer useful for doing work. To state it in another way, the potential for energy to produce useful work has been “degraded” in the energy transformation. (This will be discussed in more detail in Thermodynamics (https://legacy.cnx.org/content/m42231/latest/).

Glossary

basal metabolic rate: the total energy conversion rate of a person at rest

chemical energy: the energy in a substance stored in the bonds between atoms and molecules that can be released in a chemical reaction

conservation of mechanical energy: the rule that the sum of the kinetic energies and potential energies remains constant if only conservative forces act on and within a system

conservative force: a force that does the same work for any given initial and final configuration, regardless of the path followed

efficiency: a measure of the effectiveness of the input of energy to do work; useful energy or work divided by the total input of energy

electrical energy: the energy carried by a flow of charge

energy: the ability to do work

fossil fuels: oil, natural gas, and coal

friction: the force between surfaces that opposes one sliding on the other; friction changes mechanical energy into thermal energy

Figure 7.31 Power consumption per capita versus GDP per capita for various countries. Note the increase in energy usage with increasing GDP. (2007, credit: Frank van Mierlo, Wikimedia Commons)
gravitational potential energy: the energy an object has due to its position in a gravitational field

horsepower: an older non-SI unit of power, with 1 hp = 746 W

joule: SI unit of work and energy, equal to one newton-meter

kilowatt-hour: (kW · h) unit used primarily for electrical energy provided by electric utility companies

kinetic energy: the energy an object has by reason of its motion, equal to \( \frac{1}{2}mv^2 \) for the translational (i.e., non-rotational) motion of an object of mass \( m \) moving at speed \( v \)

law of conservation of energy: the general law that total energy is constant in any process; energy may change in form or be transferred from one system to another, but the total remains the same

mechanical energy: the sum of kinetic energy and potential energy

metabolic rate: the rate at which the body uses food energy to sustain life and to do different activities

net work: work done by the net force, or vector sum of all the forces, acting on an object

nonconservative force: a force whose work depends on the path followed between the given initial and final configurations

nuclear energy: energy released by changes within atomic nuclei, such as the fusion of two light nuclei or the fission of a heavy nucleus

potential energy: energy due to position, shape, or configuration

potential energy of a spring: the stored energy of a spring as a function of its displacement; when Hooke’s law applies, it is given by the expression \( \frac{1}{2}kx^2 \) where \( x \) is the distance the spring is compressed or extended and \( k \) is the spring constant

power: the rate at which work is done

radiant energy: the energy carried by electromagnetic waves

renewable forms of energy: those sources that cannot be used up, such as water, wind, solar, and biomass

thermal energy: the energy within an object due to the random motion of its atoms and molecules that accounts for the object’s temperature

useful work: work done on an external system

watt: (W) SI unit of power, with 1 W = 1 J/s

work: the transfer of energy by a force that causes an object to be displaced; the product of the component of the force in the direction of the displacement and the magnitude of the displacement

work-energy theorem: the result, based on Newton’s laws, that the net work done on an object is equal to its change in kinetic energy

---

**Section Summary**

**7.1 Work: The Scientific Definition**

- Work is the transfer of energy by a force acting on an object as it is displaced.
- The work \( W \) that a force \( \mathbf{F} \) does on an object is the product of the magnitude \( F \) of the force, times the magnitude \( d \) of the displacement, times the cosine of the angle \( \theta \) between them. In symbols,
  \[ W = Fd \cos \theta. \]
- The SI unit for work and energy is the joule (J), where 1 J = 1 N · m = 1 kg · m²/s².
- The work done by a force is zero if the displacement is either zero or perpendicular to the force.
- The work done is positive if the force and displacement have the same direction, and negative if they have opposite direction.
7.2 Kinetic Energy and the Work-Energy Theorem
- The net work $W_{\text{net}}$ is the work done by the net force acting on an object.
- Work done on an object transfers energy to the object.
- The translational kinetic energy of an object of mass $m$ moving at speed $v$ is $KE = \frac{1}{2}mv^2$.
- The work-energy theorem states that the net work $W_{\text{net}}$ on a system changes its kinetic energy,
  $$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$ 

7.3 Gravitational Potential Energy
- Work done against gravity in lifting an object becomes potential energy of the object-Earth system.
- The change in gravitational potential energy, $\Delta PE_g$, is $\Delta PE_g = mgh$, with $h$ being the increase in height and $g$ the acceleration due to gravity.
- The gravitational potential energy of an object near Earth’s surface is due to its position in the mass-Earth system. Only differences in gravitational potential energy, $\Delta PE_g$, have physical significance.
- As an object descends without friction, its gravitational potential energy changes into kinetic energy corresponding to increasing speed, so that $\Delta KE = -\Delta PE_g$.

7.4 Conservative Forces and Potential Energy
- A conservative force is one for which work depends only on the starting and ending points of a motion, not on the path taken.
- We can define potential energy (PE) for any conservative force, just as we defined $PE_g$ for the gravitational force.
- The potential energy of a spring is $PE_s = \frac{1}{2}kx^2$, where $k$ is the spring’s force constant and $x$ is the displacement from its undeformed position.
- Mechanical energy is defined to be $KE + PE$ for a conservative force.
- When only conservative forces act on and within a system, the total mechanical energy is constant. In equation form,
  $$\begin{align*}
  KE + PE &= \text{constant} \\
  KE_i + PE_i &= KE_f + PE_f
  \end{align*}$$
  where $i$ and $f$ denote initial and final values. This is known as the conservation of mechanical energy.

7.5 Nonconservative Forces
- A nonconservative force is one for which work depends on the path.
- Friction is an example of a nonconservative force that changes mechanical energy into thermal energy.
- Work $W_{\text{nc}}$ done by a nonconservative force changes the mechanical energy of a system. In equation form,
  $$W_{\text{nc}} = \Delta KE + \Delta PE \text{ or, equivalently, } KE_i + PE_i + W_{\text{nc}} = KE_f + PE_f.$$ 
- When both conservative and nonconservative forces act, energy conservation can be applied and used to calculate motion in terms of the known potential energies of the conservative forces and the work done by nonconservative forces, instead of finding the net work from the net force, or having to directly apply Newton’s laws.

7.6 Conservation of Energy
- The law of conservation of energy states that the total energy is constant in any process. Energy may change in form or be transferred from one system to another, but the total remains the same.
- When all forms of energy are considered, conservation of energy is written in equation form as $KE_i + PE_i + W_{\text{nc}} + OE_i = KE_f + PE_f + OE_f$, where $OE$ is all other forms of energy besides mechanical energy.
- Commonly encountered forms of energy include electric energy, chemical energy, radiant energy, nuclear energy, and thermal energy.
- Energy is often utilized to do work, but it is not possible to convert all the energy of a system to work.
- The efficiency $Eff$ of a machine or human is defined to be $Eff = \frac{W_{\text{out}}}{E_{\text{in}}}$, where $W_{\text{out}}$ is useful work output and $E_{\text{in}}$ is the energy consumed.

7.7 Power
- Power is the rate at which work is done, or in equation form, for the average power $P$ for work $W$ done over a time $t$,
\[ P = \frac{W}{t}. \]

- The SI unit for power is the watt (W), where \( 1 \text{ W} = 1 \text{ J/s} \).
- The power of many devices such as electric motors is also often expressed in horsepower (hp), where \( 1 \text{ hp} = 746 \text{ W} \).

### 7.8 Work, Energy, and Power in Humans

- The human body converts energy stored in food into work, thermal energy, and/or chemical energy that is stored in fatty tissue.
- The rate at which the body uses food energy to sustain life and to do different activities is called the metabolic rate, and the corresponding rate when at rest is called the basal metabolic rate (BMR).
- The energy included in the basal metabolic rate is divided among various systems in the body, with the largest fraction going to the liver and spleen, and the brain coming next.
- About 75% of food calories are used to sustain basic body functions included in the basal metabolic rate.
- The energy consumption of people during various activities can be determined by measuring their oxygen use, because the digestive process is basically one of oxidizing food.

### 7.9 World Energy Use

- The relative use of different fuels to provide energy has changed over the years, but fuel use is currently dominated by oil, although natural gas and solar contributions are increasing.
- Although non-renewable sources dominate, some countries meet a sizeable percentage of their electricity needs from renewable resources.
- The United States obtains only about 10% of its energy from renewable sources, mostly hydroelectric power.
- Economic well-being is dependent upon energy use, and in most countries higher standards of living, as measured by GDP (Gross Domestic Product) per capita, are matched by higher levels of energy consumption per capita.
- Even though, in accordance with the law of conservation of energy, energy can never be created or destroyed, energy that can be used to do work is always partly converted to less useful forms, such as waste heat to the environment, in all of our uses of energy for practical purposes.

### Conceptual Questions

#### 7.1 Work: The Scientific Definition

1. Give an example of something we think of as work in everyday circumstances that is not work in the scientific sense. Is energy transferred or changed in form in your example? If so, explain how this is accomplished without doing work.
2. Give an example of a situation in which there is a force and a displacement, but the force does no work. Explain why it does no work.
3. Describe a situation in which a force is exerted for a long time but does no work. Explain.

#### 7.2 Kinetic Energy and the Work-Energy Theorem

4. The person in Figure 7.32 does work on the lawn mower. Under what conditions would the mower gain energy? Under what conditions would it lose energy?

![Image](https://example.com/image7.32)

**Figure 7.32**

5. Work done on a system puts energy into it. Work done by a system removes energy from it. Give an example for each statement.
6. When solving for speed in Example 7.4, we kept only the positive root. Why?
7.3 Gravitational Potential Energy

7. In Example 7.7, we calculated the final speed of a roller coaster that descended 20 m in height and had an initial speed of 5 m/s downhill. Suppose the roller coaster had had an initial speed of 5 m/s uphill instead, and it coasted uphill, stopped, and then rolled back down to a final point 20 m below the start. We would find in that case that its final speed is the same as its initial speed. Explain in terms of conservation of energy.


7.4 Conservative Forces and Potential Energy

9. What is a conservative force?

10. The force exerted by a diving board is conservative, provided the internal friction is negligible. Assuming friction is negligible, describe changes in the potential energy of a diving board as a swimmer dives from it, starting just before the swimmer steps on the board until just after his feet leave it.

11. Define mechanical energy. What is the relationship of mechanical energy to nonconservative forces? What happens to mechanical energy if only conservative forces act?

12. What is the relationship of potential energy to conservative force?

7.6 Conservation of Energy

13. Consider the following scenario. A car for which friction is not negligible accelerates from rest down a hill, running out of gasoline after a short distance. The driver lets the car coast farther down the hill, then up and over a small crest. He then coasts down that hill into a gas station, where he brakes to a stop and fills the tank with gasoline. Identify the forms of energy the car has, and how they are changed and transferred in this series of events. (See Figure 7.33.)

14. Describe the energy transfers and transformations for a javelin, starting from the point at which an athlete picks up the javelin and ending when the javelin is stuck into the ground after being thrown.

15. Do devices with efficiencies of less than one violate the law of conservation of energy? Explain.

16. List four different forms or types of energy. Give one example of a conversion from each of these forms to another form.

17. List the energy conversions that occur when riding a bicycle.

7.7 Power

18. Most electrical appliances are rated in watts. Does this rating depend on how long the appliance is on? (When off, it is a zero-watt device.) Explain in terms of the definition of power.

19. Explain, in terms of the definition of power, why energy consumption is sometimes listed in kilowatt-hours rather than joules. What is the relationship between these two energy units?

20. A spark of static electricity, such as that you might receive from a doorknob on a cold dry day, may carry a few hundred watts of power. Explain why you are not injured by such a spark.

7.8 Work, Energy, and Power in Humans

21. Explain why it is easier to climb a mountain on a zigzag path rather than one straight up the side. Is your increase in gravitational potential energy the same in both cases? Is your energy consumption the same in both?

22. Do you do work on the outside world when you rub your hands together to warm them? What is the efficiency of this activity?

23. Shivering is an involuntary response to lowered body temperature. What is the efficiency of the body when shivering, and is this a desirable value?
24. Discuss the relative effectiveness of dieting and exercise in losing weight, noting that most athletic activities consume food energy at a rate of 400 to 500 W, while a single cup of yogurt can contain 1360 kJ (325 kcal). Specifically, is it likely that exercise alone will be sufficient to lose weight? You may wish to consider that regular exercise may increase the metabolic rate, whereas protracted dieting may reduce it.

7.9 World Energy Use

25. What is the difference between energy conservation and the law of conservation of energy? Give some examples of each.

26. If the efficiency of a coal-fired electrical generating plant is 35%, then what do we mean when we say that energy is a conserved quantity?
Problems & Exercises

7.1 Work: The Scientific Definition

1. How much work does a supermarket checkout attendant do on a can of soup he pushes 0.600 m horizontally with a force of 5.00 N? Express your answer in joules and kilocalories.

2. A 75.0-kg person climbs stairs, gaining 2.50 meters in height. Find the work done to accomplish this task.

3. (a) Calculate the work done on a 1500-kg elevator car by its cable to lift it 40.0 m at constant speed, assuming friction averages 100 N. (b) What is the work done on the elevator car by the gravitational force in this process? (c) What is the total work done on the elevator car?

4. Suppose a car travels 108 km at a speed of 30.0 m/s, and uses 2.0 gal of gasoline. Only 30% of the gasoline goes into useful work by the force that keeps the car moving at constant speed despite friction. (See Table 7.1 for the energy content of gasoline.) (a) What is the magnitude of the force exerted to keep the car moving at constant speed? (b) If the required force is directly proportional to speed, how many gallons will be used to drive 108 km at a speed of 28.0 m/s?

5. Calculate the work done by an 85.0-kg man who pushes a crate 4.00 m up along a ramp that makes an angle of 20.0º with the horizontal. (See Figure 7.34.) He exerts a force of 500 N on the crate parallel to the ramp and moves at a constant speed. Be certain to include the work he does on the crate and on his body to get up the ramp.

6. How much work is done by the boy pulling his sister 30.0 m in a wagon as shown in Figure 7.35? Assume no friction acts on the wagon.

7. A shopper pushes a grocery cart 20.0 m at constant speed on level ground, against a 35.0 N frictional force. He pushes in a direction 25.0º below the horizontal. (a) What is the work done on the cart by friction? (b) What is the work done on the cart by the gravitational force? (c) What is the work done on the cart by the shopper? (d) Find the force the shopper exerts, using energy considerations. (e) What is the total work done on the cart?

8. Suppose the ski patrol lowers a rescue sled and victim, having a total mass of 90.0 kg, down a 60.0º slope at constant speed, as shown in Figure 7.36. The coefficient of friction between the sled and the snow is 0.100. (a) How much work is done by friction as the sled moves 30.0 m along the hill? (b) How much work is done by the rope on the sled in this distance? (c) What is the work done by the gravitational force on the sled? (d) What is the total work done?

Figure 7.36 A rescue sled and victim are lowered down a steep slope.

7.2 Kinetic Energy and the Work-Energy Theorem

9. Compare the kinetic energy of a 20,000-kg truck moving at 110 km/h with that of an 80.0-kg astronaut in orbit moving at 27,500 km/h.

10. (a) How fast must a 3000-kg elephant move to have the same kinetic energy as a 65.0-kg sprinter running at 10.0 m/s? (b) Discuss how the larger energies needed for the movement of larger animals would relate to metabolic rates.

11. Confirm the value given for the kinetic energy of an aircraft carrier in Table 7.1. You will need to look up the definition of a nautical mile (1 knot = 1 nautical mile/h).

12. (a) Calculate the force needed to bring a 950-kg car to rest from a speed of 90.0 km/h in a distance of 120 m (a fairly typical distance for a non-panic stop). (b) Suppose instead the car hits a concrete abutment at full speed and is brought to a stop in 2.00 m. Calculate the force exerted on the car and compare it with the force found in part (a).

13. A car’s bumper is designed to withstand a 4.0-km/h (1.12-m/s) collision with an immovable object without damage to the body of the car. The bumper cushions the shock by absorbing the force over a distance. Calculate the magnitude of the average force on a bumper that collapses 0.200 m while bringing a 900-kg car to rest from an initial speed of 1.12 m/s.
14. Boxing gloves are padded to lessen the force of a blow. (a) Calculate the force exerted by a boxing glove on an opponent’s face, if the glove and face compress 7.50 cm during a blow in which the 7.00-kg arm and glove are brought to rest from an initial speed of 10.0 m/s. (b) Calculate the force exerted by an identical blow in the days when no gloves were used and the knuckles and face would compress only 2.00 cm. Does it seem high enough to cause damage even though it is lower than the force with no glove?

15. Using energy considerations, calculate the average force a 60.0-kg sprinter exerts backward on the track to accelerate from 2.00 to 8.00 m/s in a distance of 25.0 m, if he encounters a headwind that exerts an average force of 30.0 N against him.

7.3 Gravitational Potential Energy

16. A hydroelectric power facility (see Figure 7.37) converts the gravitational potential energy of water behind a dam to electric energy. (a) What is the gravitational potential energy relative to the generators of a lake of volume $50.0 \text{ km}^3$ (mass $= 5.00 \times 10^{13} \text{ kg}$), given that the lake has an average height of 40.0 m above the generators? (b) Compare this with the energy stored in a 9-megaton fusion bomb.

17. (a) How much gravitational potential energy (relative to the ground on which it is built) is stored in the Great Pyramid of Cheops, given that its mass is about $7 \times 10^9 \text{ kg}$ and its center of mass is 36.5 m above the surrounding ground? (b) How does this energy compare with the daily food intake of a person?

18. Suppose a 350-g kookaburra (a large kingfisher bird) picks up a 75-g snake and raises it 2.5 m from the ground to a branch. (a) How much work did the bird do on the snake? (b) How much work did it do to raise its own center of mass to the branch?

19. In Example 7.7, we found that the speed of a roller coaster that had descended 20.0 m was only slightly greater when it had an initial speed of 5.00 m/s than when it started from rest. This implies that $\Delta \text{PE} >> \text{KE}_i$. Confirm this statement by taking the ratio of $\Delta \text{PE}$ to $\text{KE}_i$. (Note that mass cancels.)

20. A 100-g toy car is propelled by a compressed spring that starts it moving. The car follows the curved track in Figure 7.38. Show that the final speed of the toy car is 0.687 m/s if its initial speed is 2.00 m/s and it coasts up the frictionless slope, gaining 0.180 m in altitude.

Figure 7.38 A toy car moves up a sloped track. (credit: Leszek Leszczynski, Flickr)

21. In a downhill ski race, surprisingly, little advantage is gained by getting a running start. (This is because the initial kinetic energy is small compared with the gain in gravitational potential energy on even small hills.) To demonstrate this, find the final speed and the time taken for a skier who skies 70.0 m along a $30^\circ$ slope neglecting friction: (a) Starting from rest. (b) Starting with an initial speed of 2.50 m/s. (c) Does the answer surprise you? Discuss why it is still advantageous to get a running start in very competitive events.

7.4 Conservative Forces and Potential Energy

22. A $5.00 \times 10^5 \text{ kg}$ subway train is brought to a stop from a speed of 0.500 m/s in 0.400 m by a large spring bumper at the end of its track. What is the force constant $k$ of the spring?

23. A pogo stick has a spring with a force constant of $2.50 \times 10^4 \text{ N/m}$, which can be compressed 12.0 cm. To what maximum height can a child jump on the stick using only the energy in the spring, if the child and stick have a total mass of 40.0 kg? Explicitly show how you follow the steps in the Problem-Solving Strategies for Energy.

7.5 Nonconservative Forces

24. A 60.0-kg skier with an initial speed of 12.0 m/s coasts up a 2.50-m-high rise as shown in Figure 7.39. Find her final speed at the top, given that the coefficient of friction between her skis and the snow is 0.0800. (Hint: Find the distance traveled up the incline assuming a straight-line path as shown in the figure.)

Figure 7.39 The skier’s initial kinetic energy is partially used in coasting to the top of a rise.
25. (a) How high a hill can a car coast up (engine disengaged) if work done by friction is negligible and its initial speed is 110 km/h? (b) If, in actuality, a 750-kg car with an initial speed of 110 km/h is observed to coast up a hill to a height 22.0 m above its starting point, how much thermal energy was generated by friction? (c) What is the average force of friction if the hill has a slope 2.5° above the horizontal?

7.6 Conservation of Energy

26. Using values from Table 7.1, how many DNA molecules could be broken by the energy carried by a single electron in the beam of an old-fashioned TV tube? (These electrons were not dangerous in themselves, but they did create dangerous x rays. Later model tube TVs had shielding that absorbed x rays before they escaped and exposed viewers.)

27. Using energy considerations and assuming negligible air resistance, show that a rock thrown from a bridge 20.0 m above water with an initial speed of 15.0 m/s strikes the water with a speed of 24.8 m/s independent of the direction thrown.

28. If the energy in fusion bombs were used to supply the energy needs of the world, how many of the 9-megaton variety would be needed for a year’s supply of energy (using data from Table 7.1)? This is not as far-fetched as it may sound—there are thousands of nuclear bombs, and their energy can be trapped in underground explosions and converted to electricity, as natural geothermal energy is.

29. (a) Use of hydrogen fusion to supply energy is a dream that may be realized in the next century. Fusion would be a relatively clean and almost limitless supply of energy, as can be seen from Table 7.1. To illustrate this, calculate how many years the present energy needs of the world could be supplied by one millionth of the oceans' hydrogen fusion energy. (b) How does this time compare with historically significant events, such as the duration of stable economic systems?

7.7 Power

30. The Crab Nebula (see Figure 7.40) pulsar is the remnant of a supernova that occurred in A.D. 1054. Using data from Table 7.3, calculate the approximate factor by which the power output of this astronomical object has declined since its explosion.

31. Suppose a star 1000 times brighter than our Sun (that is, emitting 1000 times the power) suddenly goes supernova. Using data from Table 7.3: (a) By what factor does its power output increase? (b) How many times brighter than our entire Milky Way galaxy is the supernova? (c) Based on your answers, discuss whether it should be possible to observe supernovas in distant galaxies. Note that there are on the order of \(10^{11}\) observable galaxies, the average brightness of which is somewhat less than our own galaxy.

32. A person in good physical condition can put out 100 W of useful power for several hours at a stretch, perhaps by pedaling a mechanism that drives an electric generator. Neglecting any problems of generator efficiency and practical considerations such as resting time: (a) How many people would it take to run a 4.00-kW electric clothes dryer? (b) How many people would it take to replace a large electric power plant that generates 800 MW?

33. What is the cost of operating a 3.00-W electric clock for a year if the cost of electricity is $0.0900 per kW ⋅ h?

34. A large household air conditioner may consume 15.0 kW of power. What is the cost of operating this air conditioner 3.00 h per day for 30.0 d if the cost of electricity is $0.110 per kW ⋅ h?

35. (a) What is the average power consumption in watts of an appliance that uses 5.00 kW ⋅ h of energy per day? (b) How many joules of energy does this appliance consume in a year?

36. (a) What is the average useful power output of a person who does \(6.00 \times 10^5\) J of useful work in 8.00 h? (b) Working at this rate, how long will it take this person to lift 2000 kg of bricks 1.50 m to a platform? (Work done to lift his body can be omitted because it is not considered useful output here.)

37. A 500-kg dragster accelerates from rest to a final speed of 110 m/s in 400 m (about a quarter of a mile) and encounters an average frictional force of 1200 N. What is its average power output in watts and horsepower if this takes 7.30 s?

38. (a) How long will it take an 850-kg car with a useful power output of 40.0 hp (1 hp = 746 W) to reach a speed of 15.0 m/s, neglecting friction? (b) How long will this acceleration take if the car also climbs a 3.00-m-high hill in the process?

39. (a) Find the useful power output of an elevator motor that lifts a 2500-kg load a height of 35.0 m in 12.0 s, if it also increases the speed from rest to 4.00 m/s. Note that the total mass of the counterbalanced system is 10,000 kg—so that only 2500 kg is raised in height, but the full 10,000 kg is accelerated. (b) What does it cost, if electricity is $0.0900 per kW ⋅ h?

40. (a) What is the available energy content, in joules, of a battery that operates a 2.00-W electric clock for 18 months? (b) How long can a battery that can supply \(8.00 \times 10^4\) J run a pocket calculator that consumes energy at the rate of \(1.00 \times 10^{-3}\) W?

41. (a) How long would it take a \(1.50 \times 10^5\)-kg airplane with engines that produce 100 MW of power to reach a speed of 250 m/s and an altitude of 12.0 km if air resistance were negligible? (b) If it actually takes 900 s, what is the power? (c) Given this power, what is the average force of air resistance if the airplane takes 1200 s? (Hint: You must find the distance the plane travels in 1200 s assuming constant acceleration.)

Figure 7.40 Crab Nebula (credit: ESO, via Wikimedia Commons)
42. Calculate the power output needed for a 950-kg car to climb a 2.00° slope at a constant 30.0 m/s while encountering wind resistance and friction totaling 600 N. Explicitly show how you follow the steps in the Problem-Solving Strategies for Energy.

43. (a) Calculate the power per square meter reaching Earth’s upper atmosphere from the Sun. (Take the power output of the Sun to be 4.00×10^{26} W.) (b) Part of this is absorbed and reflected by the atmosphere, so that a maximum of 1.30 kW/m² reaches Earth’s surface. Calculate the area in km² of solar energy collectors needed to replace an electric power plant that generates 750 MW if the collectors convert an average of 2.00% of the maximum power into electricity. (This small conversion efficiency is due to the devices themselves, and the fact that the sun is directly overhead only briefly.) With the same assumptions, what area would be needed to meet the United States’ energy needs (1.05×10^{20} J)? Australia’s energy needs (5.4×10^{18} J)? China’s energy needs (6.3×10^{19} J)? (These energy consumption values are from 2006.)

7.8 Work, Energy, and Power in Humans

44. (a) How long can you rapidly climb stairs (116/min) on the 93.0 kcal of energy in a 10.0-g pat of butter? (b) How many flights is this if each flight has 16 stairs?

45. (a) What is the power output in watts and horsepower of a 70.0-kg sprinter who accelerates from rest to 10.0 m/s in 3.00 s? (b) Considering the amount of power generated, do you think a well-trained athlete could do this repetitively for long periods of time?

46. Calculate the power output in watts and horsepower of a shot-putter who takes 1.20 s to accelerate the 7.27-kg shot from rest to 14.0 m/s, while raising it 0.800 m. (Do not include the power produced to accelerate his body.)

47. (a) What is the efficiency of an out-of-condition professor who does 2.10×10^{5} J of useful work while metabolizing 500 kcal of food energy? (b) How many food calories would a well-conditioned athlete metabolize in doing the same work with an efficiency of 20%?

48. Energy that is not utilized for work or heat transfer is converted to the chemical energy of body fat containing about 39 kJ/g. How many grams of fat will you gain if you eat 10,000 kJ (about 2500 kcal) one day and do nothing but sit relaxed for 16.0 h and sleep for the other 8.00 h? Use data from Table 7.5 for the energy consumption rates of these activities.

49. Using data from Table 7.5, calculate the daily energy needs of a person who sleeps for 7.00 h, walks for 2.00 h, attends classes for 4.00 h, cycles for 2.00 h, sits relaxed for 3.00 h, and studies for 6.00 h. (Studying consumes energy at the same rate as sitting in class.)

50. What is the efficiency of a subject on a treadmill who puts out work at the rate of 100 W while consuming oxygen at the rate of 2.00 L/min? (Hint: See Table 7.5.)

51. Shoveling snow can be extremely taxing because the arms have such a low efficiency in this activity. Suppose a person shoveling a footpath metabolizes food at the rate of 800 W. (a) What is her useful power output? (b) How long will it take her to lift 3000 kg of snow 1.20 m? (This could be the amount of heavy snow on 20 m of footpath.) (c) How much heat transfer in kilojoules will she generate in the process?

52. Very large forces are produced in joints when a person jumps from some height to the ground. (a) Calculate the magnitude of the force produced if an 80.0-kg person jumps from a 0.600–m-high ledge and lands stiffly, compressing joint material 1.50 cm as a result. (Be certain to include the weight of the person.) (b) In practice the knees bend almost involuntarily to help extend the distance over which you stop. Calculate the magnitude of the force produced if the stopping distance is 0.300 m. (c) Compare both forces with the weight of the person.

53. Jogging on hard surfaces with insufficiently padded shoes produces large forces in the feet and legs. (a) Calculate the magnitude of the force needed to stop the downward motion of a jogger’s leg, if his leg has a mass of 13.0 kg, a speed of 6.00 m/s, and stops in a distance of 1.50 cm. (Be certain to include the weight of the 75.0-kg jogger’s body.) (b) Compare this force with the weight of the jogger.

54. (a) Calculate the energy in kJ used by a 55.0-kg woman who does 50 deep knee bends in which her center of mass is lowered and raised 0.400 m. (She does work in both directions.) You may assume her efficiency is 20%. (b) What is the average power consumption rate in watts if she does this in 3.00 min?

Figure 7.41 Shot putter at the Dornoch Highland Gathering in 2007. (credit: John Haslam, Flickr)
55. Kanellos Kanellopoulos flew 119 km from Crete to Santorini, Greece, on April 23, 1988, in the Daedalus 88, an aircraft powered by a bicycle-type drive mechanism (see Figure 7.42). His useful power output for the 234-min trip was about 350 W. Using the efficiency for cycling from Table 7.2, calculate the food energy in kilojoules he metabolized during the flight.

Figure 7.42 The Daedalus 88 in flight. (credit: NASA photo by Beasley)

56. The swimmer shown in Figure 7.43 exerts an average horizontal backward force of 80.0 N with his arm during each 1.80 m long stroke. (a) What is his work output in each stroke? (b) Calculate the power output of his arms if he does 120 strokes per minute.

Figure 7.43

57. Mountain climbers carry bottled oxygen when at very high altitudes. (a) Assuming that a mountain climber uses oxygen at twice the rate for climbing 116 stairs per minute (because of low air temperature and winds), calculate how many liters of oxygen a climber would need for 10.0 h of climbing. (These are liters at sea level.) Note that only 40% of the inhaled oxygen is utilized; the rest is exhaled. (b) How much useful work does the climber do if he and his equipment have a mass of 90.0 kg and he gains 1000 m of altitude? (c) What is his efficiency for the 10.0-h climb?

58. The awe-inspiring Great Pyramid of Cheops was built more than 4500 years ago. Its square base, originally 230 m on a side, covered 13.1 acres, and it was 146 m high, with a mass of about $7 \times 10^9$ kg. (The pyramid’s dimensions are slightly different today due to quarrying and some sagging.) Historians estimate that 20,000 workers spent 20 years to construct it, working 12-hour days, 330 days per year. (a) Calculate the gravitational potential energy stored in the pyramid, given its center of mass is at one-fourth its height. (b) Only a fraction of the workers lifted blocks; most were involved in support services such as building ramps (see Figure 7.44), bringing food and water, and hauling blocks to the site. Calculate the efficiency of the workers who did the lifting, assuming there were 1000 of them and they consumed food energy at the rate of 300 kcal/h. What does your answer imply about how much of their work went into block-lifting, versus how much work went into friction and lifting and lowering their own bodies? (c) Calculate the mass of food that had to be supplied each day, assuming that the average worker required 3600 kcal per day and that their diet was 5% protein, 60% carbohydrate, and 35% fat. (These proportions neglect the mass of bulk and nondigestible materials consumed.)

Figure 7.44 Ancient pyramids were probably constructed using ramps as simple machines. (credit: Franck Monnier, Wikimedia Commons)

59. (a) How long can you play tennis on the 800 kJ (about 200 kcal) of energy in a candy bar? (b) Does this seem like a long time? Discuss why exercise is necessary but may not be sufficient to cause a person to lose weight.
7.9 World Energy Use

60. Integrated Concepts
(a) Calculate the force the woman in Figure 7.45 exerts to do a push-up at constant speed, taking all data to be known to three digits. (b) How much work does she do if her center of mass rises 0.240 m? (c) What is her useful power output if she does 25 push-ups in 1 min? (Should work done lowering her body be included? See the discussion of useful work in Work, Energy, and Power in Humans.

Figure 7.45 Forces involved in doing push-ups. The woman’s weight acts as a force exerted downward on her center of gravity (CG).

61. Integrated Concepts
A 75.0-kg cross-country skier is climbing a 3.00° slope at a constant speed of 2.00 m/s and encounters air resistance of 25.0 N. Find his power output for work done against the gravitational force and air resistance. (b) What average force does he exert backward on the snow to accomplish this? (c) If he continues to exert this force and to experience the same air resistance when he reaches a level area, how long will it take him to reach a velocity of 10.0 m/s?

62. Integrated Concepts
The 70.0-kg swimmer in Figure 7.43 starts a race with an initial velocity of 1.25 m/s and exerts an average force of 80.0 N backward with his arms during each 1.80 m long stroke. (a) What is his initial acceleration if water resistance is 45.0 N? (b) What is the subsequent average resistance force from the water during the 5.00 s it takes him to reach his top velocity of 2.50 m/s? (c) Discuss whether water resistance seems to increase linearly with velocity.

63. Integrated Concepts
A toy gun uses a spring with a force constant of 300 N/m to propel a 10.0-g steel ball. If the spring is compressed 7.00 cm and friction is negligible: (a) How much force is needed to compress the spring? (b) To what maximum height can the ball be shot? (c) At what angles above the horizontal may a child aim to hit a target 3.00 m away at the same height as the gun? (d) What is the gun’s maximum range on level ground?

64. Integrated Concepts
(a) What force must be supplied by an elevator cable to produce an acceleration of 0.800 m/s² against a 200-N frictional force, if the mass of the loaded elevator is 1500 kg? (b) How much work is done by the cable in lifting the elevator 20.0 m? (c) What is the final speed of the elevator if it starts from rest? (d) How much work went into thermal energy?

65. Unreasonable Results
A car advertisement claims that its 900-kg car accelerated from rest to 30.0 m/s and drove 100 km, gaining 3.00 km in altitude, on 1.0 gal of gasoline. The average force of friction including air resistance was 700 N, assuming all values are known to three significant figures. (a) Calculate the car’s efficiency. (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

66. Unreasonable Results
Body fat is metabolized, supplying 9.30 kcal/g, when dietary intake is less than needed to fuel metabolism. The manufacturers of an exercise bicycle claim that you can lose 0.500 kg of fat per day by vigorously exercising for 2.00 h per day on their machine. (a) How many kcal are supplied by the metabolism of 0.500 kg of fat? (b) Calculate the kcal/min that you would have to utilize to metabolize fat at the rate of 0.500 kg in 2.00 h. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

67. Construct Your Own Problem
Consider a person climbing and descending stairs. Construct a problem in which you calculate the long-term rate at which stairs can be climbed considering factors such as the person’s ability to generate power with his legs, the height of a single stair step. Also consider why the same person can descend stairs at a faster rate for a nearly unlimited time in spite of the fact that very similar forces are exerted going down as going up. (This points to a fundamentally different process for descending versus climbing stairs.)

68. Construct Your Own Problem
Consider humans generating electricity by pedaling a device similar to a stationary bicycle. Construct a problem in which you determine the number of people it would take to replace a large electrical generation facility. Among the things to consider are the power output that is reasonable using the legs, rest time, and the need for electricity 24 hours per day. Discuss the practical implications of your results.

69. Integrated Concepts
A 105-kg basketball player crouches down 0.400 m while waiting to jump. After exerting a force on the floor through this 0.400 m, his feet leave the floor and his center of gravity rises 0.950 m above its normal standing erect position. (a) Using energy considerations, calculate his velocity when he leaves the floor. (b) What average force did he exert on the floor? (Do not neglect the force to support his weight as well as that to accelerate him.) (c) What was his power output during the acceleration phase?
Chapter 8 | Linear Momentum and Collisions

Figure 8.1 Each rugby player has great momentum, which will affect the outcome of their collisions with each other and the ground. (credit: ozzzie, Flickr)

Chapter Outline

8.1. Linear Momentum and Force
- Define linear momentum.
- Explain the relationship between momentum and force.
- State Newton’s second law of motion in terms of momentum.
- Calculate momentum given mass and velocity.

8.2. Impulse
- Define impulse.
- Describe effects of impulses in everyday life.
- Determine the average effective force using graphical representation.
- Calculate average force and impulse given mass, velocity, and time.

8.3. Conservation of Momentum
- Describe the principle of conservation of momentum.
- Derive an expression for the conservation of momentum.
- Explain conservation of momentum with examples.
- Explain the principle of conservation of momentum as it relates to atomic and subatomic particles.

8.4. Elastic Collisions in One Dimension
- Describe an elastic collision of two objects in one dimension.
- Define internal kinetic energy.
- Derive an expression for conservation of internal kinetic energy in a one dimensional collision.
- Determine the final velocities in an elastic collision given masses and initial velocities.

8.5. Inelastic Collisions in One Dimension
- Define inelastic collision.
- Explain perfectly inelastic collision.
- Apply an understanding of collisions to sports.
- Determine recoil velocity and loss in kinetic energy given mass and initial velocity.
8.6. Collisions of Point Masses in Two Dimensions
- Discuss two dimensional collisions as an extension of one dimensional analysis.
- Define point masses.
- Derive an expression for conservation of momentum along x-axis and y-axis.
- Describe elastic collisions of two objects with equal mass.
- Determine the magnitude and direction of the final velocity given initial velocity, and scattering angle.

8.7. Introduction to Rocket Propulsion
- State Newton’s third law of motion.
- Explain the principle involved in propulsion of rockets and jet engines.
- Derive an expression for the acceleration of the rocket and discuss the factors that affect the acceleration.
- Describe the function of a space shuttle.

Introduction to Linear Momentum and Collisions

We use the term momentum in various ways in everyday language, and most of these ways are consistent with its precise scientific definition. We speak of sports teams or politicians gaining and maintaining the momentum to win. We also recognize that momentum has something to do with collisions. For example, looking at the rugby players in the photograph colliding and falling to the ground, we expect their momenta to have great effects in the resulting collisions. Generally, momentum implies a tendency to continue on course—to move in the same direction—and is associated with great mass and speed.

Momentum, like energy, is important because it is conserved. Only a few physical quantities are conserved in nature, and studying them yields fundamental insight into how nature works, as we shall see in our study of momentum.

8.1 Linear Momentum and Force

Linear Momentum

The scientific definition of linear momentum is consistent with most people’s intuitive understanding of momentum: a large, fast-moving object has greater momentum than a smaller, slower object. **Linear momentum** is defined as the product of a system’s mass multiplied by its velocity. In symbols, linear momentum is expressed as

\[ p = mv. \]  \hspace{1cm} (8.1)

Momentum is directly proportional to the object's mass and also its velocity. Thus the greater an object's mass or the greater its velocity, the greater its momentum. Momentum \( p \) is a vector having the same direction as the velocity \( v \). The SI unit for momentum is \( \text{kg} \cdot \text{m/s} \).

**Example 8.1 Calculating Momentum: A Football Player and a Football**

(a) Calculate the momentum of a 110-kg football player running at 8.00 m/s. (b) Compare the player’s momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s.

**Strategy**

No information is given regarding direction, and so we can calculate only the magnitude of the momentum, \( p \). (As usual, a symbol that is in italics is a magnitude, whereas one that is italicized, boldfaced, and has an arrow is a vector.) In both parts of this example, the magnitude of momentum can be calculated directly from the definition of momentum given in the equation, which becomes

\[ p = mv \]  \hspace{1cm} (8.3)

when only magnitudes are considered.

**Solution for (a)**

To determine the momentum of the player, substitute the known values for the player’s mass and speed into the equation.

\[ p_{\text{player}} = (110 \text{ kg})(8.00 \text{ m/s}) = 880 \text{ kg} \cdot \text{m/s} \]  \hspace{1cm} (8.4)

**Solution for (b)**

To determine the momentum of the ball, substitute the known values for the ball's mass and speed into the equation.
\[ p_{\text{ball}} = (0.410 \text{ kg})(25.0 \text{ m/s}) = 10.3 \text{ kg} \cdot \text{m/s} \]  

(8.5)

The ratio of the player’s momentum to that of the ball is

\[ \frac{p_{\text{player}}}{p_{\text{ball}}} = \frac{880}{10.3} = 85.9. \]  

(8.6)

**Discussion**

Although the ball has greater velocity, the player has a much greater mass. Thus the momentum of the player is much greater than the momentum of the football, as you might guess. As a result, the player’s motion is only slightly affected if he catches the ball. We shall quantify what happens in such collisions in terms of momentum in later sections.

**Momentum and Newton’s Second Law**

The importance of momentum, unlike the importance of energy, was recognized early in the development of classical physics. Momentum was deemed so important that it was called the “quantity of motion.” Newton actually stated his second law of motion in terms of momentum: The net external force equals the change in momentum of a system divided by the time over which it changes. Using symbols, this law is

\[ F_{\text{net}} = \frac{\Delta p}{\Delta t}. \]  

(8.7)

where \( F_{\text{net}} \) is the net external force, \( \Delta p \) is the change in momentum, and \( \Delta t \) is the change in time.

**Newton’s Second Law of Motion in Terms of Momentum**

The net external force equals the change in momentum of a system divided by the time over which it changes.

\[ F_{\text{net}} = \frac{\Delta p}{\Delta t}. \]  

(8.8)

**Making Connections: Force and Momentum**

Force and momentum are intimately related. Force acting over time can change momentum, and Newton’s second law of motion, can be stated in its most broadly applicable form in terms of momentum. Momentum continues to be a key concept in the study of atomic and subatomic particles in quantum mechanics.

This statement of Newton’s second law of motion includes the more familiar \( F_{\text{net}} = ma \) as a special case. We can derive this form as follows. First, note that the change in momentum \( \Delta p \) is given by

\[ \Delta p = \Delta (mv). \]  

(8.9)

If the mass of the system is constant, then

\[ \Delta (mv) = m\Delta v. \]  

(8.10)

So that for constant mass, Newton’s second law of motion becomes

\[ F_{\text{net}} = \frac{\Delta p}{\Delta t} = \frac{m\Delta v}{\Delta t}. \]  

(8.11)

Because \( \frac{\Delta v}{\Delta t} = a \), we get the familiar equation

\[ F_{\text{net}} = ma \]  

(8.12)

when the mass of the system is constant.

Newton’s second law of motion stated in terms of momentum is more generally applicable because it can be applied to systems where the mass is changing, such as rockets, as well as to systems of constant mass. We will consider systems with varying mass in some detail; however, the relationship between momentum and force remains useful when mass is constant, such as in the following example.

**Example 8.2 Calculating Force: Venus Williams’ Racquet**

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women’s match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams’ racquet, assuming
that the ball’s speed just after impact is 58 m/s, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms (milliseconds)?

**Strategy**

This problem involves only one dimension because the ball starts from having no horizontal velocity component before impact. Newton’s second law stated in terms of momentum is then written as

$$F_{\text{net}} = \frac{\Delta p}{\Delta t}.$$  

(8.13)

As noted above, when mass is constant, the change in momentum is given by

$$\Delta p = m\Delta v = m(v_f - v_i).$$  

(8.14)

In this example, the velocity just after impact and the change in time are given; thus, once $\Delta p$ is calculated, $F_{\text{net}} = \frac{\Delta p}{\Delta t}$ can be used to find the force.

**Solution**

To determine the change in momentum, substitute the values for the initial and final velocities into the equation above.

$$\Delta p = m(v_f - v_i)$$  

(8.15)

$$= (0.057 \text{ kg})(58 \text{ m/s} - 0 \text{ m/s})$$  

$$= 3.306 \text{ kg} \cdot \text{m/s} \approx 3.3 \text{ kg} \cdot \text{m/s}$$

Now the magnitude of the net external force can determined by using $F_{\text{net}} = \frac{\Delta p}{\Delta t}$:

$$F_{\text{net}} = \frac{\Delta p}{\Delta t} = \frac{3.306 \text{ kg} \cdot \text{m/s}}{5.0 \times 10^{-3} \text{ s}}$$

$$= 661 \text{ N} \approx 660 \text{ N},$$

where we have retained only two significant figures in the final step.

**Discussion**

This quantity was the average force exerted by Venus Williams’ racquet on the tennis ball during its brief impact (note that the ball also experienced the 0.56-N force of gravity, but that force was not due to the racquet). This problem could also be solved by first finding the acceleration and then using $F_{\text{net}} = ma$, but one additional step would be required compared with the strategy used in this example.

### 8.2 Impulse

The effect of a force on an object depends on how long it acts, as well as how great the force is. In **Example 8.1**, a very large force acting for a short time had a great effect on the momentum of the tennis ball. A small force could cause the same **change in momentum**, but it would have to act for a much longer time. For example, if the ball were thrown upward, the gravitational force (which is much smaller than the tennis racquet’s force) would eventually reverse the momentum of the ball. Quantitatively, the effect we are talking about is the change in momentum $\Delta p$.

By rearranging the equation $F_{\text{net}} = \frac{\Delta p}{\Delta t}$ to be

$$\Delta p = F_{\text{net}}\Delta t,$$

we can see how the change in momentum equals the average net external force multiplied by the time this force acts. The quantity $F_{\text{net}}\Delta t$ is given the name **impulse**. Impulse is the same as the change in momentum.

**Impulse: Change in Momentum**

Change in momentum equals the average net external force multiplied by the time this force acts.

$$\Delta p = F_{\text{net}}\Delta t$$  

(8.18)

The quantity $F_{\text{net}}\Delta t$ is given the name impulse.

There are many ways in which an understanding of impulse can save lives, or at least limbs. The dashboard padding in a car, and certainly the airbags, allow the net force on the occupants in the car to act over a much longer time when there is a
sudden stop. The momentum change is the same for an occupant, whether an air bag is deployed or not, but the force (to bring the occupant to a stop) will be much less if it acts over a larger time. Cars today have many plastic components. One advantage of plastics is their lighter weight, which results in better gas mileage. Another advantage is that a car will crumple in a collision, especially in the event of a head-on collision. A longer collision time means the force on the car will be less. Deaths during car races decreased dramatically when the rigid frames of racing cars were replaced with parts that could crumple or collapse in the event of an accident.

Bones in a body will fracture if the force on them is too large. If you jump onto the floor from a table, the force on your legs can be immense if you land stiff-legged on a hard surface. Rolling on the ground after jumping from the table, or landing with a parachute, extends the time over which the force (on you from the ground) acts.

Example 8.3 Calculating Magnitudes of Impulses: Two Billiard Balls Striking a Rigid Wall

Two identical billiard balls strike a rigid wall with the same speed, and are reflected without any change of speed. The first ball strikes perpendicular to the wall. The second ball strikes the wall at an angle of \(30^\circ\) from the perpendicular, and bounces off at an angle of \(30^\circ\) from perpendicular to the wall.

(a) Determine the direction of the force on the wall due to each ball.
(b) Calculate the ratio of the magnitudes of impulses on the two balls by the wall.

Strategy for (a)
In order to determine the force on the wall, consider the force on the ball due to the wall using Newton’s second law and then apply Newton’s third law to determine the direction. Assume the \(x\)-axis to be normal to the wall and to be positive in the initial direction of motion. Choose the \(y\)-axis to be along the wall in the plane of the second ball’s motion. The momentum direction and the velocity direction are the same.

Solution for (a)
The first ball bounces directly into the wall and exerts a force on it in the \(+x\) direction. Therefore the wall exerts a force on the ball in the \(-x\) direction. The second ball continues with the same momentum component in the \(y\) direction, but reverses its \(x\)-component of momentum, as seen by sketching a diagram of the angles involved and keeping in mind the proportionality between velocity and momentum.

These changes mean the change in momentum for both balls is in the \(-x\) direction, so the force of the wall on each ball is along the \(-x\) direction.

Strategy for (b)
Calculate the change in momentum for each ball, which is equal to the impulse imparted to the ball.

Solution for (b)
Let \(u\) be the speed of each ball before and after collision with the wall, and \(m\) the mass of each ball. Choose the \(x\)-axis and \(y\)-axis as previously described, and consider the change in momentum of the first ball which strikes perpendicular to the wall.

\[
\begin{align*}
p_{xi} &= mu; \\ p_{yi} &= 0
\end{align*}
\]

\[
\begin{align*}
p_{xf} &= -mu; \\ p_{yf} &= 0
\end{align*}
\]

Impulse is the change in momentum vector. Therefore the \(x\)-component of impulse is equal to \(-2mu\) and the \(y\)-component of impulse is equal to zero.

Now consider the change in momentum of the second ball.

\[
\begin{align*}
p_{xi} &= mu \cos 30^\circ; \\ p_{yi} &= -mu \sin 30^\circ
\end{align*}
\]

\[
\begin{align*}
p_{xf} &= -mu \cos 30^\circ; \\ p_{yf} &= -mu \sin 30^\circ
\end{align*}
\]

It should be noted here that while \(p_x\) changes sign after the collision, \(p_y\) does not. Therefore the \(x\)-component of impulse is equal to \(-2mu \cos 30^\circ\) and the \(y\)-component of impulse is equal to zero.

The ratio of the magnitudes of the impulse imparted to the balls is

\[
\frac{2mu}{2mu \cos 30^\circ} = \frac{2}{\sqrt{3}} = 1.155.
\]

Discussion
The direction of impulse and force is the same as in the case of (a); it is normal to the wall and along the negative \(x\)
-direction. Making use of Newton’s third law, the force on the wall due to each ball is normal to the wall along the positive $x$-direction.

Our definition of impulse includes an assumption that the force is constant over the time interval $\Delta t$. Forces are usually not constant. Forces vary considerably even during the brief time intervals considered. It is, however, possible to find an average effective force $F_{\text{eff}}$ that produces the same result as the corresponding time-varying force. Figure 8.2 shows a graph of what an actual force looks like as a function of time for a ball bouncing off the floor. The area under the curve has units of momentum and is equal to the impulse or change in momentum between times $t_1$ and $t_2$. That area is equal to the area inside the rectangle bounded by $F_{\text{eff}}$, $t_1$, and $t_2$. Thus the impulses and their effects are the same for both the actual and effective forces.

![Figure 8.2 A graph of force versus time with time along the $x$-axis and force along the $y$-axis for an actual force and an equivalent effective force. The areas under the two curves are equal.](image)

**Making Connections: Take-Home Investigation—Hand Movement and Impulse**

Try catching a ball while “giving” with the ball, pulling your hands toward your body. Then, try catching a ball while keeping your hands still. Hit water in a tub with your full palm. After the water has settled, hit the water again by diving your hand with your fingers first into the water. (Your full palm represents a swimmer doing a belly flop and your diving hand represents a swimmer doing a dive.) Explain what happens in each case and why. Which orientations would you advise people to avoid and why?

**Making Connections: Constant Force and Constant Acceleration**

The assumption of a constant force in the definition of impulse is analogous to the assumption of a constant acceleration in kinematics. In both cases, nature is adequately described without the use of calculus.

### 8.3 Conservation of Momentum

Momentum is an important quantity because it is conserved. Yet it was not conserved in the examples in Impulse and Linear Momentum and Force, where large changes in momentum were produced by forces acting on the system of interest. Under what circumstances is momentum conserved?

The answer to this question entails considering a sufficiently large system. It is always possible to find a larger system in which total momentum is constant, even if momentum changes for components of the system. If a football player runs into the goalpost in the end zone, there will be a force on him that causes him to bounce backward. However, the Earth also recoils—conserving momentum—because of the force applied to it through the goalpost. Because Earth is many orders of magnitude more massive than the player, its recoil is immeasurably small and can be neglected in any practical sense, but it is real nevertheless.

Consider what happens if the masses of two colliding objects are more similar than the masses of a football player and Earth—for example, one car bumping into another, as shown in Figure 8.3. Both cars are coasting in the same direction when the lead car (labeled $m_2$) is bumped by the trailing car (labeled $m_1$). The only unbalanced force on each car is the force of the collision. (Assume that the effects due to friction are negligible.) Car 1 slows down as a result of the collision, losing some momentum, while car 2 speeds up and gains some momentum. We shall now show that the total momentum of the two-car system remains constant.
A car of mass \( m_1 \) moving with a velocity of \( v_1 \) bumps into another car of mass \( m_2 \) and velocity \( v_2 \) that it is following. As a result, the first car slows down to a velocity of \( v'_1 \) and the second speeds up to a velocity of \( v'_2 \). The momentum of each car is changed, but the total momentum \( p_{\text{tot}} \) of the two cars is the same before and after the collision (if you assume friction is negligible).

Using the definition of impulse, the change in momentum of car 1 is given by

\[
\Delta p_1 = F_1 \Delta t, \tag{8.24}
\]

where \( F_1 \) is the force on car 1 due to car 2, and \( \Delta t \) is the time the force acts (the duration of the collision). Intuitively, it seems obvious that the collision time is the same for both cars, but it is only true for objects traveling at ordinary speeds. This assumption must be modified for objects travelling near the speed of light, without affecting the result that momentum is conserved.

Similarly, the change in momentum of car 2 is

\[
\Delta p_2 = F_2 \Delta t, \tag{8.25}
\]

where \( F_2 \) is the force on car 2 due to car 1, and we assume the duration of the collision \( \Delta t \) is the same for both cars. We know from Newton’s third law that \( F_2 = -F_1 \), and so

\[
\Delta p_2 = -F_1 \Delta t = -\Delta p_1. \tag{8.26}
\]

Thus, the changes in momentum are equal and opposite, and

\[
\Delta p_1 + \Delta p_2 = 0. \tag{8.27}
\]

Because the changes in momentum add to zero, the total momentum of the two-car system is constant. That is,

\[
p_1 + p_2 = \text{constant}, \tag{8.28}
\]

\[
p_1 + p_2 = p'_1 + p'_2, \tag{8.29}
\]

where \( p'_1 \) and \( p'_2 \) are the momenta of cars 1 and 2 after the collision. (We often use primes to denote the final state.)

This result—that momentum is conserved—has validity far beyond the preceding one-dimensional case. It can be similarly shown that total momentum is conserved for any isolated system, with any number of objects in it. In equation form, the conservation of momentum principle for an isolated system is written

\[
p_{\text{tot}} = \text{constant}, \tag{8.30}
\]

or

\[
p_{\text{tot}} = p'_{\text{tot}}, \tag{8.31}
\]

where \( p_{\text{tot}} \) is the total momentum (the sum of the momenta of the individual objects in the system) and \( p'_{\text{tot}} \) is the total
momentum some time later. (The total momentum can be shown to be the momentum of the center of mass of the system.) An isolated system is defined to be one for which the net external force is zero \( F_{\text{net}} = 0 \).

### Conservation of Momentum Principle

\[
\begin{align*}
\mathbf{p}_{\text{tot}} & = \text{constant} \quad \text{(8.32)} \\
\mathbf{p}_{\text{tot}} & = \mathbf{p}'_{\text{tot}} \quad \text{(isolated system)}
\end{align*}
\]

### Isolated System

An isolated system is defined to be one for which the net external force is zero \( F_{\text{net}} = 0 \).

Perhaps an easier way to see that momentum is conserved for an isolated system is to consider Newton’s second law in terms of momentum, \( F_{\text{net}} = \frac{\Delta \mathbf{p}_{\text{tot}}}{\Delta t} \). For an isolated system, \( F_{\text{net}} = 0 \); thus, \( \Delta \mathbf{p}_{\text{tot}} = 0 \), and \( \mathbf{p}_{\text{tot}} \) is constant.

We have noted that the three length dimensions in nature—\( x \), \( y \), and \( z \)—are independent, and it is interesting to note that momentum can be conserved in different ways along each dimension. For example, during projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero and momentum is unchanged. But along the vertical direction, the net vertical force is not zero and the momentum of the projectile is not conserved. (See Figure 8.4.) However, if the momentum of the projectile-Earth system is considered in the vertical direction, we find that the total momentum is conserved.

![Figure 8.4](https://legacy.cnx.org/content/col11951/1.1)

Figure 8.4 The horizontal component of a projectile’s momentum is conserved if air resistance is negligible, even in this case where a space probe separates. The forces causing the separation are internal to the system, so that the net external horizontal force \( F_{\text{net}} = 0 \) is still zero. The vertical component of the momentum is not conserved, because the net vertical force \( F_{\text{net}} = 0 \) is not zero. In the vertical direction, the space probe-Earth system needs to be considered and we find that the total momentum is conserved. The center of mass of the space probe takes the same path it would if the separation did not occur.

The conservation of momentum principle can be applied to systems as different as a comet striking Earth and a gas containing huge numbers of atoms and molecules. Conservation of momentum is violated only when the net external force is not zero. But another larger system can always be considered in which momentum is conserved by simply including the source of the external force. For example, in the collision of two cars considered above, the two-car system conserves momentum while each one-car system does not.

**Making Connections: Take-Home Investigation—Drop of Tennis Ball and a Basketball**

Hold a tennis ball side by side and in contact with a basketball. Drop the balls together. (Be careful!) What happens? Explain your observations. Now hold the tennis ball above and in contact with the basketball. What happened? Explain your observations. What do you think will happen if the basketball ball is held above and in contact with the tennis ball?

**Making Connections: Take-Home Investigation—Two Tennis Balls in a Ballistic Trajectory**

Tie two tennis balls together with a string about a foot long. Hold one ball and let the other hang down and throw it in a
ballistic trajectory. Explain your observations. Now mark the center of the string with bright ink or attach a brightly colored sticker to it and throw again. What happened? Explain your observations.

Some aquatic animals such as jellyfish move around based on the principles of conservation of momentum. A jellyfish fills its umbrella section with water and then pushes the water out resulting in motion in the opposite direction to that of the jet of water. Squids propel themselves in a similar manner but, in contrast with jellyfish, are able to control the direction in which they move by aiming their nozzle forward or backward. Typical squids can move at speeds of 8 to 12 km/h.

The ballistocardiograph (BCG) was a diagnostic tool used in the second half of the 20th century to study the strength of the heart. About once a second, your heart beats, forcing blood into the aorta. A force in the opposite direction is exerted on the rest of your body (recall Newton's third law). A ballistocardiograph is a device that can measure this reaction force. This measurement is done by using a sensor (resting on the person) or by using a moving table suspended from the ceiling. This technique can gather information on the strength of the heart beat and the volume of blood passing from the heart. However, the electrocardiogram (ECG or EKG) and the echocardiogram (cardiac ECHO or ECHO; a technique that uses ultrasound to see an image of the heart) are more widely used in the practice of cardiology.

Making Connections: Conservation of Momentum and Collision

Conservation of momentum is quite useful in describing collisions. Momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

Subatomic Collisions and Momentum

The conservation of momentum principle not only applies to the macroscopic objects, it is also essential to our explorations of atomic and subatomic particles. Giant machines hurl subatomic particles at one another, and researchers evaluate the results by assuming conservation of momentum (among other things).

On the small scale, we find that particles and their properties are invisible to the naked eye but can be measured with our instruments, and models of these subatomic particles can be constructed to describe the results. Momentum is found to be a property of all subatomic particles including massless particles such as photons that compose light. Momentum being a property of particles hints that momentum may have an identity beyond the description of an object's mass multiplied by the object's velocity. Indeed, momentum relates to wave properties and plays a fundamental role in what measurements are taken and how we take these measurements. Furthermore, we find that the conservation of momentum principle is valid when considering systems of particles. We use this principle to analyze the masses and other properties of previously undetected particles, such as the nucleus of an atom and the existence of quarks that make up particles of nuclei. Figure 8.5 below illustrates how a particle scattering backward from another implies that its target is massive and dense. Experiments seeking evidence that quarks make up protons (one type of particle that makes up nuclei) scattered high-energy electrons off of protons (nuclei of hydrogen atoms). Electrons occasionally scattered straight backward in a manner that implied a very small and very dense particle makes up the proton—this observation is considered nearly direct evidence of quarks. The analysis was based partly on the same conservation of momentum principle that works so well on the large scale.

![Figure 8.5](image)

Figure 8.5 A subatomic particle scatters straight backward from a target particle. In experiments seeking evidence for quarks, electrons were observed to occasionally scatter straight backward from a proton.

8.4 Elastic Collisions in One Dimension

Let us consider various types of two-object collisions. These collisions are the easiest to analyze, and they illustrate many of the physical principles involved in collisions. The conservation of momentum principle is very useful here, and it can be used whenever the net external force on a system is zero.
We start with the elastic collision of two objects moving along the same line—a one-dimensional problem. An elastic collision is one that also conserves internal kinetic energy. Internal kinetic energy is the sum of the kinetic energies of the objects in the system. Figure 8.6 illustrates an elastic collision in which internal kinetic energy and momentum are conserved.

Truly elastic collisions can only be achieved with subatomic particles, such as electrons striking nuclei. Macroscopic collisions can be very nearly, but not quite, elastic—some kinetic energy is always converted into other forms of energy such as heat transfer due to friction and sound. One macroscopic collision that is nearly elastic is that of two steel blocks on ice. Another nearly elastic collision is that between two carts with spring bumpers on an air track. Icy surfaces and air tracks are nearly frictionless, more readily allowing nearly elastic collisions on them.

**Elastic Collision**

An elastic collision is one that conserves internal kinetic energy.

**Internal Kinetic Energy**

Internal kinetic energy is the sum of the kinetic energies of the objects in the system.

![Diagram of an elastic one-dimensional two-object collision. Momentum and internal kinetic energy are conserved.](image)

Figure 8.6 An elastic one-dimensional two-object collision. Momentum and internal kinetic energy are conserved.

Now, to solve problems involving one-dimensional elastic collisions between two objects we can use the equations for conservation of momentum and conservation of internal kinetic energy. First, the equation for conservation of momentum for two objects in a one-dimensional collision is

$$p_1 + p_2 = p'_1 + p'_2$$

or

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2$$

(8.33)

where the primes (') indicate values after the collision. By definition, an elastic collision conserves internal kinetic energy, and so the sum of kinetic energies before the collision equals the sum after the collision. Thus,

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v'_1^2 + \frac{1}{2}m_2 v'_2^2$$

(8.34)

expresses the equation for conservation of internal kinetic energy in a one-dimensional collision.

**Example 8.4 Calculating Velocities Following an Elastic Collision**

Calculate the velocities of two objects following an elastic collision, given that...
Strategy and Concept

First, visualize what the initial conditions mean—a small object strikes a larger object that is initially at rest. This situation is slightly simpler than the situation shown in Figure 8.6 where both objects are initially moving. We are asked to find two unknowns (the final velocities \( v_1' \) and \( v_2' \)). To find two unknowns, we must use two independent equations. Because this collision is elastic, we can use the above two equations. Both can be simplified by the fact that object 2 is initially at rest, and thus \( v_2 = 0 \). Once we simplify these equations, we combine them algebraically to solve for the unknowns.

Solution

For this problem, note that \( v_2 = 0 \) and use conservation of momentum. Thus,

\[
p_1 = p_1' + p_2'
\]

or

\[
m_1 v_1 = m_1 v_1' + m_2 v_2'.
\]

Using conservation of internal kinetic energy and that \( v_2 = 0 \),

\[
\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2.
\]

Solving the first equation (momentum equation) for \( v_2' \), we obtain

\[
v_2' = \frac{m_1}{m_2} (v_1 - v_1').
\]

Substituting this expression into the second equation (internal kinetic energy equation) eliminates the variable \( v_2' \), leaving only \( v_1' \) as an unknown (the algebra is left as an exercise for the reader). There are two solutions to any quadratic equation; in this example, they are

\[
v_1' = 4.00 \text{ m/s}
\]

and

\[
v_1' = -3.00 \text{ m/s}.
\]

As noted when quadratic equations were encountered in earlier chapters, both solutions may or may not be meaningful. In this case, the first solution is the same as the initial condition. The first solution thus represents the situation before the collision and is discarded. The second solution \( (v_1' = -3.00 \text{ m/s}) \) is negative, meaning that the first object bounces backward. When this negative value of \( v_1' \) is used to find the velocity of the second object after the collision, we get

\[
v_2' = \frac{m_1}{m_2} (v_1 - v_1') = \frac{0.500 \text{ kg}}{3.50 \text{ kg}} [4.00 - (-3.00)] \text{ m/s}
\]

or

\[
v_2' = 1.00 \text{ m/s}.
\]

Discussion

The result of this example is intuitively reasonable. A small object strikes a larger one at rest and bounces backward. The larger one is knocked forward, but with a low speed. (This is like a compact car bouncing backward off a full-size SUV that is initially at rest.) As a check, try calculating the internal kinetic energy before and after the collision. You will see that the internal kinetic energy is unchanged at 4.00 J. Also check the total momentum before and after the collision; you will find it, too, is unchanged.

The equations for conservation of momentum and internal kinetic energy as written above can be used to describe any one-dimensional elastic collision of two objects. These equations can be extended to more objects if needed.

Making Connections: Take-Home Investigation—Ice Cubes and Elastic Collision

Find a few ice cubes which are about the same size and a smooth kitchen tabletop or a table with a glass top. Place the ice cubes on the surface several centimeters away from each other. Flick one ice cube toward a stationary ice cube and observe the path and velocities of the ice cubes after the collision. Try to avoid edge-on collisions and collisions with rotating ice cubes. Have you created approximately elastic collisions? Explain the speeds and directions of the ice cubes using...
momentum.

**Collision Lab**

Investigate collisions on an air hockey table. Set up your own experiments: vary the number of discs, masses and initial conditions. Is momentum conserved? Is kinetic energy conserved? Vary the elasticity and see what happens.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42163/1.14/#fs-id1167067199002)

Figure 8.7

---

### 8.5 Inelastic Collisions in One Dimension

We have seen that in an elastic collision, internal kinetic energy is conserved. An **inelastic collision** is one in which the internal kinetic energy changes (it is not conserved). This lack of conservation means that the forces between colliding objects may remove or add internal kinetic energy. Work done by internal forces may change the forms of energy within a system. For inelastic collisions, such as when colliding objects stick together, this internal work may transform some internal kinetic energy into heat transfer. Or it may convert stored energy into internal kinetic energy, such as when exploding bolts separate a satellite from its launch vehicle.

**Inelastic Collision**

An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).

Figure 8.8 shows an example of an inelastic collision. Two objects that have equal masses head toward one another at equal speeds and then stick together. Their total internal kinetic energy is initially $\frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$. The two objects come to rest after sticking together, conserving momentum. But the internal kinetic energy is zero after the collision. A collision in which the objects stick together is sometimes called a **perfectly inelastic collision** because it reduces internal kinetic energy more than does any other type of inelastic collision. In fact, such a collision reduces internal kinetic energy to the minimum it can have while still conserving momentum.

**Perfectly Inelastic Collision**

A collision in which the objects stick together is sometimes called “perfectly inelastic.”

---

**Example 8.5 Calculating Velocity and Change in Kinetic Energy: Inelastic Collision of a Puck and a Goalie**

(a) Find the recoil velocity of a 70.0-kg ice hockey goalie, originally at rest, who catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. (b) How much kinetic energy is lost during the collision? Assume friction between the ice and the puck-goalie system is negligible. (See Figure 8.9)
Figure 8.9 An ice hockey goalie catches a hockey puck and recoils backward. The initial kinetic energy of the puck is almost entirely converted to thermal energy and sound in this inelastic collision.

**Strategy**

Momentum is conserved because the net external force on the puck-goalie system is zero. We can thus use conservation of momentum to find the final velocity of the puck and goalie system. Note that the initial velocity of the goalie is zero and that the final velocity of the puck and goalie are the same. Once the final velocity is found, the kinetic energies can be calculated before and after the collision and compared as requested.

**Solution for (a)**

Momentum is conserved because the net external force on the puck-goalie system is zero.

Conservation of momentum is

\[ p_1 + p_2 = p_1' + p_2' \]

or

\[ m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'. \]

Because the goalie is initially at rest, we know \( v_2 = 0 \). Because the goalie catches the puck, the final velocities are equal, or \( v_1' = v_2' = v' \). Thus, the conservation of momentum equation simplifies to

\[ m_1 v_1 = (m_1 + m_2)v'. \]

Solving for \( v' \) yields

\[ v' = \frac{m_1}{m_1 + m_2} v_1. \]

Entering known values in this equation, we get

\[ v' = \left( \frac{0.150 \text{ kg}}{0.150 \text{ kg} + 70.0 \text{ kg}} \right)(35.0 \text{ m/s}) = 7.48 \times 10^{-2} \text{ m/s}. \]

**Discussion for (a)**

This recoil velocity is small and in the same direction as the puck’s original velocity, as we might expect.

**Solution for (b)**

Before the collision, the internal kinetic energy \( KE_{\text{int}} \) of the system is that of the hockey puck, because the goalie is initially at rest. Therefore, \( KE_{\text{int}} \) is initially

\[ KE_{\text{int}} = \frac{1}{2} m v^2 = \frac{1}{2}(0.150 \text{ kg})(35.0 \text{ m/s})^2 \]

\[ = 91.9 \text{ J.} \]

After the collision, the internal kinetic energy is

\[ KE'_{\text{int}} = \frac{1}{2}(m + M)v^2 = \frac{1}{2}(70.15 \text{ kg})(7.48 \times 10^{-2} \text{ m/s})^2 \]

\[ = 0.196 \text{ J.} \]

The change in internal kinetic energy is thus
\[
\text{KE}'_{\text{int}} - \text{KE}_{\text{int}} = 0.196 \text{ J} - 91.9 \text{ J} = -91.7 \text{ J}
\] (8.52)

where the minus sign indicates that the energy was lost.

**Discussion for (b)**

Nearly all of the initial internal kinetic energy is lost in this perfectly inelastic collision. $\text{KE}_{\text{int}}$ is mostly converted to thermal energy and sound.

During some collisions, the objects do not stick together and less of the internal kinetic energy is removed—such as happens in most automobile accidents. Alternatively, stored energy may be converted into internal kinetic energy during a collision. *Figure 8.10* shows a one-dimensional example in which two carts on an air track collide, releasing potential energy from a compressed spring. *Example 8.6* deals with data from such a collision.

*Figure 8.10* An air track is nearly frictionless, so that momentum is conserved. Motion is one-dimensional. In this collision, examined in *Example 8.6*, the potential energy of a compressed spring is released during the collision and is converted to internal kinetic energy.

Collisions are particularly important in sports and the sporting and leisure industry utilizes elastic and inelastic collisions. Let us look briefly at tennis. Recall that in a collision, it is momentum and not force that is important. So, a heavier tennis racquet will have the advantage over a lighter one. This conclusion also holds true for other sports—a lightweight bat (such as a softball bat) cannot hit a hardball very far.

The location of the impact of the tennis ball on the racquet is also important, as is the part of the stroke during which the impact occurs. A smooth motion results in the maximizing of the velocity of the ball after impact and reduces sports injuries such as tennis elbow. A tennis player tries to hit the ball on the “sweet spot” on the racquet, where the vibration and impact are minimized and the ball is able to be given more velocity. Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

**Take-Home Experiment—Bouncing of Tennis Ball**

1. Find a racquet (a tennis, badminton, or other racquet will do). Place the racquet on the floor and stand on the handle. Drop a tennis ball on the strings from a measured height. Measure how high the ball bounces. Now ask a friend to hold the racquet firmly by the handle and drop a tennis ball from the same measured height above the racquet. Measure how high the ball bounces and observe what happens to your friend’s hand during the collision. Explain your observations and measurements.

2. The coefficient of restitution ($c$) is a measure of the elasticity of a collision between a ball and an object, and is defined as the ratio of the speeds after and before the collision. A perfectly elastic collision has a $c$ of 1. For a ball bouncing off the floor (or a racquet on the floor), $c$ can be shown to be $c = (h/H)^{1/2}$ where $h$ is the height to which the ball bounces and $H$ is the height from which the ball is dropped. Determine $c$ for the cases in Part 1 and
for the case of a tennis ball bouncing off a concrete or wooden floor (\(c = 0.85\) for new tennis balls used on a tennis court).

Example 8.6 Calculating Final Velocity and Energy Release: Two Carts Collide

In the collision pictured in Figure 8.10, two carts collide inelastically. Cart 1 (denoted \(m_1\)) carries a spring which is initially compressed. During the collision, the spring releases its potential energy and converts it to internal kinetic energy. The mass of cart 1 and the spring is 0.350 kg, and the cart and the spring together have an initial velocity of 2.00 m/s. Cart 2 (denoted \(m_2\) in Figure 8.10) has a mass of 0.500 kg and an initial velocity of \(-0.500\) m/s. After the collision, cart 1 is observed to recoil with a velocity of \(-4.00\) m/s. (a) What is the final velocity of cart 2? (b) How much energy was released by the spring (assuming all of it was converted into internal kinetic energy)?

Strategy

We can use conservation of momentum to find the final velocity of cart 2, because \(F_{\text{net}} = 0\) (the track is frictionless and the force of the spring is internal). Once this velocity is determined, we can compare the internal kinetic energy before and after the collision to see how much energy was released by the spring.

Solution for (a)

As before, the equation for conservation of momentum in a two-object system is

\[
m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2.
\]

The only unknown in this equation is \(v'_2\). Solving for \(v'_2\) and substituting known values into the previous equation yields

\[
v'_2 = \frac{m_1 v_1 + m_2 v_2 - m_1 v'_1}{m_2} = \frac{(0.350 \text{ kg})(2.00 \text{ m/s}) + (0.500 \text{ kg})(-0.500 \text{ m/s}) - (0.350 \text{ kg})(-4.00 \text{ m/s})}{0.500 \text{ kg}} = 3.70 \text{ m/s}.
\]

Solution for (b)

The internal kinetic energy before the collision is

\[
\text{KE}_{\text{int}} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}(0.350 \text{ kg})(2.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(-0.500 \text{ m/s})^2 = 0.763 \text{ J}.
\]

After the collision, the internal kinetic energy is

\[
\text{KE}'_{\text{int}} = \frac{1}{2}m_1 v'_1^2 + \frac{1}{2}m_2 v'_2^2 = \frac{1}{2}(0.350 \text{ kg})(-4.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(3.70 \text{ m/s})^2 = 6.22 \text{ J}.
\]

The change in internal kinetic energy is thus

\[
\text{KE}'_{\text{int}} - \text{KE}_{\text{int}} = 6.22 \text{ J} - 0.763 \text{ J} = 5.46 \text{ J}.
\]

Discussion

The final velocity of cart 2 is large and positive, meaning that it is moving to the right after the collision. The internal kinetic energy in this collision increases by 5.46 J. That energy was released by the spring.

8.6 Collisions of Point Masses in Two Dimensions

In the previous two sections, we considered only one-dimensional collisions; during such collisions, the incoming and outgoing velocities are all along the same line. But what about collisions, such as those between billiard balls, in which objects scatter to the side? These are two-dimensional collisions, and we shall see that their study is an extension of the one-dimensional analysis already presented. The approach taken (similar to the approach in discussing two-dimensional kinematics and dynamics) is to choose a convenient coordinate system and resolve the motion into components along perpendicular axes. Resolving the motion yields a pair of one-dimensional problems to be solved simultaneously.
One complication arising in two-dimensional collisions is that the objects might rotate before or after their collision. For example, if two ice skaters hook arms as they pass by one another, they will spin in circles. We will not consider such rotation until later, and so for now we arrange things so that no rotation is possible. To avoid rotation, we consider only the scattering of point masses—that is, structureless particles that cannot rotate or spin.

We start by assuming that $\mathbf{F}_{\text{net}} = 0$, so that momentum $\mathbf{p}$ is conserved. The simplest collision is one in which one of the particles is initially at rest. (See Figure 8.11.) The best choice for a coordinate system is one with an axis parallel to the velocity of the incoming particle, as shown in Figure 8.11. Because momentum is conserved, the components of momentum along the $x$- and $y$-axes ($p_x$ and $p_y$) will also be conserved, but with the chosen coordinate system, $p_y$ is initially zero and $p_x$ is the momentum of the incoming particle. Both facts simplify the analysis. (Even with the simplifying assumptions of point masses, one particle initially at rest, and a convenient coordinate system, we still gain new insights into nature from the analysis of two-dimensional collisions.)

**Figure 8.11** A two-dimensional collision with the coordinate system chosen so that $m_2$ is initially at rest and $v_1$ is parallel to the $x$-axis. This coordinate system is sometimes called the laboratory coordinate system, because many scattering experiments have a target that is stationary in the laboratory, while particles are scattered from it to determine the particles that make-up the target and how they are bound together. The particles may not be observed directly, but their initial and final velocities are.

Along the $x$-axis, the equation for conservation of momentum is

$$p_{1x} + p_{2x} = p_{1x}' + p_{2x}'$$

(8.58)

Where the subscripts denote the particles and axes and the primes denote the situation after the collision. In terms of masses and velocities, this equation is

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v_{1x}' + m_2 v_{2x}'$$

(8.59)

But because particle 2 is initially at rest, this equation becomes

$$m_1 v_{1x} = m_1 v_{1x}' + m_2 v_{2x}'$$

(8.60)

The components of the velocities along the $x$-axis have the form $v \cos \theta$. Because particle 1 initially moves along the $x$-axis, we find $v_{1x} = v_1$.

Conservation of momentum along the $x$-axis gives the following equation:

$$m_1 v_1 = m_1 v_1' \cos \theta_1 + m_2 v_2' \cos \theta_2$$

(8.61)

where $\theta_1$ and $\theta_2$ are as shown in Figure 8.11.

**Conservation of Momentum along the $x$-axis**

$$m_1 v_1 = m_1 v_1' \cos \theta_1 + m_2 v_2' \cos \theta_2$$

(8.62)

Along the $y$-axis, the equation for conservation of momentum is

$$p_{1y} + p_{2y} = p_{1y}' + p_{2y}'$$

(8.63)

or
But \( v_{1y} \) is zero, because particle 1 initially moves along the \( x \)-axis. Because particle 2 is initially at rest, \( v_{2y} \) is also zero. The equation for conservation of momentum along the \( y \)-axis becomes
\[
0 = m_1v_{1y} + m_2v_{2y}.
\]

The components of the velocities along the \( y \)-axis have the form \( v \sin \theta \).

Thus, conservation of momentum along the \( y \)-axis gives the following equation:
\[
0 = m_1v_1' \sin \theta_1 + m_2v_2' \sin \theta_2.
\]  

Conservation of Momentum along the \( y \)-axis
\[
0 = m_1v_1' \sin \theta_1 + m_2v_2' \sin \theta_2
\]  

The equations of conservation of momentum along the \( x \)-axis and \( y \)-axis are very useful in analyzing two-dimensional collisions of particles, where one is originally stationary (a common laboratory situation). But two equations can only be used to find two unknowns, and so other data may be necessary when collision experiments are used to explore nature at the subatomic level.

**Example 8.7 Determining the Final Velocity of an Unseen Object from the Scattering of Another Object**

Suppose the following experiment is performed. A 0.250-kg object \( (m_1) \) is slid on a frictionless surface into a dark room, where it strikes an initially stationary object with mass of 0.400 kg \( (m_2) \). The 0.250-kg object emerges from the room at an angle of 45.0º with its incoming direction.

The speed of the 0.250-kg object is originally 2.00 m/s and is 1.50 m/s after the collision. Calculate the magnitude and direction of the velocity \( (v_2' \) and \( \theta_2 \)) of the 0.400-kg object after the collision.

**Strategy**

Momentum is conserved because the surface is frictionless. The coordinate system shown in Figure 8.12 is one in which \( m_2 \) is originally at rest and the initial velocity is parallel to the \( x \)-axis, so that conservation of momentum along the \( x \)- and \( y \)-axes is applicable.

Everything is known in these equations except \( v_2' \) and \( \theta_2 \), which are precisely the quantities we wish to find. We can find two unknowns because we have two independent equations: the equations describing the conservation of momentum in the \( x \)- and \( y \)-directions.

**Solution**

Solving \( m_1v_1 = m_1v_1' \cos \theta_1 + m_2v_2' \cos \theta_2 \) for \( v_2' \cos \theta_2 \) and \( 0 = m_1v_1' \sin \theta_1 + m_2v_2' \sin \theta_2 \) for \( v_2' \sin \theta_2 \)

and taking the ratio yields an equation (in which \( \theta_2 \) is the only unknown quantity. Applying the identity \( \tan \theta = \frac{\sin \theta}{\cos \theta} \), we obtain:
\[
\tan \theta_2 = \frac{v_1' \sin \theta_1}{v_1' \cos \theta_1 - v_1}.
\]

Entering known values into the previous equation gives
\[
\tan \theta_2 = \frac{(1.50 \text{ m/s})(0.7071)}{(1.50 \text{ m/s})(0.7071) - 2.00 \text{ m/s}} = -1.129.
\]

Thus,
\[
\theta_2 = \tan^{-1}(-1.129) = 311.5º \approx 312º.
\]

Angles are defined as positive in the counter clockwise direction, so this angle indicates that \( m_2 \) is scattered to the right in Figure 8.12, as expected (this angle is in the fourth quadrant). Either equation for the \( x \)- or \( y \)-axis can now be used to
solve for \( v'_2 \), but the latter equation is easiest because it has fewer terms.

\[
v'_2 = -\frac{m_1}{m_2}v'_1 \sin \theta_1 \sin \theta_2
\]  

(8.71)

Entering known values into this equation gives

\[
v'_2 = -\left(\frac{0.250 \text{ kg}}{0.400 \text{ kg}}\right)(1.50 \text{ m/s})\left(\frac{0.7071}{-0.7485}\right).
\]

Thus,

\[
v'_2 = 0.886 \text{ m/s}.
\]  

(8.72)

Thus,

\[
v'_2 = 0.886 \text{ m/s}.
\]  

(8.73)

Discussion

It is instructive to calculate the internal kinetic energy of this two-object system before and after the collision. (This calculation is left as an end-of-chapter problem.) If you do this calculation, you will find that the internal kinetic energy is less after the collision, and so the collision is inelastic. This type of result makes a physicist want to explore the system further.

\[
\text{net } F = 0
\]

Before

\[
\mathbf{p}_1 = \mathbf{p}_{\text{tot}}
\]

After

\[
\mathbf{p}'_1 + \mathbf{p}_2 = \mathbf{p}_{\text{tot}}
\]

\[
m_2 = 0.400 \text{ kg}
\]

\[
v_2 = 0
\]

\[
\theta_1 = 45^\circ
\]

\[
\theta_2 = ?
\]

\[
v'_1 = ?
\]

\[
v'_2 = ?
\]

Figure 8.12 A collision taking place in a dark room is explored in Example 8.7. The incoming object \( m_1 \) is scattered by an initially stationary object. Only the stationary object’s mass \( m_2 \) is known. By measuring the angle and speed at which \( m_1 \) emerges from the room, it is possible to calculate the magnitude and direction of the initially stationary object’s velocity after the collision.

Elastic Collisions of Two Objects with Equal Mass

Some interesting situations arise when the two colliding objects have equal mass and the collision is elastic. This situation is nearly the case with colliding billiard balls, and precisely the case with some subatomic particle collisions. We can thus get a mental image of a collision of subatomic particles by thinking about billiards (or pool). (Refer to Figure 8.11 for masses and angles.) First, an elastic collision conserves internal kinetic energy. Again, let us assume object 2 \( (m_2) \) is initially at rest. Then, the internal kinetic energy before and after the collision of two objects that have equal masses is

\[
\frac{1}{2}mv_1^2 = \frac{1}{2}mv'_1^2 + \frac{1}{2}mv'_2^2.
\]

(8.74)

Because the masses are equal, \( m_1 = m_2 = m \). Algebraic manipulation (left to the reader) of conservation of momentum in the \( x \)- and \( y \)-directions can show that

\[
\frac{1}{2}mv_1^2 = \frac{1}{2}mv'_1^2 + \frac{1}{2}mv'_2^2 + mv'_1v'_2 \cos(\theta_1 - \theta_2).
\]

(8.75)

(Remember that \( \theta_2 \) is negative here.) The two preceding equations can both be true only if

\[
mv'_1v'_2 \cos(\theta_1 - \theta_2) = 0.
\]

(8.76)
There are three ways that this term can be zero. They are

- \( v_1' = 0 \) : head-on collision; incoming ball stops
- \( v_2' = 0 \) : no collision; incoming ball continues unaffected
- \( \cos(\theta_1 - \theta_2) = 0 \) : angle of separation \( (\theta_1 - \theta_2) \) is 90° after the collision

All three of these ways are familiar occurrences in billiards and pool, although most of us try to avoid the second. If you play enough pool, you will notice that the angle between the balls is very close to 90° after the collision, although it will vary from this value if a great deal of spin is placed on the ball. (Large spin carries in extra energy and a quantity called angular momentum, which must also be conserved.) The assumption that the scattering of billiard balls is elastic is reasonable based on the correctness of the three results it produces. This assumption also implies that, to a good approximation, momentum is conserved for the two-ball system in billiards and pool. The problems below explore these and other characteristics of two-dimensional collisions.

### Connections to Nuclear and Particle Physics

Two-dimensional collision experiments have revealed much of what we know about subatomic particles, as we shall see in [Medical Applications of Nuclear Physics](https://legacy.cnx.org/content/m42646/latest/) and [Particle Physics](https://legacy.cnx.org/content/m42667/latest/). Ernest Rutherford, for example, discovered the nature of the atomic nucleus from such experiments.

### 8.7 Introduction to Rocket Propulsion

Rockets range in size from fireworks so small that ordinary people use them to immense Saturn Vs that once propelled massive payloads toward the Moon. The propulsion of all rockets, jet engines, deflating balloons, and even squids and octopuses is explained by the same physical principle—Newton’s third law of motion. Matter is forcefully ejected from a system, producing an equal and opposite reaction on what remains. Another common example is the recoil of a gun. The gun exerts a force on a bullet to accelerate it and consequently experiences an equal and opposite force, causing the gun’s recoil or kick.

#### Making Connections: Take-Home Experiment—Propulsion of a Balloon

Hold a balloon and fill it with air. Then, let the balloon go. In which direction does the air come out of the balloon and in which direction does the balloon get propelled? If you fill the balloon with water and then let the balloon go, does the balloon’s direction change? Explain your answer.

Figure 8.13 shows a rocket accelerating straight up. In part (a), the rocket has a mass \( m \) and a velocity \( v \) relative to Earth, and hence a momentum \( mv \). In part (b), a time \( \Delta t \) has elapsed in which the rocket has ejected a mass \( \Delta m \) of hot gas at a velocity \( v_e \) relative to the rocket. The remainder of the mass \( (m - \Delta m) \) now has a greater velocity \( (v + \Delta v) \). The momentum of the entire system (rocket plus expelled gas) has actually decreased because the force of gravity has acted for a time \( \Delta t \), producing a negative impulse \( \Delta p = -mg\Delta t \). (Remember that impulse is the net external force on a system multiplied by the time it acts, and it equals the change in momentum of the system.) So, the center of mass of the system is in free fall but, by rapidly expelling mass, part of the system can accelerate upward. It is a commonly held misconception that the rocket exhaust pushes on the ground. If we consider thrust; that is, the force exerted on the rocket by the exhaust gases, then a rocket’s thrust is greater in outer space than in the atmosphere or on the launch pad. In fact, gases are easier to expel into a vacuum.

By calculating the change in momentum for the entire system over \( \Delta t \), and equating this change to the impulse, the following expression can be shown to be a good approximation for the acceleration of the rocket.

\[
a = \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g \tag{8.77}
\]

“The rocket” is that part of the system remaining after the gas is ejected, and \( g \) is the acceleration due to gravity.

#### Acceleration of a Rocket

Acceleration of a rocket is

\[
a = \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g \tag{8.78}
\]

where \( a \) is the acceleration of the rocket, \( v_e \) is the exhaust velocity, \( m \) is the mass of the rocket, \( \Delta m \) is the mass of the ejected gas, and \( \Delta t \) is the time in which the gas is ejected.
A rocket’s acceleration depends on three major factors, consistent with the equation for acceleration of a rocket. First, the greater the exhaust velocity of the gases relative to the rocket, \( v_e \), the greater the acceleration is. The practical limit for \( v_e \) is about \( 2.5 \times 10^3 \text{ m/s} \) for conventional (non-nuclear) hot-gas propulsion systems. The second factor is the rate at which mass is ejected from the rocket. This is the factor \( \Delta m / \Delta t \) in the equation. The quantity \( (\Delta m / \Delta t)v_e \), with units of newtons, is called "thrust." The faster the rocket burns its fuel, the greater its thrust, and the greater its acceleration. The third factor is the mass \( m \) of the rocket. The smaller the mass is (all other factors being the same), the greater the acceleration. The rocket mass \( m \) decreases dramatically during flight because most of the rocket is fuel to begin with, so that acceleration increases continuously, reaching a maximum just before the fuel is exhausted.

**Factors Affecting a Rocket’s Acceleration**
- The greater the exhaust velocity \( v_e \) of the gases relative to the rocket, the greater the acceleration.
- The faster the rocket burns its fuel, the greater its acceleration.
- The smaller the rocket’s mass (all other factors being the same), the greater the acceleration.

**Example 8.8 Calculating Acceleration: Initial Acceleration of a Moon Launch**

A Saturn V’s mass at liftoff was \( 2.80 \times 10^6 \text{ kg} \), its fuel-burn rate was \( 1.40 \times 10^4 \text{ kg/s} \), and the exhaust velocity was \( 2.40 \times 10^3 \text{ m/s} \). Calculate its initial acceleration.

**Strategy**
This problem is a straightforward application of the expression for acceleration because \( a \) is the unknown and all of the terms on the right side of the equation are given.

**Solution**
Substituting the given values into the equation for acceleration yields
\[ a = \frac{v_e \Delta m}{m} \Delta t - g \]

\[ = \frac{2.40 \times 10^3 \text{ m/s}^2}{2.80 \times 10^6 \text{ kg}} \times (1.40 \times 10^4 \text{ kg/s}) - 9.80 \text{ m/s}^2 \]

\[ = 2.20 \text{ m/s}^2. \]

**Discussion**

This value is fairly small, even for an initial acceleration. The acceleration does increase steadily as the rocket burns fuel, because \( m \) decreases while \( v_e \) and \( \frac{\Delta m}{\Delta t} \) remain constant. Knowing this acceleration and the mass of the rocket, you can show that the thrust of the engines was \( 3.36 \times 10^7 \text{ N} \).

To achieve the high speeds needed to hop continents, obtain orbit, or escape Earth's gravity altogether, the mass of the rocket other than fuel must be as small as possible. It can be shown that, in the absence of air resistance and neglecting gravity, the final velocity of a one-stage rocket initially at rest is

\[ v = v_e \ln \frac{m_0}{m_r}, \]

where \( \ln (m_0 / m_r) \) is the natural logarithm of the ratio of the initial mass of the rocket \( m_0 \) to what is left \( m_r \) after all of the fuel is exhausted. (Note that \( v \) is actually the change in velocity, so the equation can be used for any segment of the flight. If we start from rest, the change in velocity equals the final velocity.) For example, let us calculate the mass ratio needed to escape Earth's gravity starting from rest, given that the escape velocity from Earth is about \( 11.2 \times 10^3 \text{ m/s} \), and assuming an exhaust velocity \( v_e = 2.5 \times 10^3 \text{ m/s} \).

\[ \ln \frac{m_0}{m_r} = \frac{v}{v_e} = \frac{11.2 \times 10^3 \text{ m/s}}{2.5 \times 10^3 \text{ m/s}} = 4.48 \]

Solving for \( m_0 / m_r \) gives

\[ m_0 \frac{m_0}{m_r} = e^{4.48} = 88. \]

Thus, the mass of the rocket is

\[ m_r = \frac{m_0}{88}. \]

This result means that only \( 1 / 88 \) of the mass is left when the fuel is burnt, and \( 87 / 88 \) of the initial mass was fuel. Expressed as percentages, 98.9% of the rocket is fuel, while payload, engines, fuel tanks, and other components make up only 1.10%.

Taking air resistance and gravitational force into account, the mass \( m_r \) remaining can only be about \( m_0 / 180 \). It is difficult to build a rocket in which the fuel has a mass 180 times everything else. The solution is multistage rockets. Each stage only needs to achieve part of the final velocity and is discarded after it burns its fuel. The result is that each successive stage can have smaller engines and more payload relative to its fuel. Once out of the atmosphere, the ratio of payload to fuel becomes more favorable, too.

The space shuttle was an attempt at an economical vehicle with some reusable parts, such as the solid fuel boosters and the craft itself. (See Figure 8.14) The shuttle's need to be operated by humans, however, made it at least as costly for launching satellites as expendable, unpiloted rockets. Ideally, the shuttle would only have been used when human activities were required for the success of a mission, such as the repair of the Hubble space telescope. Rockets with satellites can also be launched from airplanes. Using airplanes has the double advantage that the initial velocity is significantly above zero and a rocket can avoid most of the atmosphere's resistance.
change in momentum: the difference between the final and initial momentum; the mass times the change in velocity

conservation of momentum principle: when the net external force is zero, the total momentum of the system is conserved or constant

elastic collision: a collision that also conserves internal kinetic energy

impulse: the average net external force times the time it acts; equal to the change in momentum

inelastic collision: a collision in which internal kinetic energy is not conserved

internal kinetic energy: the sum of the kinetic energies of the objects in a system

isolated system: a system in which the net external force is zero

linear momentum: the product of mass and velocity

perfectly inelastic collision: a collision in which the colliding objects stick together

point masses: structureless particles with no rotation or spin

quark: fundamental constituent of matter and an elementary particle

second law of motion: physical law that states that the net external force equals the change in momentum of a system divided by the time over which it changes
Section Summary

8.1 Linear Momentum and Force

- Linear momentum (momentum for brevity) is defined as the product of a system's mass multiplied by its velocity.
- In symbols, linear momentum \( p \) is defined to be \( p = mv \), where \( m \) is the mass of the system and \( v \) is its velocity.
- The SI unit for momentum is \( \text{kg} \cdot \text{m/s} \).
- Newton's second law of motion in terms of momentum states that the net external force equals the change in momentum of a system divided by the time over which it changes.
  - In symbols, Newton's second law of motion is defined to be \( F_{\text{net}} = \frac{\Delta p}{\Delta t} \), where \( F_{\text{net}} \) is the net external force, \( \Delta p \) is the change in momentum, and \( \Delta t \) is the change time.

8.2 Impulse

- Impulse, or change in momentum, equals the average net external force multiplied by the time this force acts: \( \Delta p = F_{\text{net}}\Delta t \).
- Forces are usually not constant over a period of time.

8.3 Conservation of Momentum

- The conservation of momentum principle is written \( p_{\text{tot}} = \text{constant} \) or \( p_{\text{tot}} = p'_{\text{tot}} \) (isolated system), where \( p_{\text{tot}} \) is the initial total momentum and \( p'_{\text{tot}} \) is the total momentum some time later.
- An isolated system is defined to be one for which the net external force is zero \( (F_{\text{net}} = 0) \).
- During projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero.
- Conservation of momentum applies only when the net external force is zero.
- The conservation of momentum principle is valid when considering systems of particles.

8.4 Elastic Collisions in One Dimension

- An elastic collision is one that conserves internal kinetic energy.
- Conservation of kinetic energy and momentum together allow the final velocities to be calculated in terms of initial velocities and masses in one dimensional two-body collisions.

8.5 Inelastic Collisions in One Dimension

- An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).
- A collision in which the objects stick together is sometimes called perfectly inelastic because it reduces internal kinetic energy more than does any other type of inelastic collision.
- Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

8.6 Collisions of Point Masses in Two Dimensions

- The approach to two-dimensional collisions is to choose a convenient coordinate system and break the motion into components along perpendicular axes. Choose a coordinate system with the \( x \)-axis parallel to the velocity of the incoming particle.
- Two-dimensional collisions of point masses where mass 2 is initially at rest conserve momentum along the initial direction of mass 1 (the \( x \)-axis), stated by \( m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2 \) and along the direction perpendicular to the initial direction (the \( y \)-axis) stated by \( 0 = m_1 v'_{1y} + m_2 v'_{2y} \).
- The internal kinetic before and after the collision of two objects that have equal masses is \( \frac{1}{2} m v^2 = \frac{1}{2} m v'^2_1 + \frac{1}{2} m v'^2_2 + m v'_1 v'_2 \cos(\theta_1 - \theta_2) \).
- Point masses are structureless particles that cannot spin.
8.7 Introduction to Rocket Propulsion

• Newton's third law of motion states that to every action, there is an equal and opposite reaction.
• Acceleration of a rocket is \( a = \frac{v_e \Delta m}{m} \Delta t - g \).
• A rocket's acceleration depends on three main factors. They are
  1. The greater the exhaust velocity of the gases, the greater the acceleration.
  2. The faster the rocket burns its fuel, the greater its acceleration.
  3. The smaller the rocket's mass, the greater the acceleration.

Conceptual Questions

8.1 Linear Momentum and Force

1. An object that has a small mass and an object that has a large mass have the same momentum. Which object has the largest kinetic energy?
2. An object that has a small mass and an object that has a large mass have the same kinetic energy. Which mass has the largest momentum?
3. Professional Application
   Football coaches advise players to block, hit, and tackle with their feet on the ground rather than by leaping through the air. Using the concepts of momentum, work, and energy, explain how a football player can be more effective with his feet on the ground.
4. How can a small force impart the same momentum to an object as a large force?

8.2 Impulse

5. Professional Application
   Explain in terms of impulse how padding reduces forces in a collision. State this in terms of a real example, such as the advantages of a carpeted vs. tile floor for a day care center.
6. While jumping on a trampoline, sometimes you land on your back and other times on your feet. In which case can you reach a greater height and why?
7. Professional Application
   Tennis racquets have “sweet spots.” If the ball hits a sweet spot then the player's arm is not jarred as much as it would be otherwise. Explain why this is the case.

8.3 Conservation of Momentum

8. Professional Application
   If you dive into water, you reach greater depths than if you do a belly flop. Explain this difference in depth using the concept of conservation of energy. Explain this difference in depth using what you have learned in this chapter.
9. Under what circumstances is momentum conserved?
10. Can momentum be conserved for a system if there are external forces acting on the system? If so, under what conditions? If not, why not?
11. Momentum for a system can be conserved in one direction while not being conserved in another. What is the angle between the directions? Give an example.
12. Professional Application
   Explain in terms of momentum and Newton's laws how a car's air resistance is due in part to the fact that it pushes air in its direction of motion.
13. Can objects in a system have momentum while the momentum of the system is zero? Explain your answer.
14. Must the total energy of a system be conserved whenever its momentum is conserved? Explain why or why not.

8.4 Elastic Collisions in One Dimension

15. What is an elastic collision?

8.5 Inelastic Collisions in One Dimension

16. What is an inelastic collision? What is a perfectly inelastic collision?
17. Mixed-pair ice skaters performing in a show are standing motionless at arms length just before starting a routine. They reach out, clasp hands, and pull themselves together by only using their arms. Assuming there is no friction between the blades of their skates and the ice, what is their velocity after their bodies meet?
18. A small pickup truck that has a camper shell slowly coasts toward a red light with negligible friction. Two dogs in the back of the truck are moving and making various inelastic collisions with each other and the walls. What is the effect of the dogs on the motion of the center of mass of the system (truck plus entire load)? What is their effect on the motion of the truck?

8.6 Collisions of Point Masses in Two Dimensions

19. Figure 8.16 shows a cube at rest and a small object heading toward it. (a) Describe the directions (angle $\theta_1$) at which the small object can emerge after colliding elastically with the cube. How does $\theta_1$ depend on $b$, the so-called impact parameter? Ignore any effects that might be due to rotation after the collision, and assume that the cube is much more massive than the small object. (b) Answer the same questions if the small object instead collides with a massive sphere.

![Figure 8.16](image)

Figure 8.16 A small object approaches a collision with a much more massive cube, after which its velocity has the direction $\theta_1$. The angles at which the small object can be scattered are determined by the shape of the object it strikes and the impact parameter $b$.

8.7 Introduction to Rocket Propulsion

20. Professional Application

Suppose a fireworks shell explodes, breaking into three large pieces for which air resistance is negligible. How is the motion of the center of mass affected by the explosion? How would it be affected if the pieces experienced significantly more air resistance than the intact shell?

21. Professional Application

During a visit to the International Space Station, an astronaut was positioned motionless in the center of the station, out of reach of any solid object on which he could exert a force. Suggest a method by which he could move himself away from this position, and explain the physics involved.

22. Professional Application

It is possible for the velocity of a rocket to be greater than the exhaust velocity of the gases it ejects. When that is the case, the gas velocity and gas momentum are in the same direction as that of the rocket. How is the rocket still able to obtain thrust by ejecting the gases?
8.1 Linear Momentum and Force

1. (a) Calculate the momentum of a 2000-kg elephant charging a hunter at a speed of 7.50 m/s. (b) Compare the elephant's momentum with the momentum of a 0.0400-kg tranquilizer dart fired at a speed of 600 m/s. (c) What is the momentum of the 90.0-kg hunter running at 7.40 m/s after missing the elephant?

2. (a) What is the mass of a large ship that has a momentum of 1.60×10^9 kg·m/s, when the ship is moving at a speed of 48.0 km/h? (b) Compare the ship's momentum to the momentum of a 1100-kg artillery shell fired at a speed of 1200 m/s.

3. (a) At what speed would a 2.00×10^4-kg airplane have to fly to have a momentum of 1.60×10^9 kg·m/s (the same as the ship's momentum in the problem above)? (b) What is the plane's momentum when it is taking off at a speed of 60.0 m/s? (c) If the ship is an aircraft carrier that launches these airplanes with a catapult, discuss the implications of your answer to (b) as it relates to recoil effects of the catapult on the ship.

4. (a) What is the momentum of a garbage truck that is 1.20×10^4 kg and is moving at 10.0 m/s? (b) At what speed would an 8.00-kg trash can have the same momentum as the truck?

5. A runaway train car that has a mass of 15,000 kg travels at a speed of 5.4 m/s down a track. Compute the time required for a force of 1500 N to bring the car to rest.

6. The mass of Earth is 5.972×10^{24} kg and its orbital radius is an average of 1.496×10^{11} m. Calculate its linear momentum.

8.2 Impulse

7. A bullet is accelerated down the barrel of a gun by hot gases produced in the combustion of gun powder. What is the average force exerted on a 0.0300-kg bullet to accelerate it to a speed of 600.0 m/s in a time of 2.00 ms (milliseconds)?

8. Professional Application
A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seat belt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.

9. A person slaps her leg with her hand, bringing her hand to rest in 2.50 milliseconds from an initial speed of 4.00 m/s. (a) What is the average force exerted on the leg, taking the effective mass of the hand and forearm to be 1.50 kg? (b) Would the force be any different if the woman clapped her hands together at the same speed and brought them to rest in the same time? Explain why or why not.

10. Professional Application
A professional boxer hits his opponent with a 1000-N horizontal blow that lasts for 0.150 s. (a) Calculate the impulse imparted by this blow. (b) What is the opponent's final velocity, if his mass is 105 kg and he is motionless in midair when struck near his center of mass? (c) Calculate the recoil velocity of the opponent's 10.0-kg head if hit in this manner, assuming the head does not initially transfer significant momentum to the boxer's body. (d) Discuss the implications of your answers for parts (b) and (c).

11. Professional Application
Suppose a child drives a bumper car head on into the side rail, which exerts a force of 4000 N on the car for 0.200 s. (a) What impulse is imparted by this force? (b) Find the final velocity of the bumper car if its initial velocity was 2.80 m/s and the car plus driver have a mass of 200 kg. You may neglect friction between the car and floor.

12. Professional Application
One hazard of space travel is debris left by previous missions. There are several thousand objects orbiting Earth that are large enough to be detected by radar, but there are far greater numbers of very small objects, such as flakes of paint. Calculate the force exerted by a 0.100-mg chip of paint that strikes a spacecraft window at a relative speed of 4.00×10^{3} m/s, given the collision lasts 6.00×10^{-8} s.

13. Professional Application
A 75.0-kg person is riding in a car moving at 20.0 m/s when the car runs into a bridge abutment. (a) Calculate the average force on the person if he is stopped by a padded dashboard that compresses an average of 1.00 cm. (b) Calculate the average force on the person if he is stopped by an air bag that compresses an average of 15.0 cm.

14. Professional Application
Military rifles have a mechanism for reducing the recoil forces of the gun on the person firing it. An internal part recoils over a relatively large distance and is stopped by damping mechanisms in the gun. The larger distance reduces the average force needed to stop the internal part. (a) Calculate the recoil velocity of a 1.00-kg plunger that directly interacts with a 0.0200-kg bullet fired at 600 m/s from the gun. (b) If this part is stopped over a distance of 20.0 cm, what average force is exerted upon it by the gun? (c) Compare this to the force exerted on the gun if the bullet is accelerated to its velocity in 10.0 ms (milliseconds).

15. A cruise ship with a mass of 1.00×10^7 kg strikes a pier at a speed of 0.750 m/s. It comes to rest 6.00 m later, damaging the ship, the pier, and the tugboat captain's finances. Calculate the average force exerted on the pier using the concept of impulse. (Hint: First calculate the time it took to bring the ship to rest.)

16. Calculate the final speed of a 110-kg rugby player who is initially running at 8.00 m/s but collides head-on with a padded goalpost and experiences a backward force of 1.76×10^4 N for 5.50×10^{-2} s.

17. Water from a fire hose is directed horizontally against a wall at a rate of 50.0 kg/s and a speed of 42.0 m/s. Calculate the magnitude of the force exerted on the wall, assuming the water's horizontal momentum is reduced to zero.
18. A 0.450-kg hammer is moving horizontally at 7.00 m/s when it strikes a nail and comes to rest after driving the nail 1.00 cm into a board. (a) Calculate the duration of the impact. (b) What was the average force exerted on the nail?

19. Starting with the definitions of momentum and kinetic energy, derive an equation for the kinetic energy of a particle expressed as a function of its momentum.

20. A ball with an initial velocity of 10 m/s moves at an angle $60^\circ$ above the $+x$-direction. The ball hits a vertical wall and bounces off so that it is moving $60^\circ$ above the $-x$-direction with the same speed. In terms of $m$, the mass of the ball, what is the impulse delivered by the wall?

21. When serving a tennis ball, a player hits the ball when its velocity is zero (at the highest point of a vertical toss). The racquet exerts a force of 540 N on the ball for 5.00 ms, giving it a final velocity of 45.0 m/s. Using these data, find the mass of the ball.

22. A punter drops a ball from rest vertically 1 meter down onto his foot. The ball leaves the foot with a speed of 18 m/s at an angle $55^\circ$ above the horizontal. In terms of $m$, the mass of the ball, what is the impulse delivered by the foot (magnitude and direction)?

### 8.3 Conservation of Momentum

23. **Professional Application**

Train cars are coupled together by being bumped into one another. Suppose two loaded train cars are moving toward one another, the first having a mass of 150,000 kg and a velocity of 0.300 m/s, and the second having a mass of 110,000 kg and a velocity of $-0.120 \text{ m/s}$. (The minus indicates direction of motion.) What is their final velocity?

24. Suppose a clay model of a koala bear has a mass of 0.200 kg and slides on ice at a speed of 0.750 m/s. It runs into another clay model, which is initially motionless and has a mass of 0.350 kg. Both being soft clay, they naturally stick together. What is their final velocity?

25. **Professional Application**

Consider the following question: A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seatbelt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg. Would the answer to this question be different if the car with the 70-kg passenger had collided with a car that has a mass of over 400 kg and is originally at rest? Explain why the change in velocity is different in the two frames, whereas the change in kinetic energy is the same in both.

### 8.4 Elastic Collisions in One Dimension

28. Two identical objects (such as billiard balls) have a one-dimensional collision in which one is initially motionless. After the collision, the moving object is stationary and the other moves with the same speed as the other originally had. Show that both momentum and kinetic energy are conserved.

29. **Professional Application**

Two piloted satellites approach one another at a relative speed of 0.250 m/s, intending to dock. The first has a mass of $4.00 \times 10^3 \text{ kg}$, and the second a mass of $7.50 \times 10^3 \text{ kg}$. If the two satellites collide elastically rather than dock, what is their final relative velocity?

30. A 70.0-kg ice hockey goalie, originally at rest, catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. Suppose the goalie and the ice puck have an elastic collision and the puck is reflected back in the direction from which it came. What would their final velocities be in this case?

### 8.5 Inelastic Collisions in One Dimension

31. A 0.240-kg billiard ball that is moving at 3.00 m/s strikes the bumper of a pool table and bounces straight back at 2.40 m/s (80% of its original speed). The collision lasts 0.0150 s. (a) Calculate the average force exerted on the ball by the bumper. (b) How much kinetic energy in joules is lost during the collision? (c) What percent of the original energy is left?

32. During an ice show, a 60.0-kg skater leaps into the air and is caught by an initially stationary 75.0-kg skater. (a) What is their final velocity assuming negligible friction and that the 60.0-kg skater’s original horizontal velocity is 4.00 m/s? (b) How much kinetic energy is lost?

33. **Professional Application**

Using mass and speed data from Example 8.1 and assuming that the football player catches the ball with his feet off the ground with both of them moving horizontally, calculate: (a) the final velocity if the ball and player are going in the same direction and (b) the loss of kinetic energy in this case. (c) Repeat parts (a) and (b) for the situation in which the ball and the player are going in opposite directions. Might the loss of kinetic energy be related to how much it hurts to catch the pass?

34. A battleship that is $6.00 \times 10^7 \text{ kg}$ and is originally at rest fires a 1100-kg artillery shell horizontally with a velocity of 575 m/s. (a) If the shell is fired straight aft (toward the rear of the ship), there will be negligible friction opposing the ship’s recoil. Calculate its recoil velocity. (b) Calculate the increase in internal kinetic energy (that is, for the ship and the shell). This energy is less than the energy released by the gun powder—significant heat transfer occurs.

35. **Professional Application**

Two piloted satellites approaching one another, at a relative speed of 0.250 m/s, intending to dock. The first has a mass of $4.00 \times 10^3 \text{ kg}$, and the second a mass of $7.50 \times 10^3 \text{ kg}$. (a) Calculate the final velocity (after docking) by using the frame of reference in which the first satellite was originally at rest. (b) What is the loss of kinetic energy in this inelastic collision? (c) Repeat both parts by using the frame of reference in which the second satellite was originally at rest. Explain why the change in velocity is different in the two frames, whereas the change in kinetic energy is the same in both.

36. **Professional Application**

A 30,000-kg freight car is coasting at 0.850 m/s with negligible friction under a hopper that dumps 110,000 kg of scrap metal into it. (a) What is the final velocity of the loaded freight car? (b) How much kinetic energy is lost?
37. Professional Application

Space probes may be separated from their launchers by exploding bolts. (They bolt away from one another.) Suppose a 4800-kg satellite uses this method to separate from the 1500-kg remains of its launcher, and that 5000 J of kinetic energy is supplied to the two parts. What are their subsequent velocities using the frame of reference in which they were at rest before separation?

38. A 0.0250-kg bullet is accelerated from rest to a speed of 550 m/s in a 3.00-kg rifle. The pain of the rifle’s kick is much worse if you hold the gun loosely a few centimeters from your shoulder rather than holding it tightly against your shoulder. (a) Calculate the recoil velocity of the rifle if it is held loosely away from the shoulder. (b) How much kinetic energy does the rifle gain? (c) What is the recoil velocity if the rifle is held tightly against the shoulder, making the effective mass 28.0 kg? (d) How much kinetic energy is transferred to the rifle-shoulder combination? The pain is related to the amount of kinetic energy, which is significantly less in this latter situation. (e) Calculate the momentum of a 110-kg football player running at 8.00 m/s. Compare the player’s momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s. Discuss its relationship to this problem.

39. Professional Application

One of the waste products of a nuclear reactor is plutonium-239 \( ^{239}\text{Pu} \). This nucleus is radioactive and decays by splitting into a helium-4 nucleus and a uranium-235 nucleus \( ^{4}\text{He} + ^{235}\text{U} \), the latter of which is also radioactive and will itself decay some time later. The energy emitted in the plutonium decay is \( 8.40 \times 10^{-13} \text{ J} \) and is entirely converted to kinetic energy of the helium and uranium nuclei. The mass of the helium nucleus is \( 6.68 \times 10^{-27} \text{ kg} \), while that of the uranium is \( 3.92 \times 10^{-25} \text{ kg} \) (note that the ratio of the masses is 4 to 235). (a) Calculate the velocities of the two nuclei, assuming the plutonium nucleus is originally at rest. (b) How much kinetic energy does each nucleus carry away? Note that the data given here are accurate to three digits only.

40. Professional Application

The Moon’s craters are remnants of meteorite collisions. Suppose a fairly large asteroid that has a mass of 5.00 \( \times 10^{12} \text{ kg} \) (about a kilometer across) strikes the Moon at a speed of 15.0 km/s. (a) At what speed does the Moon recoil after the perfectly inelastic collision (the mass of the Moon is 7.36 \( \times 10^{22} \text{ kg} \))? (b) How much kinetic energy is lost in the collision? Such an event may have been observed by medieval English monks who reported observing a red glow and subsequent haze about the Moon. (c) In October 2009, NASA crashed a rocket into the Moon, and analyzed the plume produced by the impact. (Significant amounts of water were detected.) Answer part (a) and (b) for this real-life experiment. The mass of the rocket was 2000 kg and its speed upon impact was 9000 km/h. How does the plume produced alter these results?

41. Professional Application

Two football players collide head-on in midair while trying to catch a thrown football. The first player is 95.0 kg and has an initial velocity of 6.00 m/s, while the second player is 115 kg and has an initial velocity of -3.50 m/s. What is their velocity just after impact if they cling together?

42. What is the speed of a garbage truck that is 1.20\( \times 10^{4} \text{ kg} \) and is initially moving at 25.0 m/s just after it hits and adheres to a trash can that is 80.0 kg and is initially at rest?

43. During a circus act, an elderly performer thrills the crowd by catching a cannon ball shot at him. The cannon ball has a mass of 10.0 kg and the horizontal component of its velocity is 8.00 m/s when the 65.0-kg performer catches it. If the performer is on nearly frictionless roller skates, what is his recoil velocity?

44. (a) During an ice skating performance, an initially motionless 80.0-kg clown throws a fake barbell away. The clown’s ice skates allow her to recoil frictionlessly. If the clown recoils with a velocity of 0.500 m/s and the barbell is thrown with a velocity of 10.0 m/s, what is the mass of the barbell? (b) How much kinetic energy is gained by this maneuver? (c) Where does the kinetic energy come from?

8.6 Collisions of Point Masses in Two Dimensions

45. Two identical pucks collide on an air hockey table. One puck was originally at rest. (a) If the incoming puck has a speed of 6.00 m/s and scatters to an angle of 30.0º, what is the velocity (magnitude and direction) of the second puck? (You may use the result that \( \theta_1 - \theta_2 = 90º \) for elastic collisions of objects that have identical masses.) (b) Confirm that the collision is elastic.

46. Confirm that the results of the example Example 8.7 do conserve momentum in both the \( x \)- and \( y \)-directions.

47. A 3000-kg cannon is mounted so that it can recoil only in the horizontal direction. (a) Calculate its recoil velocity when it fires a 15.0-kg shell at 480 m/s at an angle of 20.0º above the horizontal. (b) What is the kinetic energy of the cannon? This energy is dissipated as heat transfer in shock absorbers that stop its recoil. (c) What happens to the vertical component of momentum that is imparted to the cannon when it is fired?

48. Professional Application

A 5.50-kg bowling ball moving at 9.00 m/s collides with a 0.850-kg bowling pin, which is scattered at an angle of 85.0º to the initial direction of the bowling ball and with a speed of 15.0 m/s. (a) Calculate the final velocity (magnitude and direction) of the bowling ball. (b) Is the collision elastic? (c) Linear kinetic energy is greater after the collision. Discuss how spin on the ball might be converted to linear kinetic energy in the collision.
49. Professional Application

Ernest Rutherford (the first New Zealander to be awarded the Nobel Prize in Chemistry) demonstrated that nuclei were very small and dense by scattering helium-4 nuclei \(^{4}\text{He}\) from gold-197 nuclei \(^{197}\text{Au}\). The energy of the incoming helium nucleus was \(8.00 \times 10^{-13}\) J, and the masses of the helium and gold nuclei were \(6.68 \times 10^{-27}\) kg and \(3.29 \times 10^{-25}\) kg, respectively (note that their mass ratio is 4 to 197). (a) If a helium nucleus scatters to an angle of 120° during an elastic collision with a gold nucleus, calculate the helium nucleus's final speed and the final velocity (magnitude and direction) of the gold nucleus. (b) What is the final kinetic energy of the helium nucleus?

50. Professional Application

Two cars collide at an icy intersection and stick together afterward. The first car has a mass of 1200 kg and is approaching at 8.00 m/s due south. The second car has a mass of 850 kg and is approaching at 17.0 m/s due west.

(a) Calculate the final velocity (magnitude and direction) of the cars. (b) How much kinetic energy is lost in the collision? (This energy goes into deformation of the cars.) Note that because both cars have an initial velocity, you cannot use the equations for conservation of momentum along the \(x\)-axis and \(y\)-axis; instead, you must look for other simplifying aspects.

51. Starting with equations

\[
m_1 v_1 = m_1 v_1' \cos \theta_1 + m_2 v_2' \cos \theta_2 \quad \text{and} \\
0 = m_1 v_1' \sin \theta_1 + m_2 v_2' \sin \theta_2
\]

for conservation of momentum in the \(x\)- and \(y\)-directions and assuming that one object is originally stationary, prove that for an elastic collision of two objects of equal masses,

\[
\frac{1}{2}m v_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 + m v_1' v_2' \cos (\theta_1 - \theta_2)
\]

as discussed in the text.

52. Integrated Concepts

A 90.0-kg ice hockey player hits a 0.150-kg puck, giving the puck a velocity of 45.0 m/s. If both are initially at rest and if the ice is frictionless, how far does the player recoil in the time it takes the puck to reach the goal 15.0 m away?

8.7 Introduction to Rocket Propulsion

53. Professional Application

Antiballistic missiles (ABMs) are designed to have very large accelerations so that they may intercept fast-moving incoming missiles in the short time available. What is the takeoff acceleration of a 10,000-kg ABM that expels 196 kg of gas per second at an exhaust velocity of \(2.50 \times 10^3\) m/s?

54. Professional Application

What is the acceleration of a 5000-kg rocket taking off from the Moon, where the acceleration due to gravity is only 1.6 m/s\(^2\), if the rocket expels 8.00 kg of gas per second at an exhaust velocity of \(2.20 \times 10^3\) m/s?

55. Professional Application

Calculate the increase in velocity of a 4000-kg space probe that expels 3500 kg of its mass at an exhaust velocity of \(2.00 \times 10^3\) m/s. You may assume the gravitational force is negligible at the probe's location.

56. Professional Application

Ion-propulsion rockets have been proposed for use in space. They employ atomic ionization techniques and nuclear energy sources to produce extremely high exhaust velocities, perhaps as great at \(8.00 \times 10^6\) m/s. These techniques allow a much more favorable payload-to-fuel ratio. To illustrate this fact: (a) Calculate the increase in velocity of a 20,000-kg space probe that expels only 40.0-kg of its mass at the given exhaust velocity. (b) These engines are usually designed to produce a very small thrust for a very long time—the type of engine that might be useful on a trip to the outer planets, for example. Calculate the acceleration of such an engine if it expels \(4.50 \times 10^{-6}\) kg/s at the given velocity, assuming the acceleration due to gravity is negligible.

57. Derive the equation for the vertical acceleration of a rocket.

58. Professional Application

(a) Calculate the maximum rate at which a rocket can expel gases if its acceleration cannot exceed seven times that of gravity. The mass of the rocket just as it runs out of fuel is 75,000-kg, and its exhaust velocity is \(2.40 \times 10^3\) m/s. Assume that the acceleration of gravity is the same as on Earth's surface (\(9.80\) m/s\(^2\)). (b) Why might it be necessary to limit the acceleration of a rocket?

59. Given the following data for a fire extinguisher-toy wagon rocket experiment, calculate the average exhaust velocity of the gases expelled from the extinguisher. Starting from rest, the final velocity is 10.0 m/s. The total mass is initially 75.0 kg and is 70.0 kg after the extinguisher is fired.

60. How much of a single-stage rocket that is 100,000 kg can be anything but fuel if the rocket is to have a final speed of 8.00 km/s, given that it expels gases at an exhaust velocity of \(2.20 \times 10^3\) m/s?

61. Professional Application

(a) A 5.00-kg squid initially at rest ejects 0.250-kg of fluid with a velocity of 10.0 m/s. What is the recoil velocity of the squid if the ejection is done in 0.100 s and there is a 5.00-N frictional force opposing the squid's movement. (b) How much energy is lost to work done against friction?
62. Unreasonable Results
Squids have been reported to jump from the ocean and travel 30.0 m (measured horizontally) before re-entering the water. (a) Calculate the initial speed of the squid if it leaves the water at an angle of 20.0°, assuming negligible lift from the air and negligible air resistance. (b) The squid propels itself by squirting water. What fraction of its mass would it have to eject in order to achieve the speed found in the previous part? The water is ejected at 12.0 m/s; gravitational force and friction are neglected. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

63. Construct Your Own Problem
Consider an astronaut in deep space cut free from her space ship and needing to get back to it. The astronaut has a few packages that she can throw away to move herself toward the ship. Construct a problem in which you calculate the time it takes her to get back by throwing all the packages at one time compared to throwing them one at a time. Among the things to be considered are the masses involved, the force she can exert on the packages through some distance, and the distance to the ship.

64. Construct Your Own Problem
Consider an artillery projectile striking armor plating. Construct a problem in which you find the force exerted by the projectile on the plate. Among the things to be considered are the mass and speed of the projectile and the distance over which its speed is reduced. Your instructor may also wish for you to consider the relative merits of depleted uranium versus lead projectiles based on the greater density of uranium.
Introduction to Statics and Torque

What might desks, bridges, buildings, trees, and mountains have in common—at least in the eyes of a physicist? The answer is that they are ordinarily motionless relative to the Earth. Furthermore, their acceleration is zero because they remain motionless. That means they also have something in common with a car moving at a constant velocity, because anything with a constant velocity also has an acceleration of zero. Now, the important part—Newton’s second law states that net $\mathbf{F} = ma$, and so the net external force is zero for all stationary objects and for all objects moving at constant velocity. There are forces acting, but they are balanced. That is, they are in equilibrium.
Statics

Statics is the study of forces in equilibrium, a large group of situations that makes up a special case of Newton’s second law. We have already considered a few such situations; in this chapter, we cover the topic more thoroughly, including consideration of such possible effects as the rotation and deformation of an object by the forces acting on it.

How can we guarantee that a body is in equilibrium and what can we learn from systems that are in equilibrium? There are actually two conditions that must be satisfied to achieve equilibrium. These conditions are the topics of the first two sections of this chapter.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42167/1.13/#concept-trailer-statics-torque)

9.1 The First Condition for Equilibrium

The first condition necessary to achieve equilibrium is the one already mentioned: the net external force on the system must be zero. Expressed as an equation, this is simply

\[ \text{net } \mathbf{F} = 0 \quad (9.1) \]

Note that if \( \text{net } \mathbf{F} \) is zero, then the net external force in any direction is zero. For example, the net external forces along the typical \( x \)- and \( y \)-axes are zero. This is written as

\[ \text{net } F_x = 0 \quad \text{and } \text{net } F_y = 0 \quad (9.2) \]

Figure 9.2 and Figure 9.3 illustrate situations where \( \text{net } \mathbf{F} = 0 \) for both static equilibrium (motionless), and dynamic equilibrium (constant velocity).

Figure 9.2 This motionless person is in static equilibrium. The forces acting on him add up to zero. Both forces are vertical in this case.

Figure 9.3 This car is in dynamic equilibrium because it is moving at constant velocity. There are horizontal and vertical forces, but the net external force in any direction is zero. The applied force \( F_{\text{app}} \) between the tires and the road is balanced by air friction, and the weight of the car is supported by the normal forces, here shown to be equal for all four tires.

However, it is not sufficient for the net external force of a system to be zero for a system to be in equilibrium. Consider the two situations illustrated in Figure 9.4 and Figure 9.5 where forces are applied to an ice hockey stick lying flat on ice. The net external force is zero in both situations shown in the figure; but in one case, equilibrium is achieved, whereas in the other, it is not. In Figure 9.4, the ice hockey stick remains motionless. But in Figure 9.5, with the same forces applied in different places, the stick experiences accelerated rotation. Therefore, we know that the point at which a force is applied is another factor in determining whether or not equilibrium is achieved. This will be explored further in the next section.
Figure 9.4 An ice hockey stick lying flat on ice with two equal and opposite horizontal forces applied to it. Friction is negligible, and the gravitational force is balanced by the support of the ice (a normal force). Thus, $\text{net } F = 0$. Equilibrium is achieved, which is static equilibrium in this case.

Figure 9.5 The same forces are applied at other points and the stick rotates—in fact, it experiences an accelerated rotation. Here $\text{net } F = 0$ but the system is not at equilibrium. Hence, the $\text{net } F = 0$ is a necessary—but not sufficient—condition for achieving equilibrium.

9.2 The Second Condition for Equilibrium

The second condition necessary to achieve equilibrium involves avoiding accelerated rotation (maintaining a constant angular velocity). A rotating body or system can be in equilibrium if its rate of rotation is constant and remains unchanged by the forces acting on it. To understand what factors affect rotation, let us think about what happens when you open an ordinary door by rotating it on its hinges.

Several familiar factors determine how effective you are in opening the door. See Figure 9.6. First of all, the larger the force, the more effective it is in opening the door—obviously, the harder you push, the more rapidly the door opens. Also, the point at which you push is crucial. If you apply your force too close to the hinges, the door will open slowly, if at all. Most people have been embarrassed by making this mistake and bumping up against a door when it did not open as quickly as expected. Finally, the direction in which you push is also important. The most effective direction is perpendicular to the door—we push in this direction almost instinctively.
Torque is the turning or twisting effectiveness of a force, illustrated here for door rotation on its hinges (as viewed from overhead). Torque has both magnitude and direction. (a) Counterclockwise torque is produced by this force, which means that the door will rotate in a counterclockwise due to $F$. Note that $r_\perp$ is the perpendicular distance of the pivot from the line of action of the force. (b) A smaller counterclockwise torque is produced by a smaller force $F'$ acting at the same distance from the hinges (the pivot point). (c) The same force as in (a) produces a smaller counterclockwise torque when applied at a smaller distance from the hinges. (d) The same force as in (a), but acting in the opposite direction, produces a clockwise torque. (e) A smaller counterclockwise torque is produced by the same magnitude force acting at the same point but in a different direction. Here, $\theta$ is less than 90°. (f) Torque is zero here since the force just pulls on the hinges, producing no rotation. In this case, $\theta = 0^\circ$.

The magnitude, direction, and point of application of the force are incorporated into the definition of the physical quantity called torque. **Torque** is the rotational equivalent of a force. It is a measure of the effectiveness of a force in changing or accelerating a rotation (changing the angular velocity over a period of time). In equation form, the magnitude of torque is defined to be

$$\tau = rf \sin \theta$$  \hspace{1cm} (9.3)

where $\tau$ (the Greek letter tau) is the symbol for torque, $r$ is the distance from the pivot point to the point where the force is applied, $F$ is the magnitude of the force, and $\theta$ is the angle between the force and the vector directed from the point of application to the pivot point, as seen in **Figure 9.6** and **Figure 9.7**. An alternative expression for torque is given in terms of the perpendicular lever arm $r_\perp$ as shown in **Figure 9.6** and **Figure 9.7**, which is defined as

$$r_\perp = r \sin \theta$$  \hspace{1cm} (9.4)

so that

$$\tau = r_\perp F.$$  \hspace{1cm} (9.5)
A force applied to an object can produce a torque, which depends on the location of the pivot point. (a) The three factors \( r \), \( F \), and \( \theta \) for pivot point A on a body are shown here—\( r \) is the distance from the chosen pivot point to the point where the force \( F \) is applied, and \( \theta \) is the angle between \( F \) and the vector directed from the point of application to the pivot point. If the object can rotate around point A, it will rotate counterclockwise. This means that torque is counterclockwise relative to pivot A. (b) In this case, point B is the pivot point. The torque from the applied force will cause a clockwise rotation around point B, and so it is a clockwise torque relative to B.

The perpendicular lever arm \( r \perp \) is the shortest distance from the pivot point to the line along which \( F \) acts; it is shown as a dashed line in Figure 9.6 and Figure 9.7. Note that the line segment that defines the distance \( r \perp \) is perpendicular to \( F \), as its name implies. It is sometimes easier to find or visualize \( r \perp \) than to find both \( r \) and \( \theta \). In such cases, it may be more convenient to use \( \tau = r \perp F \) rather than \( \tau = rF \sin \theta \) for torque, but both are equally valid.

The SI unit of torque is newtons times meters, usually written as \( \text{N} \cdot \text{m} \). For example, if you push perpendicular to the door with a force of 40 N at a distance of 0.800 m from the hinges, you exert a torque of 32 N·m\((0.800 \text{ m} \times 40 \text{ N} \times \sin 90^\circ)\) relative to the hinges. If you reduce the force to 20 N, the torque is reduced to 16 N·m, and so on.

The torque is always calculated with reference to some chosen pivot point. For the same applied force, a different choice for the location of the pivot will give you a different value for the torque, since both \( r \) and \( \theta \) depend on the location of the pivot. Any point in any object can be chosen to calculate the torque about that point. The object may not actually pivot about the chosen “pivot point.”

Note that for rotation in a plane, torque has two possible directions. Torque is either clockwise or counterclockwise relative to the chosen pivot point, as illustrated for points B and A, respectively, in Figure 9.7. If the object can rotate about point A, it will rotate counterclockwise, which means that the torque for the force is shown as counterclockwise relative to A. But if the object can rotate about point B, it will rotate clockwise, which means the torque for the force shown is clockwise relative to B. Also, the magnitude of the torque is greater when the lever arm is longer.

Now, the second condition necessary to achieve equilibrium is that the net external torque on a system must be zero. An external torque is one that is created by an external force. You can choose the point around which the torque is calculated. The point can be the physical pivot point of a system or any other point in space—but it must be the same point for all torques. If the second condition (net external torque on a system is zero) is satisfied for one choice of pivot point, it will also hold true for any other choice of pivot point in or out of the system of interest. (This is true only in an inertial frame of reference.) The second condition necessary to achieve equilibrium is stated in equation form as

\[
\text{net} \ \tau = 0
\]

where net means total. Torques, which are in opposite directions are assigned opposite signs. A common convention is to call counterclockwise (ccw) torques positive and clockwise (cw) torques negative.

When two children balance a seesaw as shown in Figure 9.8, they satisfy the two conditions for equilibrium. Most people have perfect intuition about seesaws, knowing that the lighter child must sit farther from the pivot and that a heavier child can keep a lighter one off the ground indefinitely.
Two children balancing a seesaw satisfy both conditions for equilibrium. The lighter child sits farther from the pivot to create a torque equal in magnitude to that of the heavier child.

Example 9.1 She Saw Torques On A Seesaw

The two children shown in Figure 9.8 are balanced on a seesaw of negligible mass. (This assumption is made to keep the example simple—more involved examples will follow.) The first child has a mass of 26.0 kg and sits 1.60 m from the pivot. (a) If the second child has a mass of 32.0 kg, how far is she from the pivot? (b) What is $F_p$, the supporting force exerted by the pivot?

Strategy

Both conditions for equilibrium must be satisfied. In part (a), we are asked for a distance; thus, the second condition (regarding torques) must be used, since the first (regarding only forces) has no distances in it. To apply the second condition for equilibrium, we first identify the system of interest to be the seesaw plus the two children. We take the supporting pivot to be the point about which the torques are calculated. We then identify all external forces acting on the system.

Solution (a)

The three external forces acting on the system are the weights of the two children and the supporting force of the pivot. Let us examine the torque produced by each. Torque is defined to be

$$\tau = rF \sin \theta$$

(9.7)

Here $\theta = 90^\circ$, so that $\sin \theta = 1$ for all three forces. That means $r_\perp = r$ for all three. The torques exerted by the three forces are first,

$$\tau_1 = r_1 w_1$$

(9.8)

second,

$$\tau_2 = -r_2 w_2$$

(9.9)

and third,

$$\tau_p = r_p F_p$$

$$= 0 \cdot F_p$$

$$= 0.$$

Note that a minus sign has been inserted into the second equation because this torque is clockwise and is therefore negative by convention. Since $F_p$ acts directly on the pivot point, the distance $r_p$ is zero. A force acting on the pivot cannot cause a rotation, just as pushing directly on the hinges of a door will not cause it to rotate. Now, the second condition for equilibrium is that the sum of the torques on both children is zero. Therefore

$$\tau_2 = -\tau_1.$$ 

(9.11)

or

$$r_2 w_2 = r_1 w_1.$$ 

(9.12)

Weight is mass times the acceleration due to gravity. Entering $mg$ for $w$, we get

$$r_2 m_2 g = r_1 m_1 g.$$ 

(9.13)
Solve this for the unknown $r_2$:

$$r_2 = \frac{m_1}{m_2} \quad (9.14)$$

The quantities on the right side of the equation are known; thus, $r_2$ is

$$r_2 = \frac{1.60 \text{ m}}{32.0 \text{ kg}} \cdot \frac{26.0 \text{ kg}}{32.0 \text{ kg}} = 1.30 \text{ m}. \quad (9.15)$$

As expected, the heavier child must sit closer to the pivot (1.30 m versus 1.60 m) to balance the seesaw.

**Solution (b)**

This part asks for a force $F_p$. The easiest way to find it is to use the first condition for equilibrium, which is

$$\text{net } F = 0. \quad (9.16)$$

The forces are all vertical, so that we are dealing with a one-dimensional problem along the vertical axis; hence, the condition can be written as

$$\text{net } F_y = 0 \quad (9.17)$$

where we again call the vertical axis the $y$-axis. Choosing upward to be the positive direction, and using plus and minus signs to indicate the directions of the forces, we see that

$$F_p - w_1 - w_2 = 0. \quad (9.18)$$

This equation yields what might have been guessed at the beginning:

$$F_p = w_1 + w_2. \quad (9.19)$$

So, the pivot supplies a supporting force equal to the total weight of the system:

$$F_p = m_1g + m_2g. \quad (9.20)$$

Entering known values gives

$$F_p = (26.0 \text{ kg} \cdot 9.80 \text{ m/s}^2) + (32.0 \text{ kg} \cdot 9.80 \text{ m/s}^2) = 568 \text{ N.} \quad (9.21)$$

**Discussion**

The two results make intuitive sense. The heavier child sits closer to the pivot. The pivot supports the weight of the two children. Part (b) can also be solved using the second condition for equilibrium, since both distances are known, but only if the pivot point is chosen to be somewhere other than the location of the seesaw’s actual pivot!

Several aspects of the preceding example have broad implications. First, the choice of the pivot as the point around which torques are calculated simplified the problem. Since $F_p$ is exerted on the pivot point, its lever arm is zero. Hence, the torque exerted by the supporting force $F_p$ is zero relative to that pivot point. The second condition for equilibrium holds for any choice of pivot point, and so we choose the pivot point to simplify the solution of the problem.

Second, the acceleration due to gravity canceled in this problem, and we were left with a ratio of masses. *This will not always be the case. Always enter the correct forces—do not jump ahead to enter some ratio of masses.*

Third, the weight of each child is distributed over an area of the seesaw, yet we treated the weights as if each force were exerted at a single point. This is not an approximation—the distances $r_1$ and $r_2$ are the distances to points directly below the center of gravity of each child. As we shall see in the next section, the mass and weight of a system can act as if they are located at a single point.

Finally, note that the concept of torque has an importance beyond static equilibrium. *Torque plays the same role in rotational motion that force plays in linear motion.* We will examine this in the next chapter.

---

**Take-Home Experiment**

Take a piece of modeling clay and put it on a table, then mash a cylinder down into it so that a ruler can balance on the round side of the cylinder while everything remains still. Put a penny 8 cm away from the pivot. Where would you need to put two pennies to balance? Three pennies?
9.3 Stability

It is one thing to have a system in equilibrium; it is quite another for it to be stable. The toy doll perched on the man’s hand in Figure 9.9, for example, is not in stable equilibrium. There are three types of equilibrium: stable, unstable, and neutral. Figures throughout this module illustrate various examples.

Figure 9.9 presents a balanced system, such as the toy doll on the man’s hand, which has its center of gravity (cg) directly over the pivot, so that the torque of the total weight is zero. This is equivalent to having the torques of the individual parts balanced about the pivot point, in this case the hand. The cgs of the arms, legs, head, and torso are labeled with smaller type.

A system is said to be in stable equilibrium if, when displaced from equilibrium, it experiences a net force or torque in a direction opposite to the direction of the displacement. For example, a marble at the bottom of a bowl will experience a restoring force when displaced from its equilibrium position. This force moves it back toward the equilibrium position. Most systems are in stable equilibrium, especially for small displacements. For another example of stable equilibrium, see the pencil in Figure 9.10.

A system is in unstable equilibrium if, when displaced, it experiences a net force or torque in the same direction as the displacement from equilibrium. A system in unstable equilibrium accelerates away from its equilibrium position if displaced even slightly. An obvious example is a ball resting on top of a hill. Once displaced, it accelerates away from the crest. See the next several figures for examples of unstable equilibrium.
Figure 9.11 If the pencil is displaced slightly to the side (counterclockwise), it is no longer in equilibrium. Its weight produces a clockwise torque that returns the pencil to its equilibrium position.

Figure 9.12 If the pencil is displaced too far, the torque caused by its weight changes direction to counterclockwise and causes the displacement to increase.

Figure 9.13 This figure shows unstable equilibrium, although both conditions for equilibrium are satisfied.
Figure 9.14 If the pencil is displaced even slightly, a torque is created by its weight that is in the same direction as the displacement, causing the displacement to increase.

A system is in neutral equilibrium if its equilibrium is independent of displacements from its original position. A marble on a flat horizontal surface is an example. Combinations of these situations are possible. For example, a marble on a saddle is stable for displacements toward the front or back of the saddle and unstable for displacements to the side. Figure 9.15 shows another example of neutral equilibrium.

Figure 9.15 (a) Here we see neutral equilibrium. The cg of a sphere on a flat surface lies directly above the point of support, independent of the position on the surface. The sphere is therefore in equilibrium in any location, and if displaced, it will remain put. (b) Because it has a circular cross section, the pencil is in neutral equilibrium for displacements perpendicular to its length.

When we consider how far a system in stable equilibrium can be displaced before it becomes unstable, we find that some systems in stable equilibrium are more stable than others. The pencil in Figure 9.10 and the person in Figure 9.16(a) are in stable equilibrium, but become unstable for relatively small displacements to the side. The critical point is reached when the cg is no longer above the base of support. Additionally, since the cg of a person’s body is above the pivots in the hips, displacements must be quickly controlled. This control is a central nervous system function that is developed when we learn to hold our bodies erect as infants. For increased stability while standing, the feet should be spread apart, giving a larger base of support. Stability is also increased by lowering one's center of gravity by bending the knees, as when a football player prepares to receive a ball or braces themselves for a tackle. A cane, a crutch, or a walker increases the stability of the user, even more as the base of support widens. Usually, the cg of a female is lower (closer to the ground) than a male. Young children have their center of gravity
between their shoulders, which increases the challenge of learning to walk.

Figure 9.16 (a) The center of gravity of an adult is above the hip joints (one of the main pivots in the body) and lies between two narrowly-separated feet. Like a pencil standing on its eraser, this person is in stable equilibrium in relation to sideways displacements, but relatively small displacements take his cg outside the base of support and make him unstable. Humans are less stable relative to forward and backward displacements because the feet are not very long. Muscles are used extensively to balance the body in the front-to-back direction. (b) While bending in the manner shown, stability is increased by lowering the center of gravity. Stability is also increased if the base is expanded by placing the feet farther apart.

Animals such as chickens have easier systems to control. Figure 9.17 shows that the cg of a chicken lies below its hip joints and between its widely separated and broad feet. Even relatively large displacements of the chicken's cg are stable and result in restoring forces and torques that return the cg to its equilibrium position with little effort on the chicken's part. Not all birds are like chickens, of course. Some birds, such as the flamingo, have balance systems that are almost as sophisticated as that of humans.

Figure 9.17 shows that the cg of a chicken is below the hip joints and lies above a broad base of support formed by widely-separated and large feet. Hence, the chicken is in very stable equilibrium, since a relatively large displacement is needed to render it unstable. The body of the chicken is supported from above by the hips and acts as a pendulum between the hips. Therefore, the chicken is stable for front-to-back displacements as well as for side-to-side displacements.

Figure 9.17 The center of gravity of a chicken is below the hip joints. The chicken is in stable equilibrium. The body of the chicken is supported from above by the hips and acts as a pendulum between them.

Engineers and architects strive to achieve extremely stable equilibriums for buildings and other systems that must withstand wind, earthquakes, and other forces that displace them from equilibrium. Although the examples in this section emphasize gravitational forces, the basic conditions for equilibrium are the same for all types of forces. The net external force must be zero, and the net torque must also be zero.

Take-Home Experiment

Stand straight with your heels, back, and head against a wall. Bend forward from your waist, keeping your heels and bottom against the wall, to touch your toes. Can you do this without toppling over? Explain why and what you need to do to be able to touch your toes without losing your balance. Is it easier for a woman to do this?

9.4 Applications of Statics, Including Problem-Solving Strategies

Statics can be applied to a variety of situations, ranging from raising a drawbridge to bad posture and back strain. We begin with a discussion of problem-solving strategies specifically used for statics. Since statics is a special case of Newton's laws, both the general problem-solving strategies and the special strategies for Newton's laws, discussed in Problem-Solving Strategies, still apply.
Problem-Solving Strategy: Static Equilibrium Situations

1. The first step is to determine whether or not the system is in static equilibrium. This condition is always the case when the acceleration of the system is zero and accelerated rotation does not occur.

2. It is particularly important to draw a free body diagram for the system of interest. Carefully label all forces, and note their relative magnitudes, directions, and points of application whenever these are known.

3. Solve the problem by applying either or both of the conditions for equilibrium (represented by the equations \( \text{net } \vec{F} = 0 \) and \( \text{net } \vec{\tau} = 0 \), depending on the list of known and unknown factors. If the second condition is involved, choose the pivot point to simplify the solution. Any pivot point can be chosen, but the most useful ones cause torques by unknown forces to be zero. (Torque is zero if the force is applied at the pivot (then \( r = 0 \)), or along a line through the pivot point (then \( \theta = 0 \)). Always choose a convenient coordinate system for projecting forces.

4. Check the solution to see if it is reasonable by examining the magnitude, direction, and units of the answer. The importance of this last step never diminishes, although in unfamiliar applications, it is usually more difficult to judge reasonableness. These judgments become progressively easier with experience.

Now let us apply this problem-solving strategy for the pole vaulter shown in the three figures below. The pole is uniform and has a mass of 5.00 kg. In Figure 9.18, the pole's cg lies halfway between the vaulter's hands. It seems reasonable that the force exerted by each hand is equal to half the weight of the pole, or 24.5 N. This obviously satisfies the first condition for equilibrium \( (\text{net } \vec{F} = 0) \). The second condition \( (\text{net } \vec{\tau} = 0) \) is also satisfied, as we can see by choosing the cg to be the pivot point. The weight exerts no torque about a pivot point located at the cg, since it is applied at that point and its lever arm is zero. The equal forces exerted by the hands are equidistant from the chosen pivot, and so they exert equal and opposite torques. Similar arguments hold for other systems where supporting forces are exerted symmetrically about the cg. For example, the four legs of a uniform table each support one-fourth of its weight.

In Figure 9.18, a pole vaulter holding a pole with its cg halfway between his hands is shown. Each hand exerts a force equal to half the weight of the pole, \( F_R = F_L = \frac{w}{2} \). (b) The pole vaulter moves the pole to his left, and the forces that the hands exert are no longer equal. See Figure 9.18. If the pole is held with its cg to the left of the person, then he must push down with his right hand and up with his left. The forces he exerts are larger here because they are in opposite directions and the cg is at a long distance from either hand.

Similar observations can be made using a meter stick held at different locations along its length.

Figure 9.18 A pole vaulter holds a pole horizontally with both hands.

Figure 9.19 A pole vaulter is holding a pole horizontally with both hands. The center of gravity is near his right hand.
Figure 9.20 A pole vaulter is holding a pole horizontally with both hands. The center of gravity is to the left side of the vaulter.

If the pole vaulter holds the pole as shown in Figure 9.19, the situation is not as simple. The total force he exerts is still equal to the weight of the pole, but it is not evenly divided between his hands. (If \( F_L = F_R \), then the torques about the cg would not be equal since the lever arms are different.) Logically, the right hand should support more weight, since it is closer to the cg. In fact, if the right hand is moved directly under the cg, it will support all the weight. This situation is exactly analogous to two people carrying a load; the one closer to the cg carries more of its weight. Finding the forces \( F_L \) and \( F_R \) is straightforward, as the next example shows.

If the pole vaulter holds the pole from near the end of the pole (Figure 9.20), the direction of the force applied by the right hand of the vaulter reverses its direction.

**Example 9.2 What Force Is Needed to Support a Weight Held Near Its CG?**

For the situation shown in Figure 9.19, calculate: (a) \( F_R \), the force exerted by the right hand, and (b) \( F_L \), the force exerted by the left hand. The hands are 0.900 m apart, and the cg of the pole is 0.600 m from the left hand.

**Strategy**

Figure 9.19 includes a free body diagram for the pole, the system of interest. There is not enough information to use the first condition for equilibrium (\( \text{net } F = 0 \)), since two of the three forces are unknown and the hand forces cannot be assumed to be equal in this case. There is enough information to use the second condition for equilibrium (\( \text{net } \tau = 0 \)) if the pivot point is chosen to be at either hand, thereby making the torque from that hand zero. We choose to locate the pivot at the left hand in this part of the problem, to eliminate the torque from the left hand.

**Solution for (a)**

There are now only two nonzero torques, those from the gravitational force (\( \tau_w \)) and from the push or pull of the right hand (\( \tau_R \)). Stating the second condition in terms of clockwise and counterclockwise torques,

\[
\text{net } \tau_{cw} = - \text{net } \tau_{ccw}. \tag{9.22}
\]

or the algebraic sum of the torques is zero.

Here this is

\[
\tau_R = - \tau_w \tag{9.23}
\]

since the weight of the pole creates a counterclockwise torque and the right hand counters with a clockwise torque. Using the definition of torque, \( \tau = rF \sin \theta \), noting that \( \theta = 90^\circ \), and substituting known values, we obtain

\[
(0.900 \text{ m})(F_R) = (0.600 \text{ m})(mg). \tag{9.24}
\]

Thus,

\[
F_R = (0.667)(5.00 \text{ kg})(9.80 \text{ m/s}^2) = 32.7 \text{ N}. \tag{9.25}
\]

**Solution for (b)**

The first condition for equilibrium is based on the free body diagram in the figure. This implies that by Newton’s second law:

\[
F_L + F_R - mg = 0 \tag{9.26}
\]
From this we can conclude:

\[ F_L + F_R = w = mg \quad (9.27) \]

Solving for \( F_L \), we obtain

\[ F_L = mg - F_R = mg - 32.7 \text{ N} = (5.00 \text{ kg})(9.80 \text{ m/s}^2) - 32.7 \text{ N} = 16.3 \text{ N} \quad (9.28) \]

**Discussion**

\( F_L \) is seen to be exactly half of \( F_R \), as we might have guessed, since \( F_L \) is applied twice as far from the cg as \( F_R \).

If the pole vaulter holds the pole as he might at the start of a run, shown in Figure 9.20, the forces change again. Both are considerably greater, and one force reverses direction.

**Take-Home Experiment**

This is an experiment to perform while standing in a bus or a train. Stand facing sideways. How do you move your body to readjust the distribution of your mass as the bus accelerates and decelerates? Now stand facing forward. How do you move your body to readjust the distribution of your mass as the bus accelerates and decelerates? Why is it easier and safer to stand facing sideways rather than forward? Note: For your safety (and those around you), make sure you are holding onto something while you carry out this activity!

**Balancing Act**

Play with objects on a teeter totter to learn about balance. Test what you've learned by trying the Balance Challenge game.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42173/1.15/#import-auto-id3233)

Figure 9.21

### 9.5 Simple Machines

Simple machines are devices that can be used to multiply or augment a force that we apply – often at the expense of a distance through which we apply the force. The word for “machine” comes from the Greek word meaning “to help make things easier.” Levers, gears, pulleys, wedges, and screws are some examples of machines. Energy is still conserved for these devices because a machine cannot do more work than the energy put into it. However, machines can reduce the input force that is needed to perform the job. The ratio of output to input force magnitudes for any simple machine is called its **mechanical advantage** (MA).

\[ MA = \frac{F_o}{F_i} \quad (9.29) \]

One of the simplest machines is the lever, which is a rigid bar pivoted at a fixed place called the fulcrum. Torques are involved in levers, since there is rotation about a pivot point. Distances from the physical pivot of the lever are crucial, and we can obtain a useful expression for the MA in terms of these distances.
A nail puller is a lever with a large mechanical advantage. The external forces on the nail puller are represented by solid arrows. The force that the nail puller applies to the nail \( F_o \) is not a force on the nail puller. The reaction force the nail exerts back on the puller \( F_n \) is an external force and is equal and opposite to \( F_o \). The perpendicular lever arms of the input and output forces are \( l_i \) and \( l_o \).

Figure 9.22 shows a lever type that is used as a nail puller. Crowbars, seesaws, and other such levers are all analogous to this one. \( F_i \) is the input force and \( F_o \) is the output force. There are three vertical forces acting on the nail puller (the system of interest) – these are \( F_i \), \( F_n \), and \( N \). \( F_n \) is the reaction force back on the system, equal and opposite to \( F_o \). (Note that \( F_o \) is not a force on the system.) \( N \) is the normal force upon the lever, and its torque is zero since it is exerted at the pivot. The torques due to \( F_i \) and \( F_n \) must be equal to each other if the nail is not moving, to satisfy the second condition for equilibrium \((\text{net } \tau = 0)\). (In order for the nail to actually move, the torque due to \( F_i \) must be ever-so-slightly greater than torque due to \( F_n \).) Hence,

\[
l_i F_i = l_o F_o \tag{9.30}
\]

Notice that \( r_i \) is the distance from the pivot point to the point where the input force \( F_i \) is applied, and \( r_o \) (not labeled on the diagram) is the distance from the pivot point to the point where the output force \( F_o \) is applied. The distances \( l_i \) and \( l_o \) are the perpendicular components of the distances from where the input and output forces are applied to the pivot, as shown in the figure. Rearranging the last equation gives

\[
\frac{F_o}{F_i} = \frac{l_i}{l_o} \tag{9.31}
\]

What interests us most here is that the magnitude of the force exerted by the nail puller, \( F_o \), is much greater than the magnitude of the input force applied to the puller at the other end, \( F_i \). For the nail puller,

\[
\text{MA} = \frac{F_o}{F_i} = \frac{l_i}{l_o} \tag{9.32}
\]

This equation is true for levers in general. For the nail puller, the MA is certainly greater than one. The longer the handle on the nail puller, the greater the force you can exert with it.

Two other types of levers that differ slightly from the nail puller are a wheelbarrow and a shovel, shown in Figure 9.23. All these lever types are similar in that only three forces are involved – the input force, the output force, and the force on the pivot – and thus their MAs are given by \( \text{MA} = \frac{F_o}{F_i} \) and \( \text{MA} = \frac{d_1}{d_2} \), with distances being measured relative to the physical pivot. The wheelbarrow and shovel differ from the nail puller because both the input and output forces are on the same side of the pivot.

In the case of the wheelbarrow, the output force or load is between the pivot (the wheel's axle) and the input or applied force. In the case of the shovel, the input force is between the pivot (at the end of the handle) and the load, but the input lever arm is shorter than the output lever arm. In this case, the MA is less than one.
Figure 9.23 (a) In the case of the wheelbarrow, the output force or load is between the pivot and the input force. The pivot is the wheel’s axle. Here, the output force is greater than the input force. Thus, a wheelbarrow enables you to lift much heavier loads than you could with your body alone. (b) In the case of the shovel, the input force is between the pivot and the load, but the input lever arm is shorter than the output lever arm. The pivot is at the handle held by the right hand. Here, the output force (supporting the shovel’s load) is less than the input force (from the hand nearest the load), because the input is exerted closer to the pivot than is the output.

Example 9.3 What is the Advantage for the Wheelbarrow?

In the wheelbarrow of Figure 9.23, the load has a perpendicular lever arm of 7.50 cm, while the hands have a perpendicular lever arm of 1.02 m. (a) What upward force must you exert to support the wheelbarrow and its load if their combined mass is 45.0 kg? (b) What force does the wheelbarrow exert on the ground?

Strategy
Here, we use the concept of mechanical advantage.

Solution
(a) In this case, \( \frac{F_o}{F_i} = \frac{l_i}{l_o} \) becomes

\[
F_i = F_o \frac{l_o}{l_i}
\]

Adding values into this equation yields

\[
F_i = (45.0 \text{ kg})(9.80 \text{ m/s}^2) \frac{0.075 \text{ m}}{1.02 \text{ m}} = 32.4 \text{ N}.
\]

(b) The free-body diagram (see Figure 9.23) gives the following normal force: \( F_i + N = W \). Therefore,

\[
N = (45.0 \text{ kg})(9.80 \text{ m/s}^2) - 32.4 \text{ N} = 409 \text{ N}.
\]

Discussion
An even longer handle would reduce the force needed to lift the load. The MA here is \( MA = 1.02/0.0750 = 13.6 \).

Another very simple machine is the inclined plane. Pushing a cart up a plane is easier than lifting the same cart straight up to the top using a ladder, because the applied force is less. However, the work done in both cases (assuming the work done by friction is negligible) is the same. Inclined lanes or ramps were probably used during the construction of the Egyptian pyramids to move large blocks of stone to the top.

A crank is a lever that can be rotated \( 360^\circ \) about its pivot, as shown in Figure 9.24. Such a machine may not look like a lever,
but the physics of its actions remain the same. The MA for a crank is simply the ratio of the radii \( r_1 / r_0 \). Wheels and gears have this simple expression for their MAs too. The MA can be greater than 1, as it is for the crank, or less than 1, as it is for the simplified car axle driving the wheels, as shown. If the axle’s radius is 2.0 cm and the wheel’s radius is 24.0 cm, then \( \text{MA} = \frac{2.0}{24.0} = 0.083 \) and the axle would have to exert a force of 12,000 N on the wheel to enable it to exert a force of 1000 N on the ground.

Figure 9.24 (a) A crank is a type of lever that can be rotated 360° about its pivot. Cranks are usually designed to have a large MA. (b) A simplified automobile axle drives a wheel, which has a much larger diameter than the axle. The MA is less than 1. (c) An ordinary pulley is used to lift a heavy load. The pulley changes the direction of the force \( T \) exerted by the cord without changing its magnitude. Hence, this machine has an MA of 1.

An ordinary pulley has an MA of 1; it only changes the direction of the force and not its magnitude. Combinations of pulleys, such as those illustrated in Figure 9.25, are used to multiply force. If the pulleys are friction-free, then the force output is approximately an integral multiple of the tension in the cable. The number of cables pulling directly upward on the system of interest, as illustrated in the figures given below, is approximately the MA of the pulley system. Since each attachment applies an external force in approximately the same direction as the others, they add, producing a total force that is nearly an integral multiple of the input force \( T \).
Figure 9.25 (a) The combination of pulleys is used to multiply force. The force is an integral multiple of tension if the pulleys are frictionless. This pulley system has two cables attached to its load, thus applying a force of approximately $2T$. This machine has $MA \approx 2$. (b) Three pulleys are used to lift a load in such a way that the mechanical advantage is about 3. Effectively, there are three cables attached to the load. (c) This pulley system applies a force of $4T$, so that it has $MA \approx 4$. Effectively, four cables are pulling on the system of interest.

### 9.6 Forces and Torques in Muscles and Joints

Muscles, bones, and joints are some of the most interesting applications of statics. There are some surprises. Muscles, for example, exert far greater forces than we might think. Figure 9.26 shows a forearm holding a book and a schematic diagram of an analogous lever system. The schematic is a good approximation for the forearm, which looks more complicated than it is, and we can get some insight into the way typical muscle systems function by analyzing it.

Muscles can only contract, so they occur in pairs. In the arm, the biceps muscle is a flexor—that is, it closes the limb. The triceps muscle is an extensor that opens the limb. This configuration is typical of skeletal muscles, bones, and joints in humans and other vertebrates. Most skeletal muscles exert much larger forces within the body than the limbs apply to the outside world. The reason is clear once we realize that most muscles are attached to bones via tendons close to joints, causing these systems to have mechanical advantages much less than one. Viewing them as simple machines, the input force is much greater than the output force, as seen in Figure 9.26.
Figure 9.26 (a) The figure shows the forearm of a person holding a book. The biceps exert a force \( F_B \) to support the weight of the forearm and the book. The triceps are assumed to be relaxed. (b) Here, you can view an approximately equivalent mechanical system with the pivot at the elbow joint as seen in Example 9.4.

**Example 9.4 Muscles Exert Bigger Forces Than You Might Think**

Calculate the force the biceps muscle must exert to hold the forearm and its load as shown in Figure 9.26, and compare this force with the weight of the forearm plus its load. You may take the data in the figure to be accurate to three significant figures.

**Strategy**

There are four forces acting on the forearm and its load (the system of interest). The magnitude of the force of the biceps is \( F_B \); that of the elbow joint is \( F_E \); that of the weights of the forearm is \( w_a \), and its load is \( w_b \). Two of these are unknown (\( F_B \) and \( F_E \)), so that the first condition for equilibrium cannot by itself yield \( F_B \). But if we use the second condition and choose the pivot to be at the elbow, then the torque due to \( F_E \) is zero, and the only unknown becomes \( F_B \).

**Solution**

The torques created by the weights are clockwise relative to the pivot, while the torque created by the biceps is counterclockwise; thus, the second condition for equilibrium (net \( \tau = 0 \)) becomes

\[
 r_2 w_a + r_3 w_b = r_1 F_B. \tag{9.35}
\]

Note that \( \sin \theta = 1 \) for all forces, since \( \theta = 90^\circ \) for all forces. This equation can easily be solved for \( F_B \) in terms of known quantities, yielding

\[
 F_B = \frac{r_2 w_a + r_3 w_b}{r_1}. \tag{9.36}
\]
In the previous example, the combined weight of the arm and its load is $(6.50 \text{ kg})(9.80 \text{ m/s}^2) = 63.7 \text{ N}$, so that the ratio of the force exerted by the biceps to the total weight is

$$\frac{F_B}{w_a + w_b} = \frac{470}{63.7} = 7.38.$$  

### Discussion

This means that the biceps muscle is exerting a force 7.38 times the weight supported.

In the above example of the biceps muscle, the angle between the forearm and upper arm is $90^\circ$. If this angle changes, the force exerted by the biceps muscle also changes. In addition, the length of the biceps muscle changes. The force the biceps muscle can exert depends upon its length; it is smaller when it is shorter than when it is stretched.

Very large forces are also created in the joints. In the previous example, the downward force $F_E$ exerted by the humerus at the elbow joint equals 407 N, or 6.38 times the total weight supported. (The calculation of $F_E$ is straightforward and is left as an end-of-chapter problem.) Because muscles can contract, but not expand beyond their resting length, joints and muscles often exert forces that act in opposite directions and thus subtract. (In the above example, the upward force of the muscle minus the downward force of the joint equals the weight supported—that is, $470 \text{ N} - 407 \text{ N} = 63 \text{ N}$, approximately equal to the weight supported.) Forces in muscles and joints are largest when their load is a long distance from the joint, as the book is in the previous example.

In racquet sports such as tennis the constant extension of the arm during game play creates large forces in this way. The mass times the lever arm of a tennis racquet is an important factor, and many players use the heaviest racquet they can handle. It is no wonder that joint deterioration and damage to the tendons in the elbow, such as “tennis elbow,” can result from repetitive motion, undue torques, and possibly poor racquet selection in such sports. Various tried techniques for holding and using a racquet or bat or stick not only increases sporting prowess but can minimize fatigue and long-term damage to the body. For example, tennis balls correctly hit at the “sweet spot” on the racquet will result in little vibration or impact force being felt in the racquet and the body—less torque as explained in *Collisions of Extended Bodies in Two Dimensions*. Twisting the hand to provide top spin on the ball or using an extended rigid elbow in a backhand stroke can also aggravate the tendons in the elbow.

Training coaches and physical therapists use the knowledge of relationships between forces and torques in the treatment of muscles and joints. In physical therapy, an exercise routine can apply a particular force and torque which can, over a period of time, revive muscles and joints. Some exercises are designed to be carried out under water, because this requires greater forces to be exerted, further strengthening muscles. However, connecting tissues in the limbs, such as tendons and cartilage as well as joints are sometimes damaged by the large forces they carry. Often, this is due to accidents, but heavily muscled athletes, such as weightlifters, can tear muscles and connecting tissue through effort alone.

The back is considerably more complicated than the arm or leg, with various muscles and joints between vertebrae, all having mechanical advantages less than 1. Back muscles must, therefore, exert very large forces, which are borne by the spinal column. Discs crushed by mere exertion are very common. The jaw is somewhat exceptional—the masseter muscles that close the jaw have a mechanical advantage greater than 1 for the back teeth, allowing us to exert very large forces with them. A cause of stress headaches is persistent clenching of teeth where the sustained large force translates into fatigue in muscles around the skull.

**Figure 9.27** shows how bad posture causes back strain. In part (a), we see a person with good posture. Note that her upper body’s cg is directly above the pivot point in the hips, which in turn is directly above the base of support at her feet. Because of this, her upper body’s weight exerts no torque about the hips. The only force needed is a vertical force at the hips equal to the weight supported. No muscle action is required, since the bones are rigid and transmit this force from the floor. This is a position of unstable equilibrium, but only small forces are needed to bring the upper body back to vertical if it is slightly displaced. Bad posture is shown in part (b); we see that the upper body’s cg is in front of the pivot in the hips. This creates a clockwise torque around the hips that is counteracted by muscles in the lower back. These muscles must exert large forces, since they have typically small mechanical advantages. (In other words, the perpendicular lever arm for the muscles is much smaller than for the cg.) Poor posture can also cause muscle strain for people sitting at their desks using computers. Special chairs are available that allow the body’s CG to be more easily situated above the seat, to reduce back pain. Prolonged muscle action produces muscle strain. Note that the cg of the entire body is still directly above the base of support in part (b) of **Figure 9.27**. This is compulsory; otherwise the person would not be in equilibrium. We lean forward for the same reason when carrying a load on our backs, to the side when carrying a load in one arm, and backward when carrying a load in front of us, as seen in Figure 9.28.
Figure 9.27 (a) Good posture places the upper body’s cg over the pivots in the hips, eliminating the need for muscle action to balance the body. (b) Poor posture requires exertion by the back muscles to counteract the clockwise torque produced around the pivot by the upper body’s weight. The back muscles have a small effective perpendicular lever arm, \( r_{b\perp} \), and must therefore exert a large force \( F_b \). Note that the legs lean backward to keep the cg of the entire body above the base of support in the feet.

You have probably been warned against lifting objects with your back. This action, even more than bad posture, can cause muscle strain and damage discs and vertebrae, since abnormally large forces are created in the back muscles and spine.

Figure 9.28 People adjust their stance to maintain balance. (a) A father carrying his son piggyback leans forward to position their overall cg above the base of support at his feet. (b) A student carrying a shoulder bag leans to the side to keep the overall cg over his feet. (c) Another student carrying a load of books in her arms leans backward for the same reason.

**Example 9.5 Do Not Lift with Your Back**

Consider the person lifting a heavy box with his back, shown in Figure 9.29. (a) Calculate the magnitude of the force \( F_B \) in the back muscles that is needed to support the upper body plus the box and compare this with his weight. The mass of the upper body is 55.0 kg and the mass of the box is 30.0 kg. (b) Calculate the magnitude and direction of the force \( F_V \) exerted by the vertebrae on the spine at the indicated pivot point. Again, data in the figure may be taken to be accurate to three significant figures.

**Strategy**

By now, we sense that the second condition for equilibrium is a good place to start, and inspection of the known values confirms that it can be used to solve for \( F_B \) — if the pivot is chosen to be at the hips. The torques created by \( \mathbf{w}_{ub} \) and \( \mathbf{w}_{box} \) are clockwise, while that created by \( \mathbf{F}_B \) is counterclockwise.

**Solution for (a)**

Using the perpendicular lever arms given in the figure, the second condition for equilibrium \( (\text{net } \tau = 0) \) becomes

\[
(0.350 \text{ m})(55.0 \text{ kg})(9.80 \text{ m/s}^2) + (0.500 \text{ m})(30.0 \text{ kg})(9.80 \text{ m/s}^2) = (0.0800 \text{ m})F_B.
\]
Solving for $F_B$ yields

$$F_B = 4.20 \times 10^3 \text{ N.} \quad (9.41)$$

The ratio of the force the back muscles exert to the weight of the upper body plus its load is

$$\frac{F_B}{w_{ub} + w_{box}} = \frac{4200 \text{ N}}{833 \text{ N}} = 5.04. \quad (9.42)$$

This force is considerably larger than it would be if the load were not present.

**Solution for (b)**

More important in terms of its damage potential is the force on the vertebrae $F_V$. The first condition for equilibrium ($\text{net } F = 0$) can be used to find its magnitude and direction. Using $y$ for vertical and $x$ for horizontal, the condition for the net external forces along those axes to be zero

$$\text{net } F_y = 0 \text{ and net } F_x = 0. \quad (9.43)$$

Starting with the vertical ($y$) components, this yields

$$F_{Vy} - w_{ub} - w_{box} - F_B \sin 29.0^\circ = 0. \quad (9.44)$$

Thus,

$$F_{Vy} = w_{ub} + w_{box} + F_B \sin 29.0^\circ = 833 \text{ N} + (4200 \text{ N}) \sin 29.0^\circ$$

yielding

$$F_{Vy} = 2.87 \times 10^3 \text{ N.} \quad (9.46)$$

Similarly, for the horizontal ($x$) components,

$$F_{Vx} - F_B \cos 29.0^\circ = 0 \quad (9.47)$$

yielding

$$F_{Vx} = 3.67 \times 10^3 \text{ N.} \quad (9.48)$$

The magnitude of $F_V$ is given by the Pythagorean theorem:

$$F_V = \sqrt{F_{Vx}^2 + F_{Vy}^2} = 4.66 \times 10^3 \text{ N.} \quad (9.49)$$

The direction of $F_V$ is

$$\theta = \tan^{-1} \left( \frac{F_{Vy}}{F_{Vx}} \right) = 38.0^\circ. \quad (9.50)$$

Note that the ratio of $F_V$ to the weight supported is

$$\frac{F_V}{w_{ub} + w_{box}} = \frac{4660 \text{ N}}{833 \text{ N}} = 5.59. \quad (9.51)$$

**Discussion**

This force is about 5.6 times greater than it would be if the person were standing erect. The trouble with the back is not so much that the forces are large—because similar forces are created in our hips, knees, and ankles—but that our spines are relatively weak. Proper lifting, performed with the back erect and using the legs to raise the body and load, creates much smaller forces in the back—in this case, about 5.6 times smaller.
center of gravity: the point where the total weight of the body is assumed to be concentrated

dynamic equilibrium: a state of equilibrium in which the net external force and torque on a system moving with constant velocity are zero

mechanical advantage: the ratio of output to input forces for any simple machine

neutral equilibrium: a state of equilibrium that is independent of a system’s displacements from its original position

perpendicular lever arm: the shortest distance from the pivot point to the line along which \( F \) lies

SI units of torque: newton times meters, usually written as N·m

stable equilibrium: a system, when displaced, experiences a net force or torque in a direction opposite to the direction of the displacement

static equilibrium: a state of equilibrium in which the net external force and torque acting on a system is zero

static equilibrium: equilibrium in which the acceleration of the system is zero and accelerated rotation does not occur

torque: turning or twisting effectiveness of a force

unstable equilibrium: a system, when displaced, experiences a net force or torque in the same direction as the displacement from equilibrium

Figure 9.29 This figure shows that large forces are exerted by the back muscles and experienced in the vertebrae when a person lifts with their back, since these muscles have small effective perpendicular lever arms. The data shown here are analyzed in the preceding example, Example 9.5.

What are the benefits of having most skeletal muscles attached so close to joints? One advantage is speed because small muscle contractions can produce large movements of limbs in a short period of time. Other advantages are flexibility and agility, made possible by the large numbers of joints and the ranges over which they function. For example, it is difficult to imagine a system with biceps muscles attached at the wrist that would be capable of the broad range of movement we vertebrates possess.

There are some interesting complexities in real systems of muscles, bones, and joints. For instance, the pivot point in many joints changes location as the joint is flexed, so that the perpendicular lever arms and the mechanical advantage of the system change, too. Thus the force the biceps muscle must exert to hold up a book varies as the forearm is flexed. Similar mechanisms operate in the legs, which explain, for example, why there is less leg strain when a bicycle seat is set at the proper height. The methods employed in this section give a reasonable description of real systems provided enough is known about the dimensions of the system. There are many other interesting examples of force and torque in the body—a few of these are the subject of end-of-chapter problems.

Glossary
Section Summary

9.1 The First Condition for Equilibrium
- Statics is the study of forces in equilibrium.
- Two conditions must be met to achieve equilibrium, which is defined to be motion without linear or rotational acceleration.
- The first condition necessary to achieve equilibrium is that the net external force on the system must be zero, so that \( \text{net } \mathbf{F} = 0 \).

9.2 The Second Condition for Equilibrium
- The second condition assures those torques are also balanced. Torque is the rotational equivalent of a force in producing a rotation and is defined to be \( \tau = rF \sin \theta \)
  where \( \tau \) is torque, \( r \) is the distance from the pivot point to the point where the force is applied, \( F \) is the magnitude of the force, and \( \theta \) is the angle between \( \mathbf{F} \) and the vector directed from the point where the force acts to the pivot point. The perpendicular lever arm \( r_\perp \) is defined to be \( r_\perp = r \sin \theta \)
  so that \( \tau = r_\perp F \).
- The perpendicular lever arm \( r_\perp \) is the shortest distance from the pivot point to the line along which \( F \) acts. The SI unit for torque is newton-meter (N·m). The second condition necessary to achieve equilibrium is that the net external torque on a system must be zero:
  \[ \text{net } \tau = 0 \]
  By convention, counterclockwise torques are positive, and clockwise torques are negative.

9.3 Stability
- A system is said to be in stable equilibrium if, when displaced from equilibrium, it experiences a net force or torque in a direction opposite the direction of the displacement.
- A system is in unstable equilibrium if, when displaced from equilibrium, it experiences a net force or torque in the same direction as the displacement from equilibrium.
- A system is in neutral equilibrium if its equilibrium is independent of displacements from its original position.

9.4 Applications of Statics, Including Problem-Solving Strategies
- Statics can be applied to a variety of situations, ranging from raising a drawbridge to bad posture and back strain. We have discussed the problem-solving strategies specifically useful for statics. Statics is a special case of Newton’s laws, both the general problem-solving strategies and the special strategies for Newton’s laws, discussed in Problem-Solving Strategies, still apply.

9.5 Simple Machines
- Simple machines are devices that can be used to multiply or augment a force that we apply — often at the expense of a distance through which we have to apply the force.
  - The ratio of output to input forces for any simple machine is called its mechanical advantage
  - A few simple machines are the lever, nail puller, wheelbarrow, crank, etc.

9.6 Forces and Torques in Muscles and Joints
- Statics plays an important part in understanding everyday strains in our muscles and bones.
  - Many lever systems in the body have a mechanical advantage of significantly less than one, as many of our muscles are attached close to joints.
  - Someone with good posture stands or sits in such a way that the person’s center of gravity lies directly above the pivot point in the hips, thereby avoiding back strain and damage to disks.

Conceptual Questions

9.1 The First Condition for Equilibrium
1. What can you say about the velocity of a moving body that is in dynamic equilibrium? Draw a sketch of such a body using clearly labeled arrows to represent all external forces on the body.
2. Under what conditions can a rotating body be in equilibrium? Give an example.
9.2 The Second Condition for Equilibrium
3. What three factors affect the torque created by a force relative to a specific pivot point?
4. A wrecking ball is being used to knock down a building. One tall unsupported concrete wall remains standing. If the wrecking ball hits the wall near the top, is the wall more likely to fall over by rotating at its base or by falling straight down? Explain your answer. How is it most likely to fall if it is struck with the same force at its base? Note that this depends on how firmly the wall is attached at its base.
5. Mechanics sometimes put a length of pipe over the handle of a wrench when trying to remove a very tight bolt. How does this help? (It is also hazardous since it can break the bolt.)

9.3 Stability
6. A round pencil lying on its side as in Figure 9.12 is in neutral equilibrium relative to displacements perpendicular to its length. What is its stability relative to displacements parallel to its length?
7. Explain the need for tall towers on a suspension bridge to ensure stable equilibrium.

9.4 Applications of Statics, Including Problem-Solving Strategies
8. When visiting some countries, you may see a person balancing a load on the head. Explain why the center of mass of the load needs to be directly above the person’s neck vertebrae.

9.5 Simple Machines
9. Scissors are like a double-lever system. Which of the simple machines in Figure 9.22 and Figure 9.23 is most analogous to scissors?
10. Suppose you pull a nail at a constant rate using a nail puller as shown in Figure 9.22. Is the nail puller in equilibrium? What if you pull the nail with some acceleration – is the nail puller in equilibrium then? In which case is the force applied to the nail puller larger and why?
11. Explain the mechanical advantage of a wheelbarrow and how it explains why moving a heavy load is easier with a wheelbarrow.
12. How does a pulley enable a person to lift a load as heavy as a piano with little effort?

9.6 Forces and Torques in Muscles and Joints
13. Why are the forces exerted on the outside world by the limbs of our bodies usually much smaller than the forces exerted by muscles inside the body?
14. Explain why the forces in our joints are several times larger than the forces we exert on the outside world with our limbs. Can these forces be even greater than muscle forces?
15. Certain types of dinosaurs were bipedal (walked on two legs). What is a good reason that these creatures invariably had long tails if they had long necks?
16. Swimmers and athletes during competition need to go through certain postures at the beginning of the race. Consider the balance of the person and why start-offs are so important for races.
17. If the maximum force the biceps muscle can exert is 1000 N, can we pick up an object that weighs 1000 N? Explain your answer.
18. Suppose the biceps muscle was attached through tendons to the upper arm close to the elbow and the forearm near the wrist. What would be the advantages and disadvantages of this type of construction for the motion of the arm?
19. Explain one of the reasons why pregnant women often suffer from back strain late in their pregnancy.
Problems & Exercises

9.2 The Second Condition for Equilibrium

1. (a) When opening a door, you push on it perpendicularly with a force of 55.0 N at a distance of 0.850 m from the hinges. What torque are you exerting relative to the hinges? (b) Does it matter if you push at the same height as the hinges?

2. When tightening a bolt, you push perpendicularly on a wrench with a force of 165 N at a distance of 0.140 m from the center of the bolt. (a) How much torque are you exerting in newton × meters (relative to the center of the bolt)? (b) Convert this torque to footpounds.

3. Two children push on opposite sides of a door during play. Both push horizontally and perpendicular to the door. One child pushes with a force of 17.5 N at a distance of 0.600 m from the hinges, and the second child pushes at a distance of 0.450 m. What force must the second child exert to keep the door from moving? Assume friction is negligible.

4. Use the second condition for equilibrium (net \( \tau = 0 \)) to calculate \( F_p \) in Example 9.1, employing any data given or solved for in part (a) of the example.

5. Repeat the seesaw problem in Example 9.1 with the center of mass of the seesaw 0.160 m to the left of the pivot (on the side of the lighter child) and assuming a mass of 12.0 kg for the seesaw. The other data given in the example remain unchanged. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium.

9.3 Stability

6. Suppose a horse leans against a wall as in Figure 9.30. Calculate the force exerted on the wall assuming that force is horizontal while using the data in the schematic representation of the situation. Note that the force exerted on the wall is equal in magnitude and opposite in direction to the force exerted on the horse, keeping it in equilibrium. The total mass of the horse and rider is 500 kg. Take the data to be accurate to three digits.

7. Two children of mass 20.0 kg and 30.0 kg sit balanced on a seesaw with the pivot point located at the center of the seesaw. If the children are separated by a distance of 3.00 m, at what distance from the pivot point is the small child sitting in order to maintain the balance?

8. (a) Calculate the magnitude and direction of the force on each foot of the horse in Figure 9.30 (two are on the ground), assuming the center of mass of the horse is midway between the hooves and ground? Note that the force exerted by the wall is horizontal.

9. A person carries a plank of wood 2.00 m long with one hand pushing down on it at one end with a force \( F_1 \) and the other hand holding it up at .500 m from the end of the plank with force \( F_2 \). If the plank has a mass of 20.0 kg and its center of gravity is at the middle of the plank, what are the magnitudes of the forces \( F_1 \) and \( F_2 \)?

10. A 17.0-m-high and 11.0-m-long wall under construction and its bracing are shown in Figure 9.31. The wall is in stable equilibrium without the bracing but can pivot at its base. Calculate the force exerted by each of the 10 braces if a strong wind exerts a horizontal force of 650 N on each square meter of the wall. Assume that the net force from the wind acts at a height halfway up the wall and that all braces exert equal forces parallel to their lengths. Neglect the thickness of the wall.

11. (a) What force must be exerted by the wind to support a 2.50-kg chicken in the position shown in Figure 9.32? (b) What is the ratio of this force to the chicken’s weight? (c) Does this support the contention that the chicken has a relatively stable construction?
12. Suppose the weight of the drawbridge in Figure 9.33 is supported entirely by its hinges and the opposite shore, so that its cables are slack. The mass of the bridge is 2500 kg. (a) What fraction of the weight is supported by the opposite shore if the point of support is directly beneath the cable attachments? (b) What is the direction and magnitude of the force the hinges exert on the bridge under these circumstances?

Figure 9.33 A small drawbridge, showing the forces on the hinges (F), its weight (W), and the tension in its wires (T).

13. Suppose a 900-kg car is on the bridge in Figure 9.33 with its center of mass halfway between the hinges and the cable attachments. (The bridge is supported by the cables and hinges only.) (a) Find the force in the cables. (b) Find the direction and magnitude of the force exerted by the hinges on the bridge.

14. A sandwich board advertising sign is constructed as shown in Figure 9.34. The sign’s mass is 8.00 kg. (a) Calculate the tension in the chain assuming no friction between the legs and the sidewalk. (b) What force is exerted by each side on the hinge?

Figure 9.34 A sandwich board advertising sign demonstrates tension.

15. (a) What minimum coefficient of friction is needed between the legs and the ground to keep the sign in Figure 9.34 in the position shown if the chain breaks? (b) What force is exerted by each side on the hinge?

16. A gymnast is attempting to perform splits. From the information given in Figure 9.35, calculate the magnitude and direction of the force exerted on each foot by the floor.

Figure 9.35 A gymnast performs full split. The center of gravity and the various distances from it are shown.

9.4 Applications of Statics, Including Problem-Solving Strategies

17. To get up on the roof, a person (mass 70.0 kg) places a 6.00-m aluminum ladder (mass 10.0 kg) against the house on a concrete pad with the base of the ladder 2.00 m from the house. The ladder rests against a plastic rain gutter, which we can assume to be frictionless. The center of mass of the ladder is 2 m from the bottom. The person is standing 3 m from the bottom. What are the magnitudes of the forces on the ladder at the top and bottom?

18. In Figure 9.20, the cg of the pole held by the pole vaulter is 2.00 m from the left hand, and the hands are 0.700 m apart. Calculate the force exerted by (a) his right hand and (b) his left hand. (c) If each hand supports half the weight of the pole in Figure 9.18, show that the second condition for equilibrium (net \( r = 0 \)) is satisfied for a pivot other than the one located at the center of gravity of the pole. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium described above.

9.5 Simple Machines

19. What is the mechanical advantage of a nail puller—similar to the one shown in Figure 9.22—where you exert a force 45 cm from the pivot and the nail is 1.8 cm on the other side? What minimum force must you exert to apply a force of 1250 N to the nail?

20. Suppose you needed to raise a 250-kg mower a distance of 6.0 cm above the ground to change a tire. If you had a 2.0-m long lever, where would you place the fulcrum if your force was limited to 300 N?

21. a) What is the mechanical advantage of a wheelbarrow, such as the one in Figure 9.23, if the center of gravity of the wheelbarrow and its load has a perpendicular lever arm of 5.50 cm, while the hands have a perpendicular lever arm of 1.02 m? (b) What upward force should you exert to support the wheelbarrow and its load if their combined mass is 55.0 kg? (c) What force does the wheel exert on the ground?

22. A typical car has an axle with 1.10 cm radius driving a tire with a radius of 27.5 cm. What is its mechanical advantage assuming the very simplified model in Figure 9.24(b)?
23. What force does the nail puller in Exercise 9.19 exert on the supporting surface? The nail puller has a mass of 2.10 kg.
24. If you used an ideal pulley of the type shown in Figure 9.25(a) to support a car engine of mass 115 kg, (a) What would be the tension in the rope? (b) What force must the ceiling supply, assuming you pull straight down on the rope? Neglect the pulley system’s mass.
25. Repeat Exercise 9.24 for the pulley shown in Figure 9.25(c), assuming you pull straight up on the rope. The pulley system’s mass is 7.00 kg.

9.6 Forces and Torques in Muscles and Joints

26. Verify that the force in the elbow joint in Example 9.4 is 407 N, as stated in the text.
27. Two muscles in the back of the leg pull on the Achilles tendon as shown in Figure 9.36. What total force do they exert?

28. The upper leg muscle (quadriceps) exerts a force of 1250 N, which is carried by a tendon over the kneecap (the patella) at the angles shown in Figure 9.37. Find the direction and magnitude of the force exerted by the kneecap on the upper leg bone (the femur).

29. A device for exercising the upper leg muscle is shown in Figure 9.38, together with a schematic representation of an equivalent lever system. Calculate the force exerted by the upper leg muscle to lift the mass at a constant speed. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium in Applications of Statistics, Including Problem-Solving Strategies.
30. A person working at a drafting board may hold her head as shown in Figure 9.39, requiring muscle action to support the head. The three major acting forces are shown. Calculate the direction and magnitude of the force supplied by the upper vertebrae \( F_V \) to hold the head stationary, assuming that this force acts along a line through the center of mass as do the weight and muscle force.

![Figure 9.39](image)

31. We analyzed the biceps muscle example with the angle between forearm and upper arm set at 90\(^\circ\). Using the same numbers as in Example 9.4, find the force exerted by the biceps muscle when the angle is 120\(^\circ\) and the forearm is in a downward position.

32. Even when the head is held erect, as in Figure 9.40, its center of mass is not directly over the principal point of support (the atlanto-occipital joint). The muscles at the back of the neck should therefore exert a force to keep the head erect. That is why your head falls forward when you fall asleep in the class. (a) Calculate the force exerted by these muscles using the information in the figure. (b) What is the force exerted by the pivot on the head?

![Figure 9.40](image)

33. A 75-kg man stands on his toes by exerting an upward force through the Achilles tendon, as in Figure 9.41. (a) What is the force in the Achilles tendon if he stands on one foot? (b) Calculate the force at the pivot of the simplified lever system shown—that force is representative of forces in the ankle joint.

![Figure 9.41](image)

34. A father lifts his child as shown in Figure 9.42. What force should the upper leg muscle exert to lift the child at a constant speed?

![Figure 9.42](image)
35. Unlike most of the other muscles in our bodies, the masseter muscle in the jaw, as illustrated in Figure 9.43, is attached relatively far from the joint, enabling large forces to be exerted by the back teeth. (a) Using the information in the figure, calculate the force exerted by the lower teeth on the bullet. (b) Calculate the force on the joint.

![Figure 9.43](image-url) A person clenching a bullet between his teeth.

36. Integrated Concepts

Suppose we replace the 4.0-kg book in Exercise 9.31 of the biceps muscle with an elastic exercise rope that obeys Hooke's Law. Assume its force constant \( k = 600 \text{ N/m} \). (a) How much is the rope stretched (past equilibrium) to provide the same force \( F_B \) as in this example? Assume the rope is held in the hand at the same location as the book. (b) What force is on the biceps muscle if the exercise rope is pulled straight up so that the forearm makes an angle of 25° with the horizontal? Assume the biceps muscle is still perpendicular to the forearm.

37. (a) What force should the woman in Figure 9.44 exert on the floor with each hand to do a push-up? Assume that she moves up at a constant speed. (b) The triceps muscle at the back of her upper arm has an effective lever arm of 1.75 cm, and it exerts force on the floor at a horizontal distance of 20.0 cm from the elbow joint. Calculate the magnitude of the force in each triceps muscle, and compare it to her weight. (c) How much work does she do if her center of mass rises 0.240 m? (d) What is her useful power output if she does 25 pushups in one minute?

![Figure 9.44](image-url) A woman doing pushups.

38. You have just planted a sturdy 2-m-tall palm tree in your front lawn for your mother’s birthday. Your brother kicks a 500 g ball, which hits the top of the tree at a speed of 5 m/s and stays in contact with it for 10 ms. The ball falls to the ground near the base of the tree and the recoil of the tree is minimal. (a) What is the force on the tree? (b) The length of the sturdy section of the root is only 20 cm. Furthermore, the soil around the roots is loose and we can assume that an effective force is applied at the tip of the 20 cm length. What is the effective force exerted by the end of the tip of the root to keep the tree from toppling? Assume the tree will be uprooted rather than bend. (c) What could you have done to ensure that the tree does not uproot easily?

39. Unreasonable Results

Suppose two children are using a uniform seesaw that is 3.00 m long and has its center of mass over the pivot. The first child has a mass of 30.0 kg and sits 1.40 m from the pivot. (a) Calculate where the second 18.0 kg child must sit to balance the seesaw. (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

40. Construct Your Own Problem

Consider a method for measuring the mass of a person’s arm in anatomical studies. The subject lies on her back, extends her relaxed arm to the side and two scales are placed below the arm. One is placed under the elbow and the other under the back of her hand. Construct a problem in which you calculate the mass of the arm and find its center of mass based on the scale readings and the distances of the scales from the shoulder joint. You must include a free body diagram of the arm to direct the analysis. Consider changing the position of the scale under the hand to provide more information, if needed. You may wish to consult references to obtain reasonable mass values.
10 ROTATIONAL MOTION AND ANGULAR MOMENTUM

Figure 10.1 The mention of a tornado conjures up images of raw destructive power. Tornadoes blow houses away as if they were made of paper and have been known to pierce tree trunks with pieces of straw. They descend from clouds in funnel-like shapes that spin violently, particularly at the bottom where they are most narrow, producing winds as high as 500 km/h. (credit: Daphne Zaras, U.S. National Oceanic and Atmospheric Administration)

Chapter Outline

10.1. Angular Acceleration
- Describe uniform circular motion.
- Explain non-uniform circular motion.
- Calculate angular acceleration of an object.
- Observe the link between linear and angular acceleration.

10.2. Kinematics of Rotational Motion
- Observe the kinematics of rotational motion.
- Derive rotational kinematic equations.
- Evaluate problem solving strategies for rotational kinematics.

10.3. Dynamics of Rotational Motion: Rotational Inertia
- Understand the relationship between force, mass and acceleration.
- Study the turning effect of force.
- Study the analogy between force and torque, mass and moment of inertia, and linear acceleration and angular acceleration.

10.4. Rotational Kinetic Energy: Work and Energy Revisited
- Derive the equation for rotational work.
- Calculate rotational kinetic energy.
- Demonstrate the Law of Conservation of Energy.

10.5. Angular Momentum and Its Conservation
- Understand the analogy between angular momentum and linear momentum.
- Observe the relationship between torque and angular momentum.
- Apply the law of conservation of angular momentum.

10.6. Collisions of Extended Bodies in Two Dimensions
- Observe collisions of extended bodies in two dimensions.
- Examine collision at the point of percussion.

10.7. Gyroscopic Effects: Vector Aspects of Angular Momentum
- Describe the right-hand rule to find the direction of angular velocity, momentum, and torque.
- Explain the gyroscopic effect.
- Study how Earth acts like a gigantic gyroscope.
Introduction to Rotational Motion and Angular Momentum

Why do tornadoes spin at all? And why do tornados spin so rapidly? The answer is that air masses that produce tornadoes are themselves rotating, and when the radii of the air masses decrease, their rate of rotation increases. An ice skater increases her spin in an exactly analogous manner as seen in Figure 10.2. The skater starts her rotation with outstretched limbs and increases her spin by pulling them in toward her body. The same physics describes the exhilarating spin of a skater and the wrenching force of a tornado.

Clearly, force, energy, and power are associated with rotational motion. These and other aspects of rotational motion are covered in this chapter. We shall see that all important aspects of rotational motion either have already been defined for linear motion or have exact analogs in linear motion. First, we look at angular acceleration—the rotational analog of linear acceleration.

10.1 Angular Acceleration

Uniform Circular Motion and Gravitation discussed only uniform circular motion, which is motion in a circle at constant speed and, hence, constant angular velocity. Recall that angular velocity \( \omega \) was defined as the time rate of change of angle \( \theta \):

\[
\omega = \frac{\Delta \theta}{\Delta t}
\]

where \( \theta \) is the angle of rotation as seen in Figure 10.3. The relationship between angular velocity \( \omega \) and linear velocity \( v \) was also defined in Rotation Angle and Angular Velocity as

\[
v = r \omega
\]

or

\[
\omega = \frac{v}{r},
\]

where \( r \) is the radius of curvature, also seen in Figure 10.3. According to the sign convention, the counter clockwise direction is considered as positive direction and clockwise direction as negative.
Angular velocity is not constant when a skater pulls in her arms, when a child starts up a merry-go-round from rest, or when a computer’s hard disk slows to a halt when switched off. In all these cases, there is an angular acceleration, in which \( \omega \) changes. The faster the change occurs, the greater the angular acceleration. Angular acceleration \( \alpha \) is defined as the rate of change of angular velocity. In equation form, angular acceleration is expressed as follows:

\[
\alpha = \frac{\Delta \omega}{\Delta t},
\]

where \( \Delta \omega \) is the change in angular velocity and \( \Delta t \) is the change in time. The units of angular acceleration are \((\text{rad/s})/\text{s}\), or \(\text{rad/s}^2\). If \( \omega \) increases, then \( \alpha \) is positive. If \( \omega \) decreases, then \( \alpha \) is negative.

**Example 10.1 Calculating the Angular Acceleration and Deceleration of a Bike Wheel**

Suppose a teenager puts her bicycle on its back and starts the rear wheel spinning from rest to a final angular velocity of 250 rpm in 5.00 s. (a) Calculate the angular acceleration in \( \text{rad/s}^2 \). (b) If she now slams on the brakes, causing an angular acceleration of \(-87.3 \text{ rad/s}^2\), how long does it take the wheel to stop?

**Strategy for (a)**

The angular acceleration can be found directly from its definition in \( \alpha = \frac{\Delta \omega}{\Delta t} \) because the final angular velocity and time are given. We see that \( \Delta \omega \) is 250 rpm and \( \Delta t \) is 5.00 s.

**Solution for (a)**

Entering known information into the definition of angular acceleration, we get

\[
\alpha = \frac{\Delta \omega}{\Delta t} = \frac{250 \text{ rpm}}{5.00 \text{ s}}.
\]

Because \( \Delta \omega \) is in revolutions per minute (rpm) and we want the standard units of \( \text{rad/s}^2 \) for angular acceleration, we need to convert \( \Delta \omega \) from rpm to rad/s:

\[
\Delta \omega = 250 \text{ rev/min} \cdot \frac{2\pi \text{ rad/rev}}{60 \text{ s}} = 26.2 \text{ rad/s}.
\]

Entering this quantity into the expression for \( \alpha \), we get

\[
\alpha = \frac{26.2 \text{ rad/s}}{5.00 \text{ s}} = 5.24 \text{ rad/s}^2.
\]

**Strategy for (b)**

In this part, we know the angular acceleration and the initial angular velocity. We can find the stoppage time by using the definition of angular acceleration and solving for \( \Delta t \), yielding

\[
\Delta t = \frac{\Delta \omega}{\alpha}.
\]

**Solution for (b)**

Here the angular velocity decreases from 26.2 rad/s (250 rpm) to zero, so that \( \Delta \omega \) is \(-26.2 \text{ rad/s} \), and \( \alpha \) is given to be \(-87.3 \text{ rad/s}^2 \). Thus,

\[
\Delta t = \frac{-26.2 \text{ rad/s}}{-87.3 \text{ rad/s}^2} = 0.300 \text{ s}.
\]

**Discussion**

Note that the angular acceleration as the girl spins the wheel is small and positive; it takes 5 s to produce an appreciable angular velocity. When she hits the brake, the angular acceleration is large and negative. The angular velocity quickly goes to zero. In both cases, the relationships are analogous to what happens with linear motion. For example, there is a large
deceleration when you crash into a brick wall—the velocity change is large in a short time interval.

If the bicycle in the preceding example had been on its wheels instead of upside-down, it would first have accelerated along the ground and then come to a stop. This connection between circular motion and linear motion needs to be explored. For example, it would be useful to know how linear and angular acceleration are related. In circular motion, linear acceleration is tangent to the circle at the point of interest, as seen in Figure 10.4. Thus, linear acceleration is called **tangential acceleration** $a_t$.

![Figure 10.4](image)

In circular motion, linear acceleration $a_t$, occurs as the magnitude of the velocity changes: $a_t$ is tangent to the motion. In the context of circular motion, linear acceleration is also called tangential acceleration $a_t$. Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction. We know from *Uniform Circular Motion and Gravitation* that in circular motion centripetal acceleration, $a_c$, refers to changes in the direction of the velocity but not its magnitude. An object undergoing circular motion experiences centripetal acceleration, as seen in Figure 10.5. Thus, $a_t$ and $a_c$ are perpendicular and independent of one another. Tangential acceleration $a_t$ is directly related to the angular acceleration $\alpha$ and is linked to an increase or decrease in the velocity, but not its direction.

![Figure 10.5](image)

Centripetal acceleration $a_c$ occurs as the direction of velocity changes; it is perpendicular to the circular motion. Centripetal and tangential acceleration are thus perpendicular to each other.

Now we can find the exact relationship between linear acceleration $a_t$ and angular acceleration $\alpha$. Because linear acceleration is proportional to a change in the magnitude of the velocity, it is defined (as it was in *One-Dimensional Kinematics*) to be

$$a_t = \frac{\Delta v}{\Delta t}. \quad (10.10)$$

For circular motion, note that $v = r\omega$, so that

$$a_t = \frac{\Delta (r\omega)}{\Delta t}. \quad (10.11)$$

The radius $r$ is constant for circular motion, and so $\Delta (r\omega) = r(\Delta \omega)$. Thus,

$$a_t = r\frac{\Delta \omega}{\Delta t}. \quad (10.12)$$
By definition, \( \alpha = \frac{\Delta \omega}{\Delta t} \). Thus,

\[
a_t = r \alpha,
\]

(10.13)

or

\[
\alpha = \frac{a_t}{r}.
\]

(10.14)

These equations mean that linear acceleration and angular acceleration are directly proportional. The greater the angular acceleration is, the larger the linear (tangential) acceleration is, and vice versa. For example, the greater the angular acceleration of a car’s drive wheels, the greater the acceleration of the car. The radius also matters. For example, the smaller a wheel, the smaller its linear acceleration for a given angular acceleration \( \alpha \).

**Example 10.2 Calculating the Angular Acceleration of a Motorcycle Wheel**

A powerful motorcycle can accelerate from 0 to 30.0 m/s (about 108 km/h) in 4.20 s. What is the angular acceleration of its 0.320-m-radius wheels? (See Figure 10.6.)

![Figure 10.6](image)

**Figure 10.6** The linear acceleration of a motorcycle is accompanied by an angular acceleration of its wheels.

**Strategy**

We are given information about the linear velocities of the motorcycle. Thus, we can find its linear acceleration \( a_t \). Then, the expression \( \alpha = \frac{a_t}{r} \) can be used to find the angular acceleration.

**Solution**

The linear acceleration is

\[
a_t = \frac{\Delta v}{\Delta t} = \frac{30.0 \text{ m/s}}{4.20 \text{ s}} = 7.14 \text{ m/s}^2.
\]

We also know the radius of the wheels. Entering the values for \( a_t \) and \( r \) into \( \alpha = \frac{a_t}{r} \), we get

\[
\alpha = \frac{a_t}{r} = \frac{7.14 \text{ m/s}^2}{0.320 \text{ m}} = 22.3 \text{ rad/s}^2.
\]

**Discussion**

Units of radians are dimensionless and appear in any relationship between angular and linear quantities.

So far, we have defined three rotational quantities— \( \theta \), \( \omega \), and \( \alpha \). These quantities are analogous to the translational quantities \( x \), \( v \), and \( a \). Table 10.1 displays rotational quantities, the analogous translational quantities, and the relationships between them.
### Table 10.1 Rotational and Translational Quantities

<table>
<thead>
<tr>
<th>Rotational</th>
<th>Translational</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>$x$</td>
<td>$\theta = \frac{x}{r}$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$v$</td>
<td>$\omega = \frac{v}{r}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$a$</td>
<td>$\alpha = \frac{a}{r}$</td>
</tr>
</tbody>
</table>

### Making Connections: Take-Home Experiment

Sit down with your feet on the ground on a chair that rotates. Lift one of your legs such that it is unbent (straightened out). Using the other leg, begin to rotate yourself by pushing on the ground. Stop using your leg to push the ground but allow the chair to rotate. From the origin where you began, sketch the angle, angular velocity, and angular acceleration of your leg as a function of time in the form of three separate graphs. Estimate the magnitudes of these quantities.

### Check Your Understanding

Angular acceleration is a vector, having both magnitude and direction. How do we denote its magnitude and direction? Illustrate with an example.

**Solution**

The magnitude of angular acceleration is $\alpha$ and its most common units are rad/s$^2$. The direction of angular acceleration along a fixed axis is denoted by $a^+$ or $a^-$ sign, just as the direction of linear acceleration in one dimension is denoted by $a^+$ or $a^-$ sign. For example, consider a gymnast doing a forward flip. Her angular momentum would be parallel to the mat and to her left. The magnitude of her angular acceleration would be proportional to her angular velocity (spin rate) and her moment of inertia about her spin axis.

### Ladybug Revolution

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug’s $x,y$ position, velocity, and acceleration using vectors or graphs.  

(This media type is not supported in this reader. Click to open media in browser) (http://legacy.cnx.org/content/m42177/1.19/LadybugMotion)

### 10.2 Kinematics of Rotational Motion

Just by using our intuition, we can begin to see how rotational quantities like $\theta$, $\omega$, and $\alpha$ are related to one another. For example, if a motorcycle wheel has a large angular acceleration for a fairly long time, it ends up spinning rapidly and rotates through many revolutions. In more technical terms, if the wheel’s angular acceleration $\alpha$ is large for a long period of time $t$, then the final angular velocity $\omega$ and angle of rotation $\theta$ are large. The wheel’s rotational motion is exactly analogous to the fact that the motorcycle’s large translational acceleration produces a large final velocity, and the distance traveled will also be large.

Kinematics is the description of motion. The **kinematics of rotational motion** describes the relationships among rotation angle, angular velocity, angular acceleration, and time. Let us start by finding an equation relating $\omega$, $\alpha$, and $t$. To determine this equation, we recall a familiar kinematic equation for translational, or straight-line, motion:

$$v = v_0 + at \quad \text{(constant } a\text{)} \quad (10.17)$$

Note that in rotational motion $a = a_t$, and we shall use the symbol $a$ for tangential or linear acceleration from now on. As in linear kinematics, we assume $a$ is constant, which means that angular acceleration $\alpha$ is also a constant, because $a = ra$. Now, let us substitute $v = r\omega$ and $a = ra$ into the linear equation above:

$$r\omega = r\omega_0 + rat \quad (10.18)$$

The radius $r$ cancels in the equation, yielding

$$\omega = \omega_0 + at \quad \text{(constant } a\text{)} \quad (10.19)$$

where $\omega_0$ is the initial angular velocity. This last equation is a kinematic relationship among $\omega$, $\alpha$, and $t$ —that is, it describes their relationship without reference to forces or masses that may affect rotation. It is also precisely analogous in form to its
translational counterpart.

Making Connections

Kinematics for rotational motion is completely analogous to translational kinematics, first presented in One-Dimensional Kinematics. Kinematics is concerned with the description of motion without regard to force or mass. We will find that translational kinematic quantities, such as displacement, velocity, and acceleration have direct analogs in rotational motion.

Starting with the four kinematic equations we developed in One-Dimensional Kinematics, we can derive the following four rotational kinematic equations (presented together with their translational counterparts):

<table>
<thead>
<tr>
<th>Rotational</th>
<th>Translational</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = \bar{\omega}t$</td>
<td>$x = \bar{v}t$</td>
</tr>
<tr>
<td>$\omega = \omega_0 + \alpha t$</td>
<td>$v = v_0 + at$ (constant $\alpha, a$)</td>
</tr>
<tr>
<td>$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$</td>
<td>$x = v_0 t + \frac{1}{2} at^2$ (constant $\alpha, a$)</td>
</tr>
<tr>
<td>$\omega^2 = \omega_0^2 + 2\alpha \theta$</td>
<td>$v^2 = v_0^2 + 2ax$ (constant $\alpha, a$)</td>
</tr>
</tbody>
</table>

In these equations, the subscript 0 denotes initial values ($\theta_0$, $x_0$, and $t_0$ are initial values), and the average angular velocity $\bar{\omega}$ and average velocity $\bar{v}$ are defined as follows:

$$\bar{\omega} = \frac{\omega_0 + \omega}{2} \quad \text{and} \quad \bar{v} = \frac{v_0 + v}{2}.$$  \hspace{1cm} (10.20)

The equations given above in Table 10.2 can be used to solve any rotational or translational kinematics problem in which $a$ and $\alpha$ are constant.

Problem-Solving Strategy for Rotational Kinematics

1. **Examine the situation to determine that rotational kinematics (rotational motion) is involved.** Rotation must be involved, but without the need to consider forces or masses that affect the motion.
2. **Identify exactly what needs to be determined in the problem (identify the unknowns).** A sketch of the situation is useful.
3. **Make a list of what is given or can be inferred from the problem as stated (identify the knowns).**
4. **Solve the appropriate equation or equations for the quantity to be determined (the unknown).** It can be useful to think in terms of a translational analog because by now you are familiar with such motion.
5. **Substitute the known values along with their units into the appropriate equation, and obtain numerical solutions complete with units.** Be sure to use units of radians for angles.
6. **Check your answer to see if it is reasonable: Does your answer make sense?**

**Example 10.3 Calculating the Acceleration of a Fishing Reel**

A deep-sea fisherman hooks a big fish that swims away from the boat pulling the fishing line from his fishing reel. The whole system is initially at rest and the fishing line unwinds from the reel at a radius of 4.50 cm from its axis of rotation. The reel is given an angular acceleration of 110 rad/s² for 2.00 s as seen in Figure 10.8.

(a) What is the final angular velocity of the reel?
(b) At what speed is fishing line leaving the reel after 2.00 s elapses?
(c) How many revolutions does the reel make?
(d) How many meters of fishing line come off the reel in this time?

**Strategy**

In each part of this example, the strategy is the same as it was for solving problems in linear kinematics. In particular, known values are identified and a relationship is then sought that can be used to solve for the unknown.

**Solution for (a)**

Here $\alpha$ and $t$ are given and $\omega$ needs to be determined. The most straightforward equation to use is $\omega = \omega_0 + \alpha t$
because the unknown is already on one side and all other terms are known. That equation states that
\[
\omega = \omega_0 + \alpha t. \tag{10.21}
\]

We are also given that \( \omega_0 = 0 \) (it starts from rest), so that
\[
\omega = 0 + (110 \text{ rad/s}^2)(2.00 \text{s}) = 220 \text{ rad/s}. \tag{10.22}
\]

**Solution for (b)**

Now that \( \omega \) is known, the speed \( v \) can most easily be found using the relationship
\[
v = r\omega, \tag{10.23}
\]
where the radius \( r \) of the reel is given to be 4.50 cm; thus,
\[
v = (0.0450 \text{ m})(220 \text{ rad/s}) = 9.90 \text{ m/s}. \tag{10.24}
\]

Note again that radians must always be used in any calculation relating linear and angular quantities. Also, because radians are dimensionless, we have \( \text{m} \times \text{rad} = \text{m} \).

**Solution for (c)**

Here, we are asked to find the number of revolutions. Because \( 1 \text{ rev} = 2\pi \text{ rad} \), we can find the number of revolutions by finding \( \theta \) in radians. We are given \( \alpha \) and \( t \), and we know \( \omega_0 \) is zero, so that \( \theta \) can be obtained using
\[
\theta = \omega_0 t + \frac{1}{2} \alpha t^2. \tag{10.25}
\]

\[
\theta = \omega_0 t + \frac{1}{2} \alpha t^2 = 0 + (0.500)(110 \text{ rad/s}^2)(2.00 \text{s})^2 = 220 \text{ rad}. \tag{10.26}
\]

Converting radians to revolutions gives
\[
\theta = (220 \text{ rad})\frac{1 \text{ rev}}{2\pi \text{ rad}} = 35.0 \text{ rev}. \tag{10.26}
\]

**Solution for (d)**

The number of meters of fishing line is \( x \), which can be obtained through its relationship with \( \theta \):
\[
x = r\theta = (0.0450 \text{ m})(220 \text{ rad}) = 9.90 \text{ m}. \tag{10.27}
\]

**Discussion**

This example illustrates that relationships among rotational quantities are highly analogous to those among linear quantities. We also see in this example how linear and rotational quantities are connected. The answers to the questions are realistic. After unwinding for two seconds, the reel is found to spin at 220 rad/s, which is 2100 rpm. (No wonder reels sometimes make high-pitched sounds.) The amount of fishing line played out is 9.90 m, about right for when the big fish bites.

---

![Figure 10.8 Fishing line coming off a rotating reel moves linearly. Example 10.3 and Example 10.4 consider relationships between rotational and linear quantities associated with a fishing reel.](image-url)
– 300 rad/s². How long does it take the reel to come to a stop?

**Strategy**

We are asked to find the time \( t \) for the reel to come to a stop. The initial and final conditions are different from those in the previous problem, which involved the same fishing reel. Now we see that the initial angular velocity is \( \omega_0 = 220 \text{ rad/s} \) and the final angular velocity \( \omega \) is zero. The angular acceleration is given to be \( \alpha = -300 \text{ rad/s}^2 \). Examining the available equations, we see all quantities but \( t \) are known in \( \omega = \omega_0 + \alpha t \), making it easiest to use this equation.

**Solution**

The equation states

\[
\omega = \omega_0 + \alpha t. \tag{10.28}
\]

We solve the equation algebraically for \( t \), and then substitute the known values as usual, yielding

\[
t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 220 \text{ rad/s}}{-300 \text{ rad/s}^2} = 0.733 \text{ s}. \tag{10.29}
\]

**Discussion**

Note that care must be taken with the signs that indicate the directions of various quantities. Also, note that the time to stop the reel is fairly small because the acceleration is rather large. Fishing lines sometimes snap because of the accelerations involved, and fishermen often let the fish swim for a while before applying brakes on the reel. A tired fish will be slower, requiring a smaller acceleration.

---

**Example 10.5 Calculating the Slow Acceleration of Trains and Their Wheels**

Large freight trains accelerate very slowly. Suppose one such train accelerates from rest, giving its 0.350-m-radius wheels an angular acceleration of \( 0.250 \text{ rad/s}^2 \). After the wheels have made 200 revolutions (assume no slippage): (a) How far has the train moved down the track? (b) What are the final angular velocity of the wheels and the linear velocity of the train?

**Strategy**

In part (a), we are asked to find \( x \), and in (b) we are asked to find \( \omega \) and \( v \). We are given the number of revolutions \( \theta \), the radius of the wheels \( r \), and the angular acceleration \( \alpha \).

**Solution for (a)**

The distance \( x \) is very easily found from the relationship between distance and rotation angle:

\[
\theta = \frac{x}{r}. \tag{10.30}
\]

Solving this equation for \( x \) yields

\[
x = r\theta. \tag{10.31}
\]

Before using this equation, we must convert the number of revolutions into radians, because we are dealing with a relationship between linear and rotational quantities:

\[
\theta = (200 \text{ rev}) \frac{2\pi \text{ rad}}{1 \text{ rev}} = 1257 \text{ rad}. \tag{10.32}
\]

Now we can substitute the known values into \( x = r\theta \) to find the distance the train moved down the track:

\[
x = r\theta = (0.350 \text{ m})(1257 \text{ rad}) = 440 \text{ m}. \tag{10.33}
\]

**Solution for (b)**

We cannot use any equation that incorporates \( t \) to find \( \omega \), because the equation would have at least two unknown values. The equation \( \omega^2 = \omega_0^2 + 2\alpha \theta \) will work, because we know the values for all variables except \( \omega \):

\[
\omega^2 = \omega_0^2 + 2\alpha \theta \tag{10.34}
\]

Taking the square root of this equation and entering the known values gives

\[
\omega = \left[ 0 + 2(0.250 \text{ rad/s}^2)(1257 \text{ rad}) \right]^{1/2}
\]

\[
= 25.1 \text{ rad/s}. \tag{10.35}
\]
We can find the linear velocity of the train, \( v \), through its relationship to \( \omega \):

\[
v = r\omega = (0.350 \text{ m})(25.1 \text{ rad/s}) = 8.77 \text{ m/s}.
\]  

(10.36)

**Discussion**

The distance traveled is fairly large and the final velocity is fairly slow (just under 32 km/h).

There is translational motion even for something spinning in place, as the following example illustrates. Figure 10.9 shows a fly on the edge of a rotating microwave oven plate. The example below calculates the total distance it travels.

![Figure 10.9](image)

**Example 10.6 Calculating the Distance Traveled by a Fly on the Edge of a Microwave Oven Plate**

A person decides to use a microwave oven to reheat some lunch. In the process, a fly accidentally flies into the microwave and lands on the outer edge of the rotating plate and remains there. If the plate has a radius of 0.15 m and rotates at 6.0 rpm, calculate the total distance traveled by the fly during a 2.0-min cooking period. (Ignore the start-up and slow-down times.)

**Strategy**

First, find the total number of revolutions \( \theta \), and then the linear distance \( x \) traveled. \( \theta = \tilde{\omega}t \) can be used to find \( \theta \) because \( \tilde{\omega} \) is given to be 6.0 rpm.

**Solution**

Entering known values into \( \theta = \tilde{\omega}t \) gives

\[
\theta = \tilde{\omega}t = (6.0 \text{ rpm})(2.0 \text{ min}) = 12 \text{ rev}.
\]  

(10.37)

As always, it is necessary to convert revolutions to radians before calculating a linear quantity like \( x \) from an angular quantity like \( \theta \):

\[
\theta = (12 \text{ rev})\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 75.4 \text{ rad}.
\]  

(10.38)

Now, using the relationship between \( x \) and \( \theta \), we can determine the distance traveled:

\[
x = r\theta = (0.15 \text{ m})(75.4 \text{ rad}) = 11 \text{ m}.
\]  

(10.39)

**Discussion**

Quite a trip (if it survives)! Note that this distance is the total distance traveled by the fly. Displacement is actually zero for complete revolutions because they bring the fly back to its original position. The distinction between total distance traveled and displacement was first noted in *One-Dimensional Kinematics*.

**Check Your Understanding**

Rotational kinematics has many useful relationships, often expressed in equation form. Are these relationships laws of physics or are they simply descriptive? (Hint: the same question applies to linear kinematics.)
Solution
Rotational kinematics (just like linear kinematics) is descriptive and does not represent laws of nature. With kinematics, we can describe many things to great precision but kinematics does not consider causes. For example, a large angular acceleration describes a very rapid change in angular velocity without any consideration of its cause.

10.3 Dynamics of Rotational Motion: Rotational Inertia

If you have ever spun a bike wheel or pushed a merry-go-round, you know that force is needed to change angular velocity as seen in Figure 10.10. In fact, your intuition is reliable in predicting many of the factors that are involved. For example, we know that a door opens slowly if we push too close to its hinges. Furthermore, we know that the more massive the door, the more slowly it opens. The first example implies that the farther the force is applied from the pivot, the greater the angular acceleration; another implication is that angular acceleration is inversely proportional to mass. These relationships should seem very similar to the familiar relationships among force, mass, and acceleration embodied in Newton’s second law of motion. There are, in fact, precise rotational analogs to both force and mass.

To develop the precise relationship among force, mass, radius, and angular acceleration, consider what happens if we exert a force $F$ on a point mass $m$ that is at a distance $r$ from a pivot point, as shown in Figure 10.11. Because the force is perpendicular to $r$, an acceleration $a = \frac{F}{m}$ is obtained in the direction of $F$. We can rearrange this equation such that $F = ma$ and then look for ways to relate this expression to expressions for rotational quantities. We note that $a = r\alpha$, and we substitute this expression into $F = ma$, yielding

$$F = mr\alpha. \quad (10.40)$$

Recall that torque is the turning effectiveness of a force. In this case, because $F$ is perpendicular to $r$, torque is simply $\tau = Fr$. So, if we multiply both sides of the equation above by $r$, we get torque on the left-hand side. That is,

$$rF = mr^2\alpha \quad (10.41)$$
or

$$\tau = mr^2\alpha. \quad (10.42)$$

This last equation is the rotational analog of Newton’s second law ($F = ma$), where torque is analogous to force, angular acceleration is analogous to translational acceleration, and $mr^2$ is analogous to mass (or inertia). The quantity $mr^2$ is called the rotational inertia or moment of inertia of a point mass $m$ a distance $r$ from the center of rotation.

**Figure 10.10** Force is required to spin the bike wheel. The greater the force, the greater the angular acceleration produced. The more massive the wheel, the smaller the angular acceleration. If you push on a spoke closer to the axle, the angular acceleration will be smaller.

**Figure 10.11** An object is supported by a horizontal frictionless table and is attached to a pivot point by a cord that supplies centripetal force. A force $F$ is applied to the object perpendicular to the radius $r$, causing it to accelerate about the pivot point. The force is kept perpendicular to $r$. 

Chapter 10 | Rotational Motion and Angular Momentum
Making Connections: Rotational Motion Dynamics

Dynamics for rotational motion is completely analogous to linear or translational dynamics. Dynamics is concerned with force and mass and their effects on motion. For rotational motion, we will find direct analogs to force and mass that behave just as we would expect from our earlier experiences.

Rotational Inertia and Moment of Inertia

Before we can consider the rotation of anything other than a point mass like the one in Figure 10.11, we must extend the idea of rotational inertia to all types of objects. To expand our concept of rotational inertia, we define the moment of inertia \( I \) of an object to be the sum of \( mr^2 \) for all the point masses of which it is composed. That is, \( I = \sum mr^2 \). Here \( I \) is analogous to \( m \) in translational motion. Because of the distance \( r \), the moment of inertia for any object depends on the chosen axis. Actually, calculating \( I \) is beyond the scope of this text except for one simple case—that of a hoop, which has all its mass at the same distance from its axis. A hoop’s moment of inertia around its axis is therefore \( MR^2 \), where \( M \) is its total mass and \( R \) its radius. (We use \( M \) and \( R \) for an entire object to distinguish them from \( m \) and \( r \) for point masses.) In all other cases, we must consult Figure 10.12 (note that the table is piece of artwork that has shapes as well as formulae) for formulas for \( I \) that have been derived from integration over the continuous body. Note that \( I \) has units of mass multiplied by distance squared (\( \text{kg} \cdot \text{m}^2 \)), as we might expect from its definition.

The general relationship among torque, moment of inertia, and angular acceleration is

\[
\text{net } \tau = I \alpha \tag{10.43}
\]

or

\[
\alpha = \frac{\text{net } \tau}{I}, \tag{10.44}
\]

where net \( \tau \) is the total torque from all forces relative to a chosen axis. For simplicity, we will only consider torques exerted by forces in the plane of the rotation. Such torques are either positive or negative and add like ordinary numbers. The relationship in \( \tau = I \alpha, \alpha = \frac{\text{net } \tau}{I} \) is the rotational analog to Newton’s second law and is very generally applicable. This equation is actually valid for any torque, applied to any object, relative to any axis.

As we might expect, the larger the torque is, the larger the angular acceleration is. For example, the harder a child pushes on a merry-go-round, the faster it accelerates. Furthermore, the more massive a merry-go-round, the slower it accelerates for the same torque. The basic relationship between moment of inertia and angular acceleration is that the larger the moment of inertia, the smaller is the angular acceleration. But there is an additional twist. The moment of inertia depends not only on the mass of an object, but also on its distribution of mass relative to the axis around which it rotates. For example, it will be much easier to accelerate a merry-go-round full of children if they stand close to its axis than if they all stand at the outer edge. The mass is the same in both cases, but the moment of inertia is much larger when the children are at the edge.

Take-Home Experiment

Cut out a circle that has about a 10 cm radius from stiff cardboard. Near the edge of the circle, write numbers 1 to 12 like hours on a clock face. Position the circle so that it can rotate freely about a horizontal axis through its center, like a wheel. (You could loosely nail the circle to a wall.) Hold the circle stationary and with the number 12 positioned at the top, attach a lump of blue putty (sticky material used for fixing posters to walls) at the number 3. How large does the lump need to be to just rotate the circle? Describe how you can change the moment of inertia of the circle. How does this change affect the amount of blue putty needed at the number 3 to just rotate the circle? Change the circle’s moment of inertia and then try rotating the circle by using different amounts of blue putty. Repeat this process several times.

Problem-Solving Strategy for Rotational Dynamics

1. **Examine the situation to determine that torque and mass are involved in the rotation.** Draw a careful sketch of the situation.

2. **Determine the system of interest.**

3. **Draw a free body diagram.** That is, draw and label all external forces acting on the system of interest.

4. **Apply \( \text{net } \tau = I \alpha, \alpha = \frac{\text{net } \tau}{I} \), the rotational equivalent of Newton’s second law, to solve the problem.** Care must be taken to use the correct moment of inertia and to consider the torque about the point of rotation.

5. **As always, check the solution to see if it is reasonable.**
Making Connections

In statics, the net torque is zero, and there is no angular acceleration. In rotational motion, net torque is the cause of angular acceleration, exactly as in Newton’s second law of motion for rotation.

Example 10.7 Calculating the Effect of Mass Distribution on a Merry-Go-Round

Consider the father pushing a playground merry-go-round in Figure 10.13. He exerts a force of 250 N at the edge of the 50.0-kg merry-go-round, which has a 1.50 m radius. Calculate the angular acceleration produced (a) when no one is on the merry-go-round and (b) when an 18.0-kg child sits 1.25 m away from the center. Consider the merry-go-round itself to be a uniform disk with negligible retarding friction.
Figure 10.13 A father pushes a playground merry-go-round at its edge and perpendicular to its radius to achieve maximum torque.

Strategy

Angular acceleration is given directly by the expression \( \alpha = \frac{\text{net } \tau}{I} \):

\[
\alpha = \frac{\tau}{I} \tag{10.45}
\]

To solve for \( \alpha \), we must first calculate the torque \( \tau \) (which is the same in both cases) and moment of inertia \( I \) (which is greater in the second case). To find the torque, we note that the applied force is perpendicular to the radius and friction is negligible, so that

\[
\tau = rF \sin \theta = (1.50 \text{ m})(250 \text{ N}) = 375 \text{ N} \cdot \text{m}. \tag{10.46}
\]

Solution for (a)

The moment of inertia of a solid disk about this axis is given in Figure 10.12 to be

\[
I = \frac{1}{2}MR^2, \tag{10.47}
\]

where \( M = 50.0 \text{ kg} \) and \( R = 1.50 \text{ m} \), so that

\[
I = (0.500)(50.0 \text{ kg})(1.50 \text{ m})^2 = 56.25 \text{ kg} \cdot \text{m}^2. \tag{10.48}
\]

Now, after we substitute the known values, we find the angular acceleration to be

\[
\alpha = \frac{\tau}{I} = \frac{375 \text{ N} \cdot \text{m}}{56.25 \text{ kg} \cdot \text{m}^2} = 6.67 \text{ rad/s}^2 \tag{10.49}
\]

Solution for (b)

We expect the angular acceleration for the system to be less in this part, because the moment of inertia is greater when the child is on the merry-go-round. To find the total moment of inertia \( I \), we first find the child’s moment of inertia \( I_c \) by considering the child to be equivalent to a point mass at a distance of 1.25 m from the axis. Then,

\[
I_c = MR^2 = (18.0 \text{ kg})(1.25 \text{ m})^2 = 28.13 \text{ kg} \cdot \text{m}^2. \tag{10.50}
\]

The total moment of inertia is the sum of moments of inertia of the merry-go-round and the child (about the same axis). To justify this sum to yourself, examine the definition of \( I \):

\[
I = I_c + I = 28.13 \text{ kg} \cdot \text{m}^2 + 56.25 \text{ kg} \cdot \text{m}^2 = 84.38 \text{ kg} \cdot \text{m}^2. \tag{10.51}
\]

Substituting known values into the equation for \( \alpha \) gives

\[
\alpha = \frac{\tau}{I} = \frac{375 \text{ N} \cdot \text{m}}{84.38 \text{ kg} \cdot \text{m}^2} = 4.44 \text{ rad/s}^2 \tag{10.52}
\]

Discussion

The angular acceleration is less when the child is on the merry-go-round than when the merry-go-round is empty, as expected. The angular accelerations found are quite large, partly due to the fact that friction was considered to be negligible. If, for example, the father kept pushing perpendicularly for 2.00 s, he would give the merry-go-round an angular velocity of 13.3 rad/s when it is empty but only 8.89 rad/s when the child is on it. In terms of revolutions per second, these angular velocities are 2.12 rev/s and 1.41 rev/s, respectively. The father would end up running at about 50 km/h in the first case. Summer Olympics, here he comes! Confirmation of these numbers is left as an exercise for the reader.
Check Your Understanding

Torque is the analog of force and moment of inertia is the analog of mass. Force and mass are physical quantities that depend on only one factor. For example, mass is related solely to the numbers of atoms of various types in an object. Are torque and moment of inertia similarly simple?

Solution

No. Torque depends on three factors: force magnitude, force direction, and point of application. Moment of inertia depends on both mass and its distribution relative to the axis of rotation. So, while the analogies are precise, these rotational quantities depend on more factors.

10.4 Rotational Kinetic Energy: Work and Energy Revisited

In this module, we will learn about work and energy associated with rotational motion. Figure 10.14 shows a worker using an electric grindstone propelled by a motor. Sparks are flying, and noise and vibration are created as layers of steel are pared from the pole. The stone continues to turn even after the motor is turned off, but it is eventually brought to a stop by friction. Clearly, the motor had to work to get the stone spinning. This work went into heat, light, sound, vibration, and considerable rotational kinetic energy.

![Figure 10.14](credit: U.S. Navy photo by Mass Communication Specialist Seaman Zachary David Bell)

Work must be done to rotate objects such as grindstones or merry-go-rounds. Work was defined in Uniform Circular Motion and Gravitation for translational motion, and we can build on that knowledge when considering work done in rotational motion. The simplest rotational situation is one in which the net force is exerted perpendicular to the radius of a disk (as shown in Figure 10.15) and remains perpendicular as the disk starts to rotate. The force is parallel to the displacement, and so the net work done is the product of the force times the arc length traveled:

\[
\text{net } W = (\text{net } F)\Delta s. \tag{10.53}
\]

To get torque and other rotational quantities into the equation, we multiply and divide the right-hand side of the equation by \( r \), and gather terms:

\[
\text{net } W = (r \text{ net } F)\frac{\Delta s}{r} = \text{net } \tau \theta. \tag{10.54}
\]

We recognize that \( r \text{ net } F = \text{net } \tau \) and \( \Delta s / r = \theta \), so that

\[
\text{net } W = (\text{net } \tau)\theta. \tag{10.55}
\]

This equation is the expression for rotational work. It is very similar to the familiar definition of translational work as force multiplied by distance. Here, torque is analogous to force, and angle is analogous to distance. The equation \( \text{net } W = (\text{net } \tau)\theta \) is valid in general, even though it was derived for a special case.

To get an expression for rotational kinetic energy, we must again perform some algebraic manipulations. The first step is to note that \( \text{net } \tau = I\alpha \), so that

\[
\text{net } W = I\alpha \theta. \tag{10.56}
\]
The net force on this disk is kept perpendicular to its radius as the force causes the disk to rotate. The net work done is thus $(\text{net } F)\Delta s$. The net work goes into rotational kinetic energy.

Making Connections

Work and energy in rotational motion are completely analogous to work and energy in translational motion, first presented in Uniform Circular Motion and Gravitation.

Now, we solve one of the rotational kinematics equations for $\alpha \theta$. We start with the equation

$$\omega^2 = \omega_0^2 + 2\alpha \theta.$$  \hspace{1cm} (10.57)

Next, we solve for $\alpha \theta$:

$$\alpha \theta = \frac{\omega^2 - \omega_0^2}{2}.$$  \hspace{1cm} (10.58)

Substituting this into the equation for net $W$ and gathering terms yields

$$\text{net } W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2.$$  \hspace{1cm} (10.59)

This equation is the work-energy theorem for rotational motion only. As you may recall, net work changes the kinetic energy of a system. Through an analogy with translational motion, we define the term $\frac{1}{2}I\omega^2$ to be rotational kinetic energy $\text{KE}_{\text{rot}}$ for an object with a moment of inertia $I$ and an angular velocity $\omega$:

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2.$$  \hspace{1cm} (10.60)

The expression for rotational kinetic energy is exactly analogous to translational kinetic energy, with $I$ being analogous to $m$ and $\omega$ to $v$. Rotational kinetic energy has important effects. Flywheels, for example, can be used to store large amounts of rotational kinetic energy in a vehicle, as seen in Figure 10.16.

Example 10.8 Calculating the Work and Energy for Spinning a Grindstone

Consider a person who spins a large grindstone by placing her hand on its edge and exerting a force through part of a revolution as shown in Figure 10.17. In this example, we verify that the work done by the torque she exerts equals the change in rotational energy. (a) How much work is done if she exerts a force of 200 N through a rotation of 1.00 rad(57.3°)
The force is kept perpendicular to the grindstone’s 0.320-m radius at the point of application, and the effects of friction are negligible. (b) What is the final angular velocity if the grindstone has a mass of 85.0 kg? (c) What is the final rotational kinetic energy? (It should equal the work.)

**Strategy**

To find the work, we can use the equation \( \text{net } W = (\text{net } \tau)\theta \). We have enough information to calculate the torque and are given the rotation angle. In the second part, we can find the final angular velocity using one of the kinematic relationships. In the last part, we can calculate the rotational kinetic energy from its expression in \( KE_{\text{rot}} = \frac{1}{2}I\omega^2 \).

**Solution for (a)**

The net work is expressed in the equation

\[
\text{net } W = (\text{net } \tau)\theta, 
\]

(10.61)

where \( \tau \) is the applied force multiplied by the radius \( (rF) \) because there is no retarding friction, and the force is perpendicular to \( r \). The angle \( \theta \) is given. Substituting the given values in the equation above yields

\[
\text{net } W = rF\theta = (0.320 \text{ m})(200 \text{ N})(1.00 \text{ rad}) 
= 64.0 \text{ N} \cdot \text{m}.
\]

Noting that \( 1 \text{ N} \cdot \text{m} = 1 \text{ J} \),

\[
\text{net } W = 64.0 \text{ J}.
\]

(10.63)

**Figure 10.17** A large grindstone is given a spin by a person grasping its outer edge.

**Solution for (b)**

To find \( \omega \) from the given information requires more than one step. We start with the kinematic relationship in the equation

\[
\omega^2 = \omega_0^2 + 2\alpha\theta.
\]

(10.64)

Note that \( \omega_0 = 0 \) because we start from rest. Taking the square root of the resulting equation gives

\[
\omega = (2\alpha\theta)^{1/2}.
\]

(10.65)

Now we need to find \( \alpha \). One possibility is

\[
\alpha = \frac{\text{net } \tau}{I},
\]

(10.66)

where the torque is

\[
\text{net } \tau = rF = (0.320 \text{ m})(200 \text{ N}) = 64.0 \text{ N} \cdot \text{m}.
\]

(10.67)

The formula for the moment of inertia for a disk is found in Figure 10.12:

\[
I = \frac{1}{2}MR^2 = 0.5(85.0 \text{ kg})(0.320 \text{ m})^2 = 4.352 \text{ kg} \cdot \text{m}^2.
\]

(10.68)

Substituting the values of torque and moment of inertia into the expression for \( \alpha \), we obtain
\[
\alpha = \frac{64.0 \text{ N} \cdot \text{m}}{4.352 \text{ kg} \cdot \text{m}^2} = 14.7 \text{ rad/s}^2
\] (10.69)

Now, substitute this value and the given value for \( \theta \) into the above expression for \( \omega \):

\[
\omega = (2\alpha \theta)^{1/2} = \left[2 \left(14.7 \text{ rad/s}^2 \cdot 1.00 \text{ rad}\right)\right]^{1/2} = 5.42 \text{ rad/s}.
\] (10.70)

**Solution for (c)**

The final rotational kinetic energy is

\[
\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2
\] (10.71)

Both \( I \) and \( \omega \) were found above. Thus,

\[
\text{KE}_{\text{rot}} = \left(0.5\right)\left(4.352 \text{ kg} \cdot \text{m}^2\right)(5.42 \text{ rad/s})^2 = 64.0 \text{ J}.
\] (10.72)

**Discussion**

The final rotational kinetic energy equals the work done by the torque, which confirms that the work done went into rotational kinetic energy. We could, in fact, have used an expression for energy instead of a kinematic relation to solve part (b). We will do this in later examples.

Helicopter pilots are quite familiar with rotational kinetic energy. They know, for example, that a point of no return will be reached if they allow their blades to slow below a critical angular velocity during flight. The blades lose lift, and it is impossible to immediately get the blades spinning fast enough to regain it. Rotational kinetic energy must be supplied to the blades to get them to rotate faster, and enough energy cannot be supplied in time to avoid a crash. Because of weight limitations, helicopter engines are too small to supply both the energy needed for lift and to replenish the rotational kinetic energy of the blades once they have slowed down. The rotational kinetic energy is put into them before takeoff and must not be allowed to drop below this crucial level. One possible way to avoid a crash is to use the gravitational potential energy of the helicopter to replenish the rotational kinetic energy of the blades by losing altitude and aligning the blades so that the helicopter is spun up in the descent. Of course, if the helicopter’s altitude is too low, then there is insufficient time for the blade to regain lift before reaching the ground.

**Problem-Solving Strategy for Rotational Energy**

1. Determine that energy or work is involved in the rotation.
2. Determine the system of interest. A sketch usually helps.
3. Analyze the situation to determine the types of work and energy involved.
4. For closed systems, mechanical energy is conserved. That is, \( \text{KE}_i + \text{PE}_i = \text{KE}_f + \text{PE}_f \). Note that \( \text{KE}_i \) and \( \text{KE}_f \) may each include translational and rotational contributions.
5. For open systems, mechanical energy may not be conserved, and other forms of energy (referred to previously as \( \text{OE} \)), such as heat transfer, may enter or leave the system. Determine what they are, and calculate them as necessary.
6. Eliminate terms wherever possible to simplify the algebra.
7. Check the answer to see if it is reasonable.

**Example 10.9 Calculating Helicopter Energies**

A typical small rescue helicopter, similar to the one in Figure 10.18, has four blades, each is 4.00 m long and has a mass of 50.0 kg. The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg. (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm. (b) Calculate the translational kinetic energy of the helicopter when it flies at 20.0 m/s, and compare it with the rotational energy in the blades. (c) To what height could the helicopter be raised if all of the rotational kinetic energy could be used to lift it?

**Strategy**

Rotational and translational kinetic energies can be calculated from their definitions. The last part of the problem relates to the idea that energy can change form, in this case from rotational kinetic energy to gravitational potential energy.

**Solution for (a)**

The rotational kinetic energy is
\[ KE_{\text{rot}} = \frac{1}{2} I \omega^2. \]  

(10.73)

We must convert the angular velocity to radians per second and calculate the moment of inertia before we can find \( KE_{\text{rot}} \).

The angular velocity \( \omega \) is

\[ \omega = \frac{300 \text{ rev}}{1.00 \text{ min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} \cdot \frac{1.00 \text{ min}}{60.0 \text{ s}} = 31.4 \text{ rad/s}. \]  

(10.74)

The moment of inertia of one blade will be that of a thin rod rotated about its end, found in Figure 10.12. The total \( I \) is four times this moment of inertia, because there are four blades. Thus,

\[ I = 4 \left( \frac{M \ell^2}{3} \right) = 4 \times \left( \frac{50.0 \text{ kg} \cdot (4.00 \text{ m})^2}{3} \right) = 1067 \text{ kg} \cdot \text{m}^2. \]  

(10.75)

Entering \( \omega \) and \( I \) into the expression for rotational kinetic energy gives

\[ KE_{\text{rot}} = 0.5 \left( 1067 \text{ kg} \cdot \text{m}^2 \right) (31.4 \text{ rad/s})^2 \]  

\[ = 5.26 \times 10^5 \text{ J}. \]  

(10.76)

**Solution for (b)**

Translational kinetic energy was defined in *Uniform Circular Motion and Gravitation*. Entering the given values of mass and velocity, we obtain

\[ KE_{\text{trans}} = \frac{1}{2} m v^2 = (0.5)(1000 \text{ kg})(20.0 \text{ m/s})^2 = 2.00 \times 10^5 \text{ J}. \]  

(10.77)

To compare kinetic energies, we take the ratio of translational kinetic energy to rotational kinetic energy. This ratio is

\[ \frac{2.00 \times 10^5 \text{ J}}{5.26 \times 10^5 \text{ J}} = 0.380. \]  

(10.78)

**Solution for (c)**

At the maximum height, all rotational kinetic energy will have been converted to gravitational energy. To find this height, we equate those two energies:

\[ KE_{\text{rot}} = PE_{\text{grav}} \]  

(10.79)

or

\[ \frac{1}{2} I \omega^2 = mgh. \]  

(10.80)

We now solve for \( h \) and substitute known values into the resulting equation

\[ h = \frac{\frac{1}{2} I \omega^2}{mg} = \frac{5.26 \times 10^5 \text{ J}}{(1000 \text{ kg})(9.80 \text{ m/s}^2)} = 53.7 \text{ m}. \]  

(10.81)

**Discussion**

The ratio of translational energy to rotational kinetic energy is only 0.380. This ratio tells us that most of the kinetic energy of the helicopter is in its spinning blades—something you probably would not suspect. The 53.7 m height to which the helicopter could be raised with the rotational kinetic energy is also impressive, again emphasizing the amount of rotational kinetic energy in the blades.
The first image shows how helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades. The second image shows a helicopter from the Auckland Westpac Rescue Helicopter Service. Over 50,000 lives have been saved since its operations beginning in 1973. Here, a water rescue operation is shown. (credit: 111 Emergency, Flickr)

Making Connections

Conservation of energy includes rotational motion, because rotational kinetic energy is another form of $KE$. Uniform Circular Motion and Gravitation has a detailed treatment of conservation of energy.

How Thick Is the Soup? Or Why Don’t All Objects Roll Downhill at the Same Rate?

One of the quality controls in a tomato soup factory consists of rolling filled cans down a ramp. If they roll too fast, the soup is too thin. Why should cans of identical size and mass roll down an incline at different rates? And why should the thickest soup roll the slowest?

The easiest way to answer these questions is to consider energy. Suppose each can starts down the ramp from rest. Each can starting from rest means each starts with the same gravitational potential energy $PE_{grav}$, which is converted entirely to $KE$, provided each rolls without slipping. $KE$, however, can take the form of $KE_{trans}$ or $KE_{rot}$, and total $KE$ is the sum of the two. If a can rolls down a ramp, it puts part of its energy into rotation, leaving less for translation. Thus, the can goes slower than it would if it slid down. Furthermore, the thin soup does not rotate, whereas the thick soup does, because it sticks to the can. The thick soup thus puts more of the can’s original gravitational potential energy into rotation than the thin soup, and the can rolls more slowly, as seen in Figure 10.19.

Assuming no losses due to friction, there is only one force doing work—gravity. Therefore the total work done is the change in kinetic energy. As the cans start moving, the potential energy is changing into kinetic energy. Conservation of energy gives

$$PE_i = KE_f,$$  \hspace{1cm} (10.82)

More specifically,
\[ PE_{\text{grav}} = KE_{\text{trans}} + KE_{\text{rot}} \]  
\[ \text{or} \]
\[ mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2. \]

So, the initial \( mgh \) is divided between translational kinetic energy and rotational kinetic energy; and the greater \( I \) is, the less energy goes into translation. If the can slides down without friction, then \( \omega = 0 \) and all the energy goes into translation; thus, the can goes faster.

### Take-Home Experiment
Locate several cans each containing different types of food. First, predict which can will win the race down an inclined plane and explain why. See if your prediction is correct. You could also do this experiment by collecting several empty cylindrical containers of the same size and filling them with different materials such as wet or dry sand.

### Example 10.10 Calculating the Speed of a Cylinder Rolling Down an Incline

Calculate the final speed of a solid cylinder that rolls down a 2.00-m-high incline. The cylinder starts from rest, has a mass of 0.750 kg, and has a radius of 4.00 cm.

**Strategy**
We can solve for the final velocity using conservation of energy, but we must first express rotational quantities in terms of translational quantities to end up with \( v \) as the only unknown.

**Solution**
Conservation of energy for this situation is written as described above:
\[ mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2. \]  

Before we can solve for \( v \), we must get an expression for \( I \) from Figure 10.12. Because \( v \) and \( \omega \) are related (note here that the cylinder is rolling without slipping), we must also substitute the relationship \( \omega = v/R \) into the expression. These substitutions yield
\[ mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v^2}{R^2}\right). \]

Interestingly, the cylinder’s radius \( R \) and mass \( m \) cancel, yielding
\[ gh = \frac{1}{2}v^2 + \frac{1}{4}v^2 = \frac{3}{4}v^2. \]

Solving algebraically, the equation for the final velocity \( v \) gives
\[ v = \left( \frac{4gh}{3} \right)^{1/2}. \]

Substituting known values into the resulting expression yields
\[ v = \left[ \frac{4(9.80 \text{ m/s}^2)(2.00 \text{ m})}{3} \right]^{1/2} = 5.11 \text{ m/s}. \]

**Discussion**
Because \( m \) and \( R \) cancel, the result \( v = \left( \frac{4}{3}gh \right)^{1/2} \) is valid for any solid cylinder, implying that all solid cylinders will roll down an incline at the same rate independent of their masses and sizes. (Rolling cylinders down inclines is what Galileo actually did to show that objects fall at the same rate independent of mass.) Note that if the cylinder slid without friction down the incline without rolling, then the entire gravitational potential energy would go into translational kinetic energy. Thus, \( \frac{1}{2}mv^2 = mgh \) and \( v = (2gh)^{1/2} \), which is 22\% greater than \( \left( \frac{4gh}{3} \right)^{1/2} \). That is, the cylinder would go faster at the bottom.
Check Your Understanding

Analogy of Rotational and Translational Kinetic Energy

Is rotational kinetic energy completely analogous to translational kinetic energy? What, if any, are their differences? Give an example of each type of kinetic energy.

Solution

Yes, rotational and translational kinetic energy are exact analogs. They both are the energy of motion involved with the coordinated (non-random) movement of mass relative to some reference frame. The only difference between rotational and translational kinetic energy is that translational is straight line motion while rotational is not. An example of both kinetic and translational kinetic energy is found in a bike tire while being ridden down a bike path. The rotational motion of the tire means it has rotational kinetic energy while the movement of the bike along the path means the tire also has translational kinetic energy. If you were to lift the front wheel of the bike and spin it while the bike is stationary, then the wheel would have only rotational kinetic energy relative to the Earth.

My Solar System

Build your own system of heavenly bodies and watch the gravitational ballet. With this orbit simulator, you can set initial positions, velocities, and masses of 2, 3, or 4 bodies, and then see them orbit each other.

10.5 Angular Momentum and Its Conservation

Why does Earth keep on spinning? What started it spinning to begin with? And how does an ice skater manage to spin faster and faster simply by pulling her arms in? Why does she not have to exert a torque to spin faster? Questions like these have answers based in angular momentum, the rotational analog to linear momentum.

By now the pattern is clear—every rotational phenomenon has a direct translational analog. It seems quite reasonable, then, to define angular momentum \( L \) as

\[
L = I \omega.
\]  

(10.90)

This equation is an analog to the definition of linear momentum as \( p = mv \). Units for linear momentum are \( \text{kg} \cdot \text{m/s} \) while units for angular momentum are \( \text{kg} \cdot \text{m}^2/\text{s} \). As we would expect, an object that has a large moment of inertia \( I \), such as Earth, has a very large angular momentum. An object that has a large angular velocity \( \omega \), such as a centrifuge, also has a rather large angular momentum.

Making Connections

Angular momentum is completely analogous to linear momentum, first presented in Uniform Circular Motion and Gravitation. It has the same implications in terms of carrying rotation forward, and it is conserved when the net external torque is zero. Angular momentum, like linear momentum, is also a property of the atoms and subatomic particles.

Example 10.11 Calculating Angular Momentum of the Earth

Strategy

No information is given in the statement of the problem; so we must look up pertinent data before we can calculate \( L = I \omega \). First, according to Figure 10.12, the formula for the moment of inertia of a sphere is

\[
I = \frac{2MR^2}{5}.
\]  

(10.91)

so that

\[
L = I \omega = \frac{2MR^2 \omega}{5}.
\]  

(10.92)

Earth’s mass \( M \) is \( 5.979 \times 10^{24} \text{ kg} \) and its radius \( R \) is \( 6.376 \times 10^6 \text{ m} \). The Earth’s angular velocity \( \omega \) is, of course, exactly one revolution per day, but we must convert \( \omega \) to radians per second to do the calculation in SI units.

Solution

Substituting known information into the expression for \( L \) and converting \( \omega \) to radians per second gives
\[ L = 0.4 \times (5.979 \times 10^{24} \text{ kg}) \times (6.376 \times 10^6 \text{ m})^2 \times \left( \frac{1 \text{ rev}}{d} \right) \]
\[ = 9.72 \times 10^{37} \text{ kg} \cdot \text{m}^2 \cdot \text{rev/d}. \]

Substituting \(2\pi\ \text{rad}\) for \(1\ \text{rev}\) and \(8.64 \times 10^4 \text{ s}\) for 1 day gives
\[ L = \left( 9.72 \times 10^{37} \text{ kg} \cdot \text{m}^2 \right) \times \left( \frac{2\pi \text{ rad/rev}}{8.64 \times 10^4 \text{ s/d}} \right) \times (1 \text{ rev/d}) \]
\[ = 7.07 \times 10^{33} \text{ kg} \cdot \text{m}^2/\text{s}. \]

**Discussion**

This number is large, demonstrating that Earth, as expected, has a tremendous angular momentum. The answer is approximate, because we have assumed a constant density for Earth in order to estimate its moment of inertia.

When you push a merry-go-round, spin a bike wheel, or open a door, you exert a torque. If the torque you exert is greater than opposing torques, then the rotation accelerates, and angular momentum increases. The greater the net torque, the more rapid the increase in \(L\). The relationship between torque and angular momentum is

\[ \text{net } \tau = \frac{\Delta L}{\Delta t}. \]

This expression is exactly analogous to the relationship between force and linear momentum, \(F = \frac{\Delta p}{\Delta t}\). The equation \(\text{net } \tau = \frac{\Delta L}{\Delta t}\) is very fundamental and broadly applicable. It is, in fact, the rotational form of Newton’s second law.
Example 10.12 Calculating the Torque Putting Angular Momentum Into a Lazy Susan

Figure 10.21 shows a Lazy Susan food tray being rotated by a person in quest of sustenance. Suppose the person exerts a 2.50 N force perpendicular to the lazy Susan’s 0.260-m radius for 0.150 s. (a) What is the final angular momentum of the lazy Susan if it starts from rest, assuming friction is negligible? (b) What is the final angular velocity of the lazy Susan, given that its mass is 4.00 kg and assuming its moment of inertia is that of a disk?

Figure 10.21 A partygoer exerts a torque on a lazy Susan to make it rotate. The equation \( \tau = \frac{\Delta L}{\Delta t} \) gives the relationship between torque and the angular momentum produced.

Strategy

We can find the angular momentum by solving \( \tau = \frac{\Delta L}{\Delta t} \) for \( \Delta L \), and using the given information to calculate the torque. The final angular momentum equals the change in angular momentum, because the lazy Susan starts from rest. That is, \( \Delta L = L \). To find the final velocity, we must calculate \( \omega \) from the definition of \( L \) in \( L = I \omega \).

Solution for (a)

Solving \( \tau = \frac{\Delta L}{\Delta t} \) for \( \Delta L \) gives

\[ \Delta L = (\text{net } \tau) \Delta t. \] (10.96)

Because the force is perpendicular to \( r \), we see that \( \text{net } \tau = rF \), so that

\[ L = rF \Delta t = (0.260 \text{ m})(2.50 \text{ N})(0.150 \text{ s}) \]

\[ = 9.75 \times 10^{-2} \text{ kg} \cdot \text{m}^2/\text{s}. \] (10.97)

Solution for (b)

The final angular velocity can be calculated from the definition of angular momentum,

\[ L = I \omega. \] (10.98)

Solving for \( \omega \) and substituting the formula for the moment of inertia of a disk into the resulting equation gives

\[ \omega = \frac{L}{I} = \frac{L}{\frac{1}{2}MR^2}. \] (10.99)

And substituting known values into the preceding equation yields

\[ \omega = \frac{9.75 \times 10^{-2} \text{ kg} \cdot \text{m}^2/\text{s}}{(0.500)(4.00 \text{ kg})(0.260 \text{ m})^2} = 0.721 \text{ rad/s}. \] (10.100)

Discussion

Note that the imparted angular momentum does not depend on any property of the object but only on torque and time. The final angular velocity is equivalent to one revolution in 8.71 s (determination of the time period is left as an exercise for the reader), which is about right for a lazy Susan.

Example 10.13 Calculating the Torque in a Kick

The person whose leg is shown in Figure 10.22 kicks his leg by exerting a 2000-N force with his upper leg muscle. The effective perpendicular lever arm is 2.20 cm. Given the moment of inertia of the lower leg is 1.25 kg \cdot m^2, (a) find the angular acceleration of the leg. (b) Neglecting the gravitational force, what is the rotational kinetic energy of the leg after it has rotated through 57.3\(^\circ\) (1.00 rad)?
The muscle in the upper leg gives the lower leg an angular acceleration and imparts rotational kinetic energy to it by exerting a torque about the knee. $\mathbf{F}$ is a vector that is perpendicular to $\mathbf{r}$. This example examines the situation.

**Strategy**

The angular acceleration can be found using the rotational analog to Newton's second law, or $\alpha = \text{net } \tau / I$. The moment of inertia $I$ is given and the torque can be found easily from the given force and perpendicular lever arm. Once the angular acceleration $\alpha$ is known, the final angular velocity and rotational kinetic energy can be calculated.

**Solution to (a)**

From the rotational analog to Newton's second law, the angular acceleration $\alpha$ is

$$\alpha = \frac{\text{net } \tau}{I}. \quad (10.101)$$

Because the force and the perpendicular lever arm are given and the leg is vertical so that its weight does not create a torque, the net torque is thus

$$\text{net } \tau = r_\perp F = (0.0220 \text{ m})(2000 \text{ N}) = 44.0 \text{ N} \cdot \text{m}. \quad (10.102)$$

Substituting this value for the torque and the given value for the moment of inertia into the expression for $\alpha$ gives

$$\alpha = \frac{44.0 \text{ N} \cdot \text{m}}{1.25 \text{ kg} \cdot \text{m}^2} = 35.2 \text{ rad/s}^2. \quad (10.103)$$

**Solution to (b)**

The final angular velocity can be calculated from the kinematic expression

$$\omega^2 = \omega_0^2 + 2\alpha \theta \quad (10.104)$$

or

$$\omega^2 = 2\alpha \theta \quad (10.105)$$

because the initial angular velocity is zero. The kinetic energy of rotation is

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2 \quad (10.106)$$

so it is most convenient to use the value of $\omega^2$ just found and the given value for the moment of inertia. The kinetic energy is then

$$\text{KE}_{\text{rot}} = 0.5\left(1.25 \text{ kg} \cdot \text{m}^2\right)(70.4 \text{ rad}^2/\text{s}^2) = 44.0 \text{ J}. \quad (10.107)$$

**Discussion**

These values are reasonable for a person kicking his leg starting from the position shown. The weight of the leg can be neglected in part (a) because it exerts no torque when the center of gravity of the lower leg is directly beneath the pivot in the knee. In part (b), the force exerted by the upper leg is so large that its torque is much greater than that created by the weight of the lower leg as it rotates. The rotational kinetic energy given to the lower leg is enough that it could give a ball a significant velocity by transferring some of this energy in a kick.
Making Connections: Conservation Laws

Angular momentum, like energy and linear momentum, is conserved. This universally applicable law is another sign of underlying unity in physical laws. Angular momentum is conserved when net external torque is zero, just as linear momentum is conserved when the net external force is zero.

Conservation of Angular Momentum

We can now understand why Earth keeps on spinning. As we saw in the previous example, \( \Delta L = (\text{net } \tau) \Delta t \). This equation means that, to change angular momentum, a torque must act over some period of time. Because Earth has a large angular momentum, a large torque acting over a long time is needed to change its rate of spin. So what external torques are there? Tidal friction exerts torque that is slowing Earth’s rotation, but tens of millions of years must pass before the change is very significant. Recent research indicates the length of the day was 18 h some 900 million years ago. Only the tides exert significant retarding torques on Earth, and so it will continue to spin, although ever more slowly, for many billions of years.

What we have here is, in fact, another conservation law. If the net torque is \( 0 \), then angular momentum is constant or conserved. We can see this rigorously by considering \( \text{net } \tau = \frac{\Delta L}{\Delta t} \) for the situation in which the net torque is zero. In that case,

\[
\text{net } \tau = 0
\]

implying that

\[
\frac{\Delta L}{\Delta t} = 0. \tag{10.109}
\]

If the change in angular momentum \( \Delta L \) is zero, then the angular momentum is constant; thus,

\[
L = \text{constant } (\text{net } \tau = 0) \tag{10.110}
\]

or

\[
L = L' (\text{net } \tau = 0). \tag{10.111}
\]

These expressions are the law of conservation of angular momentum. Conservation laws are as scarce as they are important.

An example of conservation of angular momentum is seen in Figure 10.23, in which an ice skater is executing a spin. The net torque on her is very close to zero, because there is relatively little friction between her skates and the ice and because the friction is exerted very close to the pivot point. (Both \( F \) and \( r \) are small, and so \( \tau \) is negligibly small.) Consequently, she can spin for quite some time. She can do something else, too. She can increase her rate of spin by pulling her arms and legs in. Why does pulling her arms and legs in increase her rate of spin? The answer is that her angular momentum is constant, so that

\[
L = L'. \tag{10.112}
\]

Expressing this equation in terms of the moment of inertia,

\[
I\omega = I'\omega'. \tag{10.113}
\]

where the primed quantities refer to conditions after she has pulled in her arms and reduced her moment of inertia. Because \( I' \) is smaller, the angular velocity \( \omega' \) must increase to keep the angular momentum constant. The change can be dramatic, as the following example shows.

Figure 10.23 (a) An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is negligibly small. In the next image, her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.
Example 10.14 Calculating the Angular Momentum of a Spinning Skater

Suppose an ice skater, such as the one in Figure 10.23, is spinning at 0.800 rev/s with her arms extended. She has a moment of inertia of $2.34 \text{ kg} \cdot \text{m}^2$ with her arms extended and of $0.363 \text{ kg} \cdot \text{m}^2$ with her arms close to her body. (These moments of inertia are based on reasonable assumptions about a 60.0-kg skater.) (a) What is her angular velocity in revolutions per second after she pulls in her arms? (b) What is her rotational kinetic energy before and after she does this?

**Strategy**

In the first part of the problem, we are looking for the skater’s angular velocity $\omega'$ after she has pulled in her arms. To find this quantity, we use the conservation of angular momentum and note that the moments of inertia and initial angular velocity are given. To find the initial and final kinetic energies, we use the definition of rotational kinetic energy given by

$$ KE_{\text{rot}} = \frac{1}{2}I\omega^2. $$

(a) **Solution for (a)**

Because torque is negligible (as discussed above), the conservation of angular momentum given in $L = L'$ is applicable. Thus,

$$ L = L' $$

or

$$ I\omega = I'\omega' $$

Solving for $\omega'$ and substituting known values into the resulting equation gives

$$ \omega' = \frac{I}{I'} \omega = \left( \frac{2.34 \text{ kg} \cdot \text{m}^2}{0.363 \text{ kg} \cdot \text{m}^2} \right)(0.800 \text{ rev/s}) $$

$$ = 5.16 \text{ rev/s}. $$

(b) **Solution for (b)**

Rotational kinetic energy is given by

$$ KE_{\text{rot}} = \frac{1}{2}I\omega^2. $$

The initial value is found by substituting known values into the equation and converting the angular velocity to rad/s:

$$ KE_{\text{rot}} = (0.5)(2.34 \text{ kg} \cdot \text{m}^2)(0.800 \text{ rev/s})(2\pi \text{ rad/rev})^2 $$

$$ = 29.6 \text{ J}. $$

The final rotational kinetic energy is

$$ KE_{\text{rot}}' = \frac{1}{2}I'\omega'^2. $$

Substituting known values into this equation gives

$$ KE_{\text{rot}}' = (0.5)(0.363 \text{ kg} \cdot \text{m}^2)(5.16 \text{ rev/s})(2\pi \text{ rad/rev})^2 $$

$$ = 191 \text{ J}. $$

**Discussion**

In both parts, there is an impressive increase. First, the final angular velocity is large, although most world-class skaters can achieve spin rates about this great. Second, the final kinetic energy is much greater than the initial kinetic energy. The increase in rotational kinetic energy comes from work done by the skater in pulling in her arms. This work is internal work that depletes some of the skater’s food energy.

There are several other examples of objects that increase their rate of spin because something reduced their moment of inertia. Tornadoes are one example. Storm systems that create tornadoes are slowly rotating. When the radius of rotation narrows, even in a local region, angular velocity increases, sometimes to the furious level of a tornado. Earth is another example. Our planet was born from a huge cloud of gas and dust, the rotation of which came from turbulence in an even larger cloud. Gravitational forces caused the cloud to contract, and the rotation rate increased as a result. (See Figure 10.24.)
The Solar System coalesced from a cloud of gas and dust that was originally rotating. The orbital motions and spins of the planets are in the same direction as the original spin and conserve the angular momentum of the parent cloud.

In case of human motion, one would not expect angular momentum to be conserved when a body interacts with the environment as its foot pushes off the ground. Astronauts floating in space aboard the International Space Station have no angular momentum relative to the inside of the ship if they are motionless. Their bodies will continue to have this zero value no matter how they twist about as long as they do not give themselves a push off the side of the vessel.

Check Your Understanding

Is angular momentum completely analogous to linear momentum? What, if any, are their differences?

Solution

Yes, angular and linear momentums are completely analogous. While they are exact analogs they have different units and are not directly inter-convertible like forms of energy are.

10.6 Collisions of Extended Bodies in Two Dimensions

Bowling pins are sent flying and spinning when hit by a bowling ball—angular momentum as well as linear momentum and energy have been imparted to the pins. (See Figure 10.25). Many collisions involve angular momentum. Cars, for example, may spin and collide on ice or a wet surface. Baseball pitchers throw curves by putting spin on the baseball. A tennis player can put a lot of top spin on the tennis ball which causes it to dive down onto the court once it crosses the net. We now take a brief look at what happens when objects that can rotate collide.

Consider the relatively simple collision shown in Figure 10.26, in which a disk strikes and adheres to an initially motionless stick nailed at one end to a frictionless surface. After the collision, the two rotate about the nail. There is an unbalanced external force on the system at the nail. This force exerts no torque because its lever arm \( r \) is zero. Angular momentum is therefore conserved in the collision. Kinetic energy is not conserved, because the collision is inelastic. It is possible that momentum is not conserved either because the force at the nail may have a component in the direction of the disk’s initial velocity. Let us examine a case of rotation in a collision in Example 10.15.

Figure 10.25 The bowling ball causes the pins to fly, some of them spinning violently. (credit: Tinou Bao, Flickr)
Figure 10.26 (a) A disk slides toward a motionless stick on a frictionless surface. (b) The disk hits the stick at one end and adheres to it, and they rotate together, pivoting around the nail. Angular momentum is conserved for this inelastic collision because the surface is frictionless and the unbalanced external force at the nail exerts no torque.

Example 10.15 Rotation in a Collision

Suppose the disk in Figure 10.26 has a mass of 50.0 g and an initial velocity of 30.0 m/s when it strikes the stick that is 1.20 m long and 2.00 kg.

(a) What is the angular velocity of the two after the collision?
(b) What is the kinetic energy before and after the collision?
(c) What is the total linear momentum before and after the collision?

Strategy for (a)

We can answer the first question using conservation of angular momentum as noted. Because angular momentum is \( I \omega \), we can solve for angular velocity.

Solution for (a)

Conservation of angular momentum states

\[
L = L',
\]

where primed quantities stand for conditions after the collision and both momenta are calculated relative to the pivot point. The initial angular momentum of the system of stick-disk is that of the disk just before it strikes the stick. That is,

\[
L = I_0 \omega,
\]

where \( I \) is the moment of inertia of the disk and \( \omega \) is its angular velocity around the pivot point. Now, \( I = mr^2 \) (taking the disk to be approximately a point mass) and \( \omega = v/r \), so that

\[
L = mr^2 \frac{v}{r} = mvr.
\]

After the collision,

\[
L' = I_0 \omega'.
\]

It is \( \omega' \) that we wish to find. Conservation of angular momentum gives

\[
I' \omega' = mvr.
\]

Rearranging the equation yields

\[
\omega' = \frac{mvr}{I'},
\]

where \( I' \) is the moment of inertia of the stick and disk stuck together, which is the sum of their individual moments of inertia about the nail. Figure 10.12 gives the formula for a rod rotating around one end to be \( I = Mr^2/3 \). Thus,

\[
I' = mr^2 + \frac{M}{3} r^2 = (m + M/3)^2.
\]

Entering known values in this equation yields,

\[
I' = (0.0500 \text{ kg} + 0.667 \text{ kg})(1.20 \text{ m})^2 = 1.032 \text{ kg} \cdot \text{m}^2.
\]

The value of \( I' \) is now entered into the expression for \( \omega' \), which yields

\[
\omega' = \frac{mvr}{I'} = \frac{(0.0500 \text{ kg})(30.0 \text{ m/s})(1.20 \text{ m})}{1.032 \text{ kg} \cdot \text{m}^2} = 1.744 \text{ rad/s} \approx 1.74 \text{ rad/s}.
\]
Strategy for (b)
The kinetic energy before the collision is the incoming disk’s translational kinetic energy, and after the collision, it is the rotational kinetic energy of the two stuck together.

Solution for (b)
First, we calculate the translational kinetic energy by entering given values for the mass and speed of the incoming disk.

\[
KE = \frac{1}{2}mv^2 = (0.500)(0.0500 \text{ kg})(30.0 \text{ m/s})^2 = 22.5 \text{ J}
\]  
(10.131)

After the collision, the rotational kinetic energy can be found because we now know the final angular velocity and the final moment of inertia. Thus, entering the values into the rotational kinetic energy equation gives

\[
KE' = \frac{1}{2}I'\omega'^2 = (0.5)(1.032 \text{ kg} \cdot \text{m}^2)(1.744 \text{rad/s})^2
\]

(10.132)

\[
= 1.57 \text{ J}
\]

Strategy for (c)
The linear momentum before the collision is that of the disk. After the collision, it is the sum of the disk’s momentum and that of the center of mass of the stick.

Solution of (c)
Before the collision, then, linear momentum is

\[
p = mv = (0.0500 \text{ kg})(30.0 \text{ m/s}) = 1.50 \text{ kg} \cdot \text{m/s}
\]  
(10.133)

After the collision, the disk and the stick’s center of mass move in the same direction. The total linear momentum is that of the disk moving at a new velocity \(v' = \frac{r}{2}\omega'\) plus that of the stick’s center of mass, which moves at half this speed because \(v_{CM} = \frac{r}{2}v' = \frac{\omega'}{2}\). Thus,

\[
p' = mv' + Mv_{CM} = mv' + \frac{Mv'}{2}
\]

(10.134)

Gathering similar terms in the equation yields,

\[
p' = \left(m + \frac{M}{2}\right)v'
\]

(10.135)

so that

\[
p' = \left(m + \frac{M}{2}\right)\omega'.
\]

(10.136)

Substituting known values into the equation,

\[
p' = (1.050 \text{ kg})(1.20 \text{ m})(1.744 \text{ rad/s}) = 2.20 \text{ kg} \cdot \text{m/s}
\]  
(10.137)

Discussion
First note that the kinetic energy is less after the collision, as predicted, because the collision is inelastic. More surprising is that the momentum after the collision is actually greater than before the collision. This result can be understood if you consider how the nail affects the stick and vice versa. Apparently, the stick pushes backward on the nail when first struck by the disk. The nail’s reaction (consistent with Newton’s third law) is to push forward on the stick, imparting momentum to it in the same direction in which the disk was initially moving, thereby increasing the momentum of the system.

The above example has other implications. For example, what would happen if the disk hit very close to the nail? Obviously, a force would be exerted on the nail in the forward direction. So, when the stick is struck at the end farthest from the nail, a backward force is exerted on the nail, and when it is hit at the end nearest the nail, a forward force is exerted on the nail. Thus, striking it at a certain point in between produces no force on the nail. This intermediate point is known as the percussion point.

An analogous situation occurs in tennis as seen in Figure 10.27. If you hit a ball with the end of your racquet, the handle is pulled away from your hand. If you hit a ball much farther down, for example, on the shaft of the racquet, the handle is pushed into your palm. And if you hit the ball at the racquet’s percussion point (what some people call the “sweet spot”), then little or no force is exerted on your hand, and there is less vibration, reducing chances of a tennis elbow. The same effect occurs for a baseball bat.
Figure 10.27 A disk hitting a stick is compared to a tennis ball being hit by a racquet. (a) When the ball strikes the racquet near the end, a backward force is exerted on the hand. (b) When the racquet is struck much farther down, a forward force is exerted on the hand. (c) When the racquet is struck at the percussion point, no force is delivered to the hand.

Check Your Understanding

Is rotational kinetic energy a vector? Justify your answer.

Solution
No, energy is always scalar whether motion is involved or not. No form of energy has a direction in space and you can see that rotational kinetic energy does not depend on the direction of motion just as linear kinetic energy is independent of the direction of motion.
Angular momentum is a vector and, therefore, has direction as well as magnitude. Torque affects both the direction and the magnitude of angular momentum. What is the direction of the angular momentum of a rotating object like the disk in Figure 10.28? The figure shows the right-hand rule used to find the direction of both angular momentum and angular velocity. Both \( \mathbf{L} \) and \( \mathbf{\omega} \) are vectors—each has direction and magnitude. Both can be represented by arrows. The right-hand rule defines both to be perpendicular to the plane of rotation in the direction shown. Because angular momentum is related to angular velocity by \( \mathbf{L} = I \mathbf{\omega} \), the direction of \( \mathbf{L} \) is the same as the direction of \( \mathbf{\omega} \). Notice in the figure that both point along the axis of rotation.

Figure 10.28 Figure (a) shows a disk rotating counterclockwise when viewed from above. Figure (b) shows the right-hand rule. The direction of angular velocity \( \mathbf{\omega} \) size and angular momentum \( \mathbf{L} \) are defined to be the direction in which the thumb of your right hand points when you curl your fingers in the direction of the disk’s rotation as shown.

Now, recall that torque changes angular momentum as expressed by

\[
\text{net } \tau = \frac{\Delta \mathbf{L}}{\Delta t}
\]

This equation means that the direction of \( \Delta \mathbf{L} \) is the same as the direction of the torque \( \tau \) that creates it. This result is illustrated in Figure 10.29, which shows the direction of torque and the angular momentum it creates.

Let us now consider a bicycle wheel with a couple of handles attached to it, as shown in Figure 10.30. (This device is popular in demonstrations among physicists, because it does unexpected things.) With the wheel rotating as shown, its angular momentum is to the woman’s left. Suppose the person holding the wheel tries to rotate it as in the figure. Her natural expectation is that the wheel will rotate in the direction she pushes it—but what happens is quite different. The forces exerted create a torque that is horizontal toward the person, as shown in Figure 10.30(a). This torque creates a change in angular momentum \( \mathbf{L} \) in the same direction, perpendicular to the original angular momentum \( \mathbf{L} \), thus changing the direction of \( \mathbf{L} \) but not the magnitude of \( \mathbf{L} \).

Figure 10.30 shows how \( \Delta \mathbf{L} \) and \( \mathbf{L} \) add, giving a new angular momentum with direction that is inclined more toward the person than before. The axis of the wheel has thus moved perpendicular to the forces exerted on it, instead of in the expected direction.

Figure 10.29 In figure (a), the torque is perpendicular to the plane formed by \( \mathbf{F} \) and \( \mathbf{r} \) and is the direction your right thumb would point to if you curled your fingers in the direction of \( \mathbf{F} \). Figure (b) shows that the direction of the torque is the same as that of the angular momentum it produces.
In figure (a), a person holding the spinning bike wheel lifts it with her right hand and pushes down with her left hand in an attempt to rotate the wheel. This action creates a torque directly toward her. This torque causes a change in angular momentum $\Delta L$ in exactly the same direction. Figure (b) shows a vector diagram depicting how $\Delta L$ and $L$ add, producing a new angular momentum pointing more toward the person. The wheel moves toward the person, perpendicular to the forces she exerts on it.

This same logic explains the behavior of gyroscopes. Figure 10.31 shows the two forces acting on a spinning gyroscope. The torque produced is perpendicular to the angular momentum, thus the direction of the torque is changed, but not its magnitude. The gyroscope precesses around a vertical axis, since the torque is always horizontal and perpendicular to $L$. If the gyroscope is not spinning, it acquires angular momentum in the direction of the torque ($L = \Delta L$), and it rotates around a horizontal axis, falling over just as we would expect.

Earth itself acts like a gigantic gyroscope. Its angular momentum is along its axis and points at Polaris, the North Star. But Earth is slowly precessing (once in about 26,000 years) due to the torque of the Sun and the Moon on its nonspherical shape.

Check Your Understanding

Rotational kinetic energy is associated with angular momentum? Does that mean that rotational kinetic energy is a vector?
Angular acceleration

Angular momentum

Change in angular velocity

Kinematics of rotational motion

Law of conservation of angular momentum

Moment of inertia

Right-hand rule

Rotational inertia

Rotational kinetic energy

Tangential acceleration

Torque

Work-energy theorem

Solution

No, energy is always a scalar whether motion is involved or not. No form of energy has a direction in space and you can see that rotational kinetic energy does not depend on the direction of motion just as linear kinetic energy is independent of the direction of motion.

Glossary

Angular acceleration: the rate of change of angular velocity with time

Angular momentum: the product of moment of inertia and angular velocity

Change in angular velocity: the difference between final and initial values of angular velocity

Kinematics of rotational motion: describes the relationships among rotation angle, angular velocity, angular acceleration, and time

Law of conservation of angular momentum: angular momentum is conserved, i.e., the initial angular momentum is equal to the final angular momentum when no external torque is applied to the system

Moment of inertia: mass times the square of perpendicular distance from the rotation axis; for a point mass, it is \( I = mr^2 \) and, because any object can be built up from a collection of point masses, this relationship is the basis for all other moments of inertia

Right-hand rule: direction of angular velocity \( \omega \) and angular momentum \( L \) in which the thumb of your right hand points when you curl your fingers in the direction of the disk’s rotation

Rotational inertia: resistance to change of rotation. The more rotational inertia an object has, the harder it is to rotate

Rotational kinetic energy: the kinetic energy due to the rotation of an object. This is part of its total kinetic energy

Tangential acceleration: the acceleration in a direction tangent to the circle at the point of interest in circular motion

Torque: the turning effectiveness of a force

Work-energy theorem: if one or more external forces act upon a rigid object, causing its kinetic energy to change from \( KE_1 \) to \( KE_2 \), then the work \( W \) done by the net force is equal to the change in kinetic energy

Section Summary

10.1 Angular Acceleration

- Uniform circular motion is the motion with a constant angular velocity \( \omega = \frac{\Delta \theta}{\Delta t} \).
- In non-uniform circular motion, the velocity changes with time and the rate of change of angular velocity (i.e. angular acceleration) is \( \alpha = \frac{\Delta \omega}{\Delta t} \).
- Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction, given as \( a_t = \frac{\Delta v}{\Delta t} \).
- For circular motion, note that \( v = r\omega \), so that
  \[
  a_t = \frac{\Delta (r\omega)}{\Delta t}.
  \]
- The radius \( r \) is constant for circular motion, and so \( \Delta (r\omega) = r\Delta \omega \). Thus,
  \[
  a_t = r\frac{\Delta \omega}{\Delta t}.
  \]
- By definition, \( \Delta \omega / \Delta t = \alpha \). Thus,
  \[
  a_t = ra
  \]
  or
  \[
  \alpha = \frac{a_t}{r}.
  \]

10.2 Kinematics of Rotational Motion

- Kinematics is the description of motion.
- The kinematics of rotational motion describes the relationships among rotation angle, angular velocity, angular acceleration, and time.
• Starting with the four kinematic equations we developed in the One-Dimensional Kinematics, we can derive the four rotational kinematic equations (presented together with their translational counterparts) seen in Table 10.2.

• In these equations, the subscript 0 denotes initial values (\(x_0\) and \(t_0\) are initial values), and the average angular velocity \(\bar{\omega}\) and average velocity \(\bar{v}\) are defined as follows:

\[
\bar{\omega} = \frac{\omega_0 + \omega}{2} \quad \text{and} \quad \bar{v} = \frac{v_0 + v}{2}.
\]

10.3 Dynamics of Rotational Motion: Rotational Inertia

• The farther the force is applied from the pivot, the greater is the angular acceleration; angular acceleration is inversely proportional to mass.

• If we exert a force \(F\) on a point mass \(m\) that is at a distance \(r\) from a pivot point and because the force is perpendicular to \(r\), an acceleration \(a = \frac{F}{m}\) is obtained in the direction of \(F\). We can rearrange this equation such that \(F = ma\), and then look for ways to relate this expression to expressions for rotational quantities. We note that \(a = r\alpha\), and we substitute this expression into \(F = ma\), yielding \(F = mra\)

or

\[
\tau = mr^2\alpha.
\]

• The moment of inertia \(I\) of an object is the sum of \(MR^2\) for all the point masses of which it is composed. That is,

\[
I = \sum mR^2.
\]

• The general relationship among torque, moment of inertia, and angular acceleration is

\[
\tau = I\alpha
\]

or

\[
\alpha = \frac{\text{net} \ \tau}{I}.
\]

10.4 Rotational Kinetic Energy: Work and Energy Revisited

• The rotational kinetic energy \(KE_{rot}\) for an object with a moment of inertia \(I\) and an angular velocity \(\omega\) is given by

\[
KE_{rot} = \frac{1}{2}I\omega^2.
\]

• Helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades.

• Work and energy in rotational motion are completely analogous to work and energy in translational motion.

• The equation for the work-energy theorem for rotational motion is,

\[
\text{net} \ W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2.
\]

10.5 Angular Momentum and Its Conservation

• Every rotational phenomenon has a direct translational analog, likewise angular momentum \(L\) can be defined as \(L = I\omega\).

• This equation is an analog to the definition of linear momentum as \(p = mv\). The relationship between torque and angular momentum is net \(\tau = \frac{\Delta L}{\Delta t}\).

• Angular momentum, like energy and linear momentum, is conserved. This universally applicable law is another sign of underlying unity in physical laws. Angular momentum is conserved when net external torque is zero, just as linear momentum is conserved when the net external force is zero.

10.6 Collisions of Extended Bodies in Two Dimensions

• Angular momentum \(L\) is analogous to linear momentum and is given by \(L = I\omega\).
Angular momentum is changed by torque, following the relationship \( \text{net } \tau = \frac{\Delta L}{\Delta t} \).

Angular momentum is conserved if the net torque is zero \( L = \text{constant} \) (net \( \tau = 0 \)) or \( L = L' \) (net \( \tau = 0 \)). This equation is known as the law of conservation of angular momentum, which may be conserved in collisions.

10.7 Gyroscopic Effects: Vector Aspects of Angular Momentum

- Torque is perpendicular to the plane formed by \( r \) and \( F \) and is the direction your right thumb would point if you curled the fingers of your right hand in the direction of \( F \). The direction of the torque is thus the same as that of the angular momentum it produces.
- The gyroscope precesses around a vertical axis, since the torque is always horizontal and perpendicular to \( L \). If the gyroscope is not spinning, it acquires angular momentum in the direction of the torque (\( L = \Delta L \)), and it rotates about a horizontal axis, falling over just as we would expect.
- Earth itself acts like a gigantic gyroscope. Its angular momentum is along its axis and points at Polaris, the North Star.

Conceptual Questions

10.1 Angular Acceleration

1. Analogies exist between rotational and translational physical quantities. Identify the rotational term analogous to each of the following: acceleration, force, mass, work, translational kinetic energy, linear momentum, impulse.

2. Explain why centripetal acceleration changes the direction of velocity in circular motion but not its magnitude.

3. In circular motion, a tangential acceleration can change the magnitude of the velocity but not its direction. Explain your answer.

4. Suppose a piece of food is on the edge of a rotating microwave oven plate. Does it experience nonzero tangential acceleration, centripetal acceleration, or both when: (a) The plate starts to spin? (b) The plate rotates at constant angular velocity? (c) The plate slows to a halt?

10.3 Dynamics of Rotational Motion: Rotational Inertia

5. The moment of inertia of a long rod spun around an axis through one end perpendicular to its length is \( ML^2/3 \). Why is this moment of inertia greater than it would be if you spun a point mass \( M \) at the location of the center of mass of the rod (at \( L/2 \))? (That would be \( ML^2/4 \).)

6. Why is the moment of inertia of a hoop that has a mass \( M \) and a radius \( R \) greater than the moment of inertia of a disk that has the same mass and radius? Why is the moment of inertia of a spherical shell that has a mass \( M \) and a radius \( R \) greater than that of a solid sphere that has the same mass and radius?

7. Give an example in which a small force exerts a large torque. Give another example in which a large force exerts a small torque.

8. While reducing the mass of a racing bike, the greatest benefit is realized from reducing the mass of the tires and wheel rims. Why does this allow a racer to achieve greater accelerations than would an identical reduction in the mass of the bicycle’s frame?

Figure 10.32 The image shows a side view of a racing bicycle. Can you see evidence in the design of the wheels on this racing bicycle that their moment of inertia has been purposely reduced? (credit: Jesús Rodríguez)

9. A ball slides up a frictionless ramp. It is then rolled without slipping and with the same initial velocity up another frictionless ramp (with the same slope angle). In which case does it reach a greater height, and why?
10.4 Rotational Kinetic Energy: Work and Energy Revisited

10. Describe the energy transformations involved when a yo-yo is thrown downward and then climbs back up its string to be caught in the user’s hand.

11. What energy transformations are involved when a dragster engine is revved, its clutch let out rapidly, its tires spun, and it starts to accelerate forward? Describe the source and transformation of energy at each step.

12. The Earth has more rotational kinetic energy now than did the cloud of gas and dust from which it formed. Where did this energy come from?

Figure 10.33 An immense cloud of rotating gas and dust contracted under the influence of gravity to form the Earth and in the process rotational kinetic energy increased. (credit: NASA)

10.5 Angular Momentum and Its Conservation

13. When you start the engine of your car with the transmission in neutral, you notice that the car rocks in the opposite sense of the engine’s rotation. Explain in terms of conservation of angular momentum. Is the angular momentum of the car conserved for long (for more than a few seconds)?

14. Suppose a child walks from the outer edge of a rotating merry-go-round to the inside. Does the angular velocity of the merry-go-round increase, decrease, or remain the same? Explain your answer.

Figure 10.34 A child may jump off a merry-go-round in a variety of directions.

15. Suppose a child gets off a rotating merry-go-round. Does the angular velocity of the merry-go-round increase, decrease, or remain the same if: (a) He jumps off radially? (b) He jumps backward to land motionless? (c) He jumps straight up and hangs onto an overhead tree branch? (d) He jumps off forward, tangential to the edge? Explain your answers. (Refer to Figure 10.34).

16. Helicopters have a small propeller on their tail to keep them from rotating in the opposite direction of their main lifting blades. Explain in terms of Newton’s third law why the helicopter body rotates in the opposite direction to the blades.

17. Whenever a helicopter has two sets of lifting blades, they rotate in opposite directions (and there will be no tail propeller). Explain why it is best to have the blades rotate in opposite directions.

18. Describe how work is done by a skater pulling in her arms during a spin. In particular, identify the force she exerts on each arm to pull it in and the distance each moves, noting that a component of the force is in the direction moved. Why is angular momentum not increased by this action?

19. When there is a global heating trend on Earth, the atmosphere expands and the length of the day increases very slightly. Explain why the length of a day increases.
20. Nearly all conventional piston engines have flywheels on them to smooth out engine vibrations caused by the thrust of individual piston firings. Why does the flywheel have this effect?

21. Jet turbines spin rapidly. They are designed to fly apart if something makes them seize suddenly, rather than transfer angular momentum to the plane's wing, possibly tearing it off. Explain how flying apart conserves angular momentum without transferring it to the wing.

22. An astronaut tightens a bolt on a satellite in orbit. He rotates in a direction opposite to that of the bolt, and the satellite rotates in the same direction as the bolt. Explain why. If a handhold is available on the satellite, can this counter-rotation be prevented? Explain your answer.

23. Competitive divers pull their limbs in and curl up their bodies when they do flips. Just before entering the water, they fully extend their limbs to enter straight down. Explain the effect of both actions on their angular velocities. Also explain the effect on their angular momenta.

24. Draw a free body diagram to show how a diver gains angular momentum when leaving the diving board.

25. In terms of angular momentum, what is the advantage of giving a football or a rifle bullet a spin when throwing or releasing it?

26. Describe two different collisions—one in which angular momentum is conserved, and the other in which it is not. Which condition determines whether or not angular momentum is conserved in a collision?

27. Suppose an ice hockey puck strikes a hockey stick that lies flat on the ice and is free to move in any direction. Which quantities are likely to be conserved: angular momentum, linear momentum, or kinetic energy (assuming the puck and stick are very resilient)?

28. While driving his motorcycle at highway speed, a physics student notices that pulling back lightly on the right handlebar tips the cycle to the left and produces a left turn. Explain why this happens.

10.6 Collisions of Extended Bodies in Two Dimensions

26. Describe two different collisions—one in which angular momentum is conserved, and the other in which it is not. Which condition determines whether or not angular momentum is conserved in a collision?

27. Suppose an ice hockey puck strikes a hockey stick that lies flat on the ice and is free to move in any direction. Which quantities are likely to be conserved: angular momentum, linear momentum, or kinetic energy (assuming the puck and stick are very resilient)?

28. While driving his motorcycle at highway speed, a physics student notices that pulling back lightly on the right handlebar tips the cycle to the left and produces a left turn. Explain why this happens.

10.7 Gyroscopic Effects: Vector Aspects of Angular Momentum

29. While driving his motorcycle at highway speed, a physics student notices that pulling back lightly on the right handlebar tips the cycle to the left and produces a left turn. Explain why this happens.
30. Gyroscopes used in guidance systems to indicate directions in space must have an angular momentum that does not change in direction. Yet they are often subjected to large forces and accelerations. How can the direction of their angular momentum be constant when they are accelerated?
10.1 Angular Acceleration

1. At its peak, a tornado is 60.0 m in diameter and carries 500 km/h winds. What is its angular velocity in revolutions per second?

2. Integrated Concepts

An ultracentrifuge accelerates from rest to 100,000 rpm in 2.00 min. (a) What is its angular acceleration in rad/s$^2$? (b) What is the tangential acceleration of a point 9.50 cm from the axis of rotation? (c) What is the radial acceleration in m/s$^2$ and multiples of $g$ of this point at full rpm?

3. Integrated Concepts

You have a grindstone (a disk) that is 90.0 kg, has a 0.340-m radius, and is turning at 90.0 rpm, and you press a steel axe against it with a radial force of 20.0 N. (a) Assuming the kinetic coefficient of friction between steel and stone is 0.20, calculate the angular acceleration of the grindstone. (b) How many turns will the stone make before coming to rest?

4. Unreasonable Results

You are told that a basketball player spins the ball with an angular acceleration of 100 rad/s$^2$. (a) What is the ball's final angular velocity if the ball starts from rest and the acceleration lasts 2.00 s? (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?

10.2 Kinematics of Rotational Motion

5. With the aid of a string, a gyroscope is accelerated from rest to 32 rad/s in 0.40 s. (a) What is its angular acceleration in rad/s$^2$? (b) How many revolutions does it go through in the process?

6. Suppose a piece of dust finds itself on a CD. If the spin rate of the CD is 500 rpm, and the piece of dust is 4.3 cm from the center, what is the total distance traveled by the dust in 3 minutes? (Ignore accelerations due to getting the CD rotating.)

7. A gyroscope slows from an initial rate of 32.0 rad/s at a rate of 0.700 rad/s$^2$. (a) How long does it take to come to rest? (b) How many revolutions does it make before stopping?

8. During a very quick stop, a car decelerates at 7.00 m/s$^2$. (a) What is the angular acceleration of its 0.280-m-radius tires, assuming they do not slip on the pavement? (b) How many revolutions do the tires make before coming to rest, given their initial angular velocity is 95.0 rad/s? (c) How long does the car take to stop completely? (d) What distance does the car travel in this time? (e) What was the car's initial velocity? (f) Do the values obtained seem reasonable, considering that this stop happens very quickly?

9. Everyday application: Suppose a yo-yo has a center shaft that has a 0.250 cm radius and that its string is being pulled. (a) If the string is stationary and the yo-yo accelerates away from it at a rate of 1.50 m/s$^2$, what is the angular acceleration of the yo-yo? (b) What is the angular velocity after 0.750 s if it starts from rest? (c) The outside radius of the yo-yo is 3.50 cm. What is the tangential acceleration of a point on its edge?

10.3 Dynamics of Rotational Motion: Rotational Inertia

10. This problem considers additional aspects of example Calculating the Effect of Mass Distribution on a Merry-Go-Round. (a) How long does it take the father to give the merry-go-round an angular velocity of 1.50 rad/s? (b) How many revolutions must he go through to generate this velocity? (c) If he exerts a slowing force of 300 N at a radius of 1.35 m, how long would it take him to stop them?

11. Calculate the moment of inertia of a skater given the following information. (a) The 60.0-kg skater is approximated as a cylinder that has a 0.110-m radius. (b) The skater with arms extended is approximated as a cylinder that is 52.5 kg, has a 0.110-m radius, and has two 0.900-m-long arms which are 3.75 kg each and extend straight out from the cylinder like rods rotated about their ends.

12. The triceps muscle in the back of the upper arm extends the forearm. This muscle in a professional boxer exerts a force of 2.00x$10^3$ N with an effective perpendicular lever arm of 3.00 cm, producing an angular acceleration of the forearm of 120 rad/s$^2$. What is the moment of inertia of the boxer’s forearm?
13. A soccer player extends her lower leg in a kicking motion by exerting a force with the muscle above the knee in the front of her leg. She produces an angular acceleration of 30.00 rad/s² and her lower leg has a moment of inertia of 0.750 kg · m². What is the force exerted by the muscle if its effective perpendicular lever arm is 1.90 cm?

14. Suppose you exert a force of 180 N tangential to a 0.280-m-radius 75.0-kg grindstone (a solid disk).
(a) What torque is exerted? (b) What is the angular acceleration assuming negligible opposing friction? (c) What is the angular acceleration if there is an opposing frictional force of 20.0 N exerted 1.50 cm from the axis?

15. Consider the 12.0 kg motorcycle wheel shown in Figure 10.38. Assume it to be approximately an annular ring with an inner radius of 0.280 m and an outer radius of 0.330 m. The motorcycle is on its center stand, so that the wheel can spin freely. (a) If the drive chain exerts a force of 2200 N at a radius of 5.00 cm, what is the angular acceleration of the wheel? (b) What is the tangential acceleration of a point on the outer edge of the tire? (c) How long, starting from rest, does it take to reach an angular velocity of 80.0 rad/s?

16. Zorch, an archenemy of Superman, decides to slow Earth's rotation to once per 28.0 h by exerting an opposing force at and parallel to the equator. Superman is not immediately concerned, because he knows Zorch can only exert a force of \(4.00 \times 10^7\) N (a little greater than a Saturn V rocket's thrust). How long must Zorch push with this force to accomplish his goal? (This period gives Superman time to devote to other villains.) Explicitly show how you follow the steps found in Problem-Solving Strategy for Rotational Dynamics.

17. An automobile engine can produce 200 N·m of torque. Calculate the angular acceleration produced if 95.0% of this torque is applied to the drive shaft, axle, and rear wheels of a car, given the following information. The car is suspended so that the wheels can turn freely. Each wheel acts like a 15.0 kg disk that has a 0.180 m radius. The walls of each tire act like a 2.00-kg annular ring that has inside radius of 0.180 m and outside radius of 0.320 m. The tread of each tire acts like a 10.0-kg hoop of radius 0.330 m. The 14.0-kg axle acts like a rod that has a 2.00-cm radius. The 30.0-kg drive shaft acts like a rod that has a 3.20-cm radius.

18. Starting with the formula for the moment of inertia of a rod rotated around an axis through one end perpendicular to its length \(I = Mℓ^2 / 3\), prove that the moment of inertia of a rod rotated about an axis through its center perpendicular to its length is \(I = Mℓ^2 / 12\). You will find the graphics in Figure 10.12 useful in visualizing these rotations.

19. Unreasonable Results
A gymnast doing a forward flip lands on the mat and exerts a 500-N · m torque to slow her angular velocity to zero. Her initial angular velocity is 10.0 rad/s, and her moment of inertia is 0.050 kg · m². (a) What time is required for her to exactly stop her spin? (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?

20. Unreasonable Results
An advertisement claims that an 800-kg car is aided by its 20.0-kg flywheel, which can accelerate the car from rest to a speed of 30.0 m/s. The flywheel is a disk with a 0.150-m radius. (a) Calculate the angular velocity the flywheel must have if 95.0% of its rotational energy is used to get the car up to speed. (b) What is unreasonable about the result? (c) Which premise is unreasonable or which premises are inconsistent?

10.4 Rotational Kinetic Energy: Work and Energy Revisited

21. This problem considers energy and work aspects of Example 10.7—use data from that example as needed. (a) Calculate the rotational kinetic energy in the merry-go-round plus child when they have an angular velocity of 20.0 rpm. (b) Using energy considerations, find the number of revolutions the father will have to push to achieve this angular velocity starting from rest. (c) Again, using energy considerations, calculate the force the father must exert to stop the merry-go-round in two revolutions.

22. What is the final velocity of a hoop that rolls without slipping down a 5.00-m-high hill, starting from rest?

23. (a) Calculate the rotational kinetic energy of Earth on its axis. (b) What is the rotational kinetic energy of Earth in its orbit around the Sun?

24. Calculate the rotational kinetic energy in the motorcycle wheel (Figure 10.38) if its angular velocity is 120 rad/s. Assume \(M = 12.0\) kg, \(R_1 = 0.280\) m, and \(R_2 = 0.330\) m.

25. A baseball pitcher throws the ball in a motion where there is rotation of the forearm about the elbow joint as well as other movements. If the linear velocity of the ball relative to the elbow joint is 20.0 m/s at a distance of 0.480 m from the joint and the moment of inertia of the forearm is 0.500 kg · m², what is the rotational kinetic energy of the forearm?

26. While punting a football, a kicker rotates his leg about the hip joint. The moment of inertia of the leg is 3.75 kg · m² and its rotational kinetic energy is 175 J. (a) What is the angular velocity of the leg? (b) What is the velocity of tip of the punter’s shoe if it is 1.05 m from the hip joint? (c) Explain how the football can be given a velocity greater than the tip of the shoe (necessary for a decent kick distance).
27. A bus contains a 1500 kg flywheel (a disk that has a 0.600 m radius) and has a total mass of 10,000 kg. (a) Calculate the angular velocity the flywheel must have to contain enough energy to take the bus from rest to a speed of 20.0 m/s, assuming 90.0% of the rotational kinetic energy can be transformed into translational energy. (b) How high a hill can the bus climb with this stored energy and still have a speed of 3.00 m/s at the top of the hill? Explicitly show how you follow the steps in the Problem-Solving Strategy for Rotational Energy.

28. A ball with an initial velocity of 8.00 m/s rolls up a hill without slipping. Treating the ball as a spherical shell, calculate the vertical height it reaches. (b) Repeat the calculation for the same ball if it slides up the hill without rolling.

29. While exercising in a fitness center, a man lies face down on a bench and lifts a weight with one lower leg by contacting the muscles in the back of the upper leg. (a) Find the angular acceleration produced given the mass lifted is 10.0 kg at a distance of 28.0 cm from the knee joint, the moment of inertia of the lower leg is 0.900 kg \( \cdot \) m\(^2\), the muscle force is 1500 N, and its effective perpendicular lever arm is 3.00 cm. (b) How much work is done if the leg rotates through an angle of 20.0º with a constant force exerted by the muscle?

30. To develop muscle tone, a woman lifts a 2.00-kg weight held in her hand. She uses her biceps muscle to flex the lower arm through an angle of 60.0º. (a) What is the angular acceleration if the weight is 24.0 cm from the elbow joint, her forearm has a moment of inertia of 0.250 kg \( \cdot \) m\(^2\), and the net force she exerts is 750 N at an effective perpendicular lever arm of 2.00 cm? (b) How much work does she do?

31. Consider two cylinders that start down identical inclines from rest except that one is frictionless. Thus one cylinder rolls without slipping, while the other slides frictionlessly without rolling. They both travel a short distance at the bottom and then start up another incline. (a) Show that they both reach the same height on the other incline, and that this height is equal to their original height. (b) Find the ratio of the time the rolling cylinder takes to reach the height on the second incline to the time the sliding cylinder takes to reach the height on the second incline. (c) Explain why the time for the rolling motion is greater than that for the sliding motion.

32. What is the moment of inertia of an object that rolls without slipping down a 2.00-m-high incline starting from rest, and has a final velocity of 6.00 m/s? Express the moment of inertia as a multiple of \( MR^2 \), where \( M \) is the mass of the object and \( R \) is its radius.

33. Suppose a 200-kg motorcycle has two wheels like, the one described in Problem 10.15 and is heading toward a hill at a speed of 30.0 m/s. (a) How high can it coast up the hill, if you neglect friction? (b) How much energy is lost to friction if the motorcycle only gains an altitude of 35.0 m before coming to rest?

34. In softball, the pitcher throws with the arm fully extended (straight at the elbow). In a fast pitch the ball leaves the hand with a speed of 139 km/h. (a) Find the rotational kinetic energy of the pitcher’s arm and ball together given that the arm's moment of inertia is 0.720 kg \( \cdot \) m\(^2\) and the ball leaves the hand at a distance of 0.600 m from the pivot at the shoulder. (b) What force did the muscles exert to cause the arm to rotate if their effective perpendicular lever arm is 4.00 cm and the ball is 0.156 kg?

35. Construct Your Own Problem
Consider the work done by a spinning skater pulling her arms in to increase her rate of spin. Construct a problem in which you calculate the work done with a “force multiplied by distance” calculation and compare it to the skater’s increase in kinetic energy.

10.5 Angular Momentum and Its Conservation

36. (a) Calculate the angular momentum of the Earth in its orbit around the Sun.
(b) Compare this angular momentum with the angular momentum of Earth on its axis.

37. (a) What is the angular momentum of the Moon in its orbit around Earth?
(b) How does this angular momentum compare with the angular momentum of the Moon on its axis? Remember that the Moon keeps one side toward Earth at all times.
(c) Discuss whether the values found in parts (a) and (b) seem consistent with the fact that tidal effects with Earth have caused the Moon to rotate with one side always facing Earth.

38. Suppose you start an antique car by exerting a force of 300 N on its crank for 0.250 s. What angular momentum is given to the engine if the handle of the crank is 0.300 m from the pivot and the force is exerted to create maximum torque the entire time?

39. A playground merry-go-round has a mass of 120 kg and a radius of 1.80 m and it is rotating with an angular velocity of 0.500 rev/s. What is its angular velocity after a 22.0-kg child gets onto it by grabbing its outer edge? The child is initially at rest.

40. Three children are riding on the edge of a merry-go-round that is 100 kg, has a 1.60-m radius, and is spinning at 20.0 rpm. The children have masses of 22.0, 28.0, and 33.0 kg. If the child who has a mass of 28.0 kg moves to the center of the merry-go-round, what is the new angular velocity in rpm?

41. (a) Calculate the angular momentum of an ice skater spinning at 6.00 rev/s given his moment of inertia is 0.400 kg \( \cdot \) m\(^2\). (b) He reduces his rate of spin (his angular velocity) by extending his arms and increasing his moment of inertia. Find the value of his moment of inertia if his angular velocity decreases to 1.25 rev/s. (c) Suppose instead he keeps his arms in and allows friction of the ice to slow him to 3.00 rev/s. What average torque was exerted if this takes 15.0 s?
42. Construct Your Own Problem
Consider the Earth-Moon system. Construct a problem in which you calculate the total angular momentum of the system including the spins of the Earth and the Moon on their axes and the orbital angular momentum of the Earth-Moon system in its nearly monthly rotation. Calculate what happens to the Moon’s orbital radius if the Earth’s rotation decreases due to tidal drag. Among the things to be considered are the amount by which the Earth’s rotation slows and the fact that the Moon will continue to have one side always facing the Earth.

10.6 Collisions of Extended Bodies in Two Dimensions

43. Repeat Example 10.15 in which the disk strikes and adheres to the stick 0.100 m from the nail.

44. Repeat Example 10.15 in which the disk originally spins clockwise at 1000 rpm and has a radius of 1.50 cm.

45. Twin skaters approach one another as shown in Figure 10.39 and lock hands. (a) Calculate their final angular velocity, given each had an initial speed of 2.50 m/s relative to the ice. Each has a mass of 70.0 kg, and each has a center of mass located 0.800 m from their locked hands. You may approximate their moments of inertia to be that of point masses at this radius. (b) Compare the initial kinetic energy and final kinetic energy.

46. Suppose a 0.250-kg ball is thrown at 15.0 m/s to a motionless person standing on ice who catches it with an outstretched arm as shown in Figure 10.40.
(a) Calculate the final linear velocity of the person, given his mass is 70.0 kg.
(b) What is his angular velocity if each arm is 5.00 kg? You may treat the ball as a point mass and treat the person’s arms as uniform rods (each has a length of 0.900 m) and the rest of his body as a uniform cylinder of radius 0.180 m. Neglect the effect of the ball on his center of mass so that his center of mass remains in his geometrical center.
(c) Compare the initial and final total kinetic energies.

47. Repeat Example 10.15 in which the stick is free to have translational motion as well as rotational motion.

Figure 10.39 Twin skaters approach each other with identical speeds. Then, the skaters lock hands and spin.

Figure 10.40 The figure shows the overhead view of a person standing motionless on ice about to catch a ball. Both arms are outstretched. After catching the ball, the skater recoils and rotates.
10.7 Gyroscopic Effects: Vector Aspects of Angular Momentum

48. Integrated Concepts

The axis of Earth makes a 23.5° angle with a direction perpendicular to the plane of Earth’s orbit. As shown in Figure 10.41, this axis precesses, making one complete rotation in 25,780 y.

(a) Calculate the change in angular momentum in half this time.

(b) What is the average torque producing this change in angular momentum?

(c) If this torque were created by a single force (it is not) acting at the most effective point on the equator, what would its magnitude be?

Figure 10.41 The Earth’s axis slowly precesses, always making an angle of 23.5° with the direction perpendicular to the plane of Earth’s orbit. The change in angular momentum for the two shown positions is quite large, although the magnitude \( \mathbf{L} \) is unchanged.
Chapter 11 | Oscillatory Motion and Waves

Chapter Outline

11.1. Hooke’s Law: Stress and Strain Revisited
   - Explain Newton’s third law of motion with respect to stress and deformation.
   - Describe the restoration of force and displacement.
   - Calculate the energy in Hooke’s Law of deformation, and the stored energy in a spring.

11.2. Period and Frequency in Oscillations
   - Observe the vibrations of a guitar string.
   - Determine the frequency of oscillations.

11.3. Simple Harmonic Motion: A Special Periodic Motion
   - Describe a simple harmonic oscillator.
   - Explain the link between simple harmonic motion and waves.

11.4. The Simple Pendulum
   - Measure acceleration due to gravity.

11.5. Energy and the Simple Harmonic Oscillator
   - Determine the maximum speed of an oscillating system.

11.6. Uniform Circular Motion and Simple Harmonic Motion
   - Compare simple harmonic motion with uniform circular motion.

11.7. Damped Harmonic Motion
   - Compare and discuss underdamped and overdamped oscillating systems.
   - Explain critically damped system.

11.8. Forced Oscillations and Resonance
   - Observe resonance of a paddle ball on a string.
   - Observe amplitude of a damped harmonic oscillator.

11.9. Waves
   - State the characteristics of a wave.
   - Calculate the velocity of wave propagation.

11.10. Superposition and Interference
   - Explain standing waves.
   - Describe the mathematical representation of overtones and beat frequency.

11.11. Energy in Waves: Intensity
   - Calculate the intensity and the power of rays and waves.
Introduction to Oscillatory Motion and Waves

What do an ocean buoy, a child in a swing, the cone inside a speaker, a guitar, atoms in a crystal, the motion of chest cavities, and the beating of hearts all have in common? They all **oscillate**—that is, they move back and forth between two points. Many systems oscillate, and they have certain characteristics in common. All oscillations involve force and energy. You push a child in a swing to get the motion started. The energy of atoms vibrating in a crystal can be increased with heat. You put energy into a guitar string when you pluck it.

Some oscillations create **waves**. A guitar creates sound waves. You can make water waves in a swimming pool by slapping the water with your hand. Some, such as water waves, are visible. Some, such as sound waves, are not. But *every wave is a disturbance that moves from its source and carries energy*. Other examples of waves include earthquakes and visible light. Even subatomic particles, such as electrons, can behave like waves.

By studying oscillatory motion and waves, we shall find that a small number of underlying principles describe all of them and that wave phenomena are more common than you have ever imagined. We begin by studying the type of force that underlies the simplest oscillations and waves. We will then expand our exploration of oscillatory motion and waves to include concepts such as simple harmonic motion, uniform circular motion, and damped harmonic motion. Finally, we will explore what happens when two or more waves share the same space, in the phenomena known as superposition and interference.

(This media type is not supported in this reader. Click to open media in browser.)

### 11.1 Hooke's Law: Stress and Strain Revisited

![Figure 11.2](http://legacy.cnx.org/content/m42239/1.9/#concept-trailer-harmonic-motion)

**Figure 11.2** When displaced from its vertical equilibrium position, this plastic ruler oscillates back and forth because of the restoring force opposing displacement. When the ruler is on the left, there is a force to the right, and vice versa.

Newton’s first law implies that an object oscillating back and forth is experiencing forces. Without force, the object would move in a straight line at a constant speed rather than oscillate. Consider, for example, plucking a plastic ruler to the left as shown in **Figure 11.2**. The deformation of the ruler creates a force in the opposite direction, known as a **restoring force**. Once released, the restoring force causes the ruler to move back toward its stable equilibrium position, where the net force on it is zero. However, by the time the ruler gets there, it gains momentum and continues to move to the right, producing the opposite deformation. It is then forced to the left, back through equilibrium, and the process is repeated until dissipative forces dampen the motion. These forces remove mechanical energy from the system, gradually reducing the motion until the ruler comes to rest.

The simplest oscillations occur when the restoring force is directly proportional to displacement. When stress and strain were covered in **Newton’s Third Law of Motion**, the name was given to this relationship between force and displacement was Hooke’s law:

\[ F = -kx. \]  

(11.1)

Here, \( F \) is the restoring force, \( x \) is the displacement from equilibrium or deformation, and \( k \) is a constant related to the difficulty in deforming the system. The minus sign indicates the restoring force is in the direction opposite to the displacement.
Figure 11.3 (a) The plastic ruler has been released, and the restoring force is returning the ruler to its equilibrium position. (b) The net force is zero at the equilibrium position, but the ruler has momentum and continues to move to the right. (c) The restoring force is in the opposite direction. It stops the ruler and moves it back toward equilibrium again. (d) Now the ruler has momentum to the left. (e) In the absence of damping (caused by frictional forces), the ruler reaches its original position. From there, the motion will repeat itself.

The force constant $k$ is related to the rigidity (or stiffness) of a system—the larger the force constant, the greater the restoring force, and the stiffer the system. The units of $k$ are newtons per meter (N/m). For example, $k$ is directly related to Young’s modulus when we stretch a string. Figure 11.4 shows a graph of the absolute value of the restoring force versus the displacement for a system that can be described by Hooke’s law—a simple spring in this case. The slope of the graph equals the force constant $k$ in newtons per meter. A common physics laboratory exercise is to measure restoring forces created by springs, determine if they follow Hooke’s law, and calculate their force constants if they do.

![Graph showing the relationship between force and displacement](image)

**Figure 11.4** (a) A graph of absolute value of the restoring force versus displacement is displayed. The fact that the graph is a straight line means that the system obeys Hooke’s law. The slope of the graph is the force constant $k$. (b) The data in the graph were generated by measuring the displacement of a spring from equilibrium while supporting various weights. The restoring force equals the weight supported, if the mass is stationary.
Example 11.1 How Stiff Are Car Springs?

Figure 11.5 The mass of a car increases due to the introduction of a passenger. This affects the displacement of the car on its suspension system. (credit: exfordy on Flickr)

What is the force constant for the suspension system of a car that settles 1.20 cm when an 80.0-kg person gets in?

Strategy
Consider the car to be in its equilibrium position $x = 0$ before the person gets in. The car then settles down 1.20 cm, which means it is displaced to a position $x = -1.20 \times 10^{-2}$ m. At that point, the springs supply a restoring force $F$ equal to the person's weight $w = mg = (80.0 \text{ kg})(9.80 \text{ m/s}^2) = 784 \text{ N}$. We take this force to be $F$ in Hooke's law. Knowing $F$ and $x$, we can then solve the force constant $k$.

Solution
1. Solve Hooke's law, $F = -kx$, for $k$:
\[ k = \frac{-F}{x}. \]

Substitute known values and solve $k$:
\[ k = \frac{-784 \text{ N}}{-1.20 \times 10^{-2} \text{ m}} = 6.53 \times 10^4 \text{ N/m}. \]

Discussion
Note that $F$ and $x$ have opposite signs because they are in opposite directions—the restoring force is up, and the displacement is down. Also, note that the car would oscillate up and down when the person got in if it were not for damping (due to frictional forces) provided by shock absorbers. Bouncing cars are a sure sign of bad shock absorbers.

Energy in Hooke's Law of Deformation

In order to produce a deformation, work must be done. That is, a force must be exerted through a distance, whether you pluck a guitar string or compress a car spring. If the only result is deformation, and no work goes into thermal, sound, or kinetic energy, then all the work is initially stored in the deformed object as some form of potential energy. The potential energy stored in a spring is $PE_{el} = \frac{1}{2}kx^2$. Here, we generalize the idea to elastic potential energy for a deformation of any system that can be described by Hooke's law. Hence,
\[ PE_{el} = \frac{1}{2}kx^2, \tag{11.4} \]

where $PE_{el}$ is the elastic potential energy stored in any deformed system that obeys Hooke's law and has a displacement $x$ from equilibrium and a force constant $k$.

It is possible to find the work done in deforming a system in order to find the energy stored. This work is performed by an applied
force $F_{\text{app}}$. The applied force is exactly opposite to the restoring force (action-reaction), and so $F_{\text{app}} = kx$. Figure 11.6 shows a graph of the applied force versus deformation $x$ for a system that can be described by Hooke’s law. Work done on the system is force multiplied by distance, which equals the area under the curve or $(1/2)kx^2$ (Method A in the figure). Another way to determine the work is to note that the force increases linearly from 0 to $kx$, so that the average force is $(1/2)kx$, the distance moved is $x$, and thus $W = F_{\text{app}}d = [(1/2)(kx)](x) = (1/2)kx^2$ (Method B in the figure).

Figure 11.6 A graph of applied force versus distance for the deformation of a system that can be described by Hooke’s law is displayed. The work done on the system equals the area under the graph or the area of the triangle, which is half its base multiplied by its height, or $W = (1/2)kx^2$.

Example 11.2 Calculating Stored Energy: A Tranquilizer Gun Spring

We can use a toy gun’s spring mechanism to ask and answer two simple questions: (a) How much energy is stored in the spring of a tranquilizer gun that has a force constant of 50.0 N/m and is compressed 0.150 m? (b) If you neglect friction and the mass of the spring, at what speed will a 2.00-g projectile be ejected from the gun?

Figure 11.7 (a) In this image of the gun, the spring is uncompressed before being cocked. (b) The spring has been compressed a distance $x$, and the projectile is in place. (c) When released, the spring converts elastic potential energy $PE_{el}$ into kinetic energy.

Strategy for a
(a): The energy stored in the spring can be found directly from elastic potential energy equation, because $k$ and $x$ are given.

Solution for a
Entering the given values for $k$ and $x$ yields
PE\_cl = \frac{1}{2}kx^2 = \frac{1}{2}(50.0 \text{ N/m})(0.150 \text{ m})^2 = 0.563 \text{ N} \cdot \text{m}

= 0.563 \text{ J} \tag{11.5}

**Strategy for b**

Because there is no friction, the potential energy is converted entirely into kinetic energy. The expression for kinetic energy can be solved for the projectile’s speed.

**Solution for b**

1. Identify known quantities:

\[ \text{KE}_f = \text{PE}_{cl} \quad \text{or} \quad \frac{1}{2}mv^2 = \frac{1}{2}kx^2 = \text{PE}_{cl} = 0.563 \text{ J} \quad \tag{11.6} \]

2. Solve for \( v \):

\[ v = \left[ \frac{2\text{PE}_{cl}}{m} \right]^{1/2} = \left[ \frac{2(0.563 \text{ J})}{0.002 \text{ kg}} \right]^{1/2} = 23.7(\text{J/kg})^{1/2} \quad \tag{11.7} \]

3. Convert units: 23.7 m/s

**Discussion**

(a) and (b): This projectile speed is impressive for a tranquilizer gun (more than 80 km/h). The numbers in this problem seem reasonable. The force needed to compress the spring is small enough for an adult to manage, and the energy imparted to the dart is small enough to limit the damage it might do. Yet, the speed of the dart is great enough for it to travel an acceptable distance.

---

### Check Your Understanding

Envision holding the end of a ruler with one hand and deforming it with the other. When you let go, you can see the oscillations of the ruler. In what way could you modify this simple experiment to increase the rigidity of the system?

**Solution**

You could hold the ruler at its midpoint so that the part of the ruler that oscillates is half as long as in the original experiment.

---

### Check Your Understanding

If you apply a deforming force on an object and let it come to equilibrium, what happened to the work you did on the system?

**Solution**

It was stored in the object as potential energy.

---

### 11.2 Period and Frequency in Oscillations

When you pluck a guitar string, the resulting sound has a steady tone and lasts a long time. Each successive vibration of the string takes the same time as the previous one. We define **periodic motion** to be a motion that repeats itself at regular time intervals, such as exhibited by the guitar string or by an object on a spring moving up and down. The time to complete one oscillation remains constant and is called the **period** \( T \). Its units are usually seconds, but may be any convenient unit of time.

The word period refers to the time for some event whether repetitive or not, but we shall be primarily interested in periodic motion, which is by definition repetitive. A concept closely related to period is the **frequency** of an event. For example, if you get a paycheck twice a month, the frequency of payment is two per month and the period between checks is half a month. **Frequency**
\( f \) is defined to be the number of events per unit time. For periodic motion, frequency is the number of oscillations per unit time. The relationship between frequency and period is

\[
f = \frac{1}{T}.
\]  

(11.8)

The SI unit for frequency is the cycle per second, which is defined to be a hertz (Hz):

\[
1 \text{ Hz} = \frac{1\text{ cycle}}{s} \quad \text{or} \quad 1 \text{ Hz} = \frac{1}{s}
\]

(11.9)

A cycle is one complete oscillation. Note that a vibration can be a single or multiple event, whereas oscillations are usually repetitive for a significant number of cycles.

**Example 11.3 Determine the Frequency of Two Oscillations: Medical Ultrasound and the Period of Middle C**

We can use the formulas presented in this module to determine both the frequency based on known oscillations and the oscillation based on a known frequency. Let’s try one example of each. (a) A medical imaging device produces ultrasound by oscillating with a period of 0.400 µs. What is the frequency of this oscillation? (b) The frequency of middle C on a typical musical instrument is 264 Hz. What is the time for one complete oscillation?

**Strategy**

Both questions (a) and (b) can be answered using the relationship between period and frequency. In question (a), the period \( T \) is given and we are asked to find frequency \( f \). In question (b), the frequency \( f \) is given and we are asked to find the period \( T \).

**Solution a**

1. Substitute 0.400 µs for \( T \) in \( f = \frac{1}{T} \):

\[
f = \frac{1}{T} = \frac{1}{0.400 \times 10^{-6}} \text{ s}.
\]

Solve to find

\[
f = 2.50 \times 10^6 \text{ Hz}.
\]

(11.10)

(11.11)

**Discussion a**

The frequency of sound found in (a) is much higher than the highest frequency that humans can hear and, therefore, is called ultrasound. Appropriate oscillations at this frequency generate ultrasound used for noninvasive medical diagnoses, such as observations of a fetus in the womb.

**Solution b**

1. Identify the known values:
   - The time for one complete oscillation is the period \( T \):

   \[
f = \frac{1}{T}.
\]

   (11.12)

2. Solve for \( T \):

\[
T = \frac{1}{f}.
\]

(11.13)

3. Substitute the given value for the frequency into the resulting expression:

\[
T = \frac{1}{f} = \frac{1}{264 \text{ Hz}} = \frac{1}{264 \text{ cycles/s}} = 3.79 \times 10^{-3} \text{ s} = 3.79 \text{ ms}.
\]

(11.14)

**Discussion b**

The period found in (b) is the time per cycle, but this value is often quoted as simply the time in convenient units (ms or milliseconds in this case).

**Check Your Understanding**

Identify an event in your life (such as receiving a paycheck) that occurs regularly. Identify both the period and frequency of this event.

**Solution**

I visit my parents for dinner every other Sunday. The frequency of my visits is 26 per calendar year. The period is two weeks.
11.3 Simple Harmonic Motion: A Special Periodic Motion

The oscillations of a system in which the net force can be described by Hooke's law are of special importance, because they are very common. They are also the simplest oscillatory systems. Simple Harmonic Motion (SHM) is the name given to oscillatory motion for a system where the net force can be described by Hooke's law, and such a system is called a simple harmonic oscillator. If the net force can be described by Hooke's law and there is no damping (by friction or other non-conservative forces), then a simple harmonic oscillator will oscillate with equal displacement on either side of the equilibrium position, as shown for an object on a spring in Figure 11.9. The maximum displacement from equilibrium is called the amplitude, $X$. The units for amplitude and displacement are the same, but depend on the type of oscillation. For the object on the spring, the units of amplitude and displacement are meters; whereas for sound oscillations, they have units of pressure (and other types of oscillations have yet other units). Because amplitude is the maximum displacement, it is related to the energy in the oscillation.

Take-Home Experiment: SHM and the Marble

Find a bowl or basin that is shaped like a hemisphere on the inside. Place a marble inside the bowl and tilt the bowl periodically so the marble rolls from the bottom of the bowl to equally high points on the sides of the bowl. Get a feel for the force required to maintain this periodic motion. What is the restoring force and what role does the force you apply play in the simple harmonic motion (SHM) of the marble?

Figure 11.9 An object attached to a spring sliding on a frictionless surface is an uncomplicated simple harmonic oscillator. When displaced from equilibrium, the object performs simple harmonic motion that has an amplitude, $X$, and a period, $T$. The object's maximum speed occurs as it passes through equilibrium. The stiffer the spring is, the smaller the period, $T$. The greater the mass of the object is, the greater the period, $T$.

What is so significant about simple harmonic motion? One special thing is that the period, $T$, and frequency, $f$, of a simple harmonic oscillator are independent of amplitude. The string of a guitar, for example, will oscillate with the same frequency whether plucked gently or hard. Because the period is constant, a simple harmonic oscillator can be used as a clock.

Two important factors do affect the period of a simple harmonic oscillator. The period is related to how stiff the system is. A very stiff object has a large force constant, $k$, which causes the system to have a smaller period. For example, you can adjust a diving board's stiffness—the stiffer it is, the faster it vibrates, and the shorter its period. Period also depends on the mass of the oscillating system. The more massive the system is, the longer the period. For example, a heavy person on a diving board bounces up and down more slowly than a light one.

In fact, the mass, $m$, and the force constant, $k$, are the only factors that affect the period and frequency of simple harmonic motion.

**Period of Simple Harmonic Oscillator**

The period of a simple harmonic oscillator is given by
\[ T = 2\pi \sqrt{\frac{m}{k}} \]  
(11.15)

and, because \( f = 1 / T \), the frequency of a simple harmonic oscillator is

\[ f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \]  
(11.16)

Note that neither \( T \) nor \( f \) has any dependence on amplitude.

**Take-Home Experiment: Mass and Ruler Oscillations**

Find two identical wooden or plastic rulers. Tape one end of each ruler firmly to the edge of a table so that the length of each ruler that protrudes from the table is the same. On the free end of one ruler tape a heavy object such as a few large coins. Pluck the ends of the rulers at the same time and observe which one undergoes more cycles in a time period, and measure the period of oscillation of each of the rulers.

**Example 11.4 Calculate the Frequency and Period of Oscillations: Bad Shock Absorbers in a Car**

If the shock absorbers in a car go bad, then the car will oscillate at the least provocation, such as when going over bumps in the road and after stopping (See Figure 11.10). Calculate the frequency and period of these oscillations for such a car if the car’s mass (including its load) is 900 kg and the force constant \( k \) of the suspension system is \( 6.53 \times 10^4 \) N/m.

**Strategy**

The frequency of the car’s oscillations will be that of a simple harmonic oscillator as given in the equation \( f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \). The mass and the force constant are both given.

**Solution**

1. Enter the known values of \( k \) and \( m \):

\[ f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{6.53 \times 10^4 \text{ N/m}}{900 \text{ kg}}} \]  
(11.17)

2. Calculate the frequency:

\[ \frac{1}{2\pi} \sqrt{72.6 / \text{s}^2} = 1.3656 / \text{s}^{-1} \approx 1.36 / \text{s}^{-1} = 1.36 \text{ Hz}. \]  
(11.18)

3. You could use \( T = 2\pi \sqrt{\frac{m}{k}} \) to calculate the period, but it is simpler to use the relationship \( T = 1 / f \) and substitute the value just found for \( f \):

\[ T = \frac{1}{f} = \frac{1}{1.356 \text{ Hz}} = 0.738 \text{ s}. \]  
(11.19)

**Discussion**

The values of \( T \) and \( f \) both seem about right for a bouncing car. You can observe these oscillations if you push down hard on the end of a car and let go.

**The Link between Simple Harmonic Motion and Waves**

If a time-exposure photograph of the bouncing car were taken as it drove by, the headlight would make a wavelike streak, as shown in Figure 11.10. Similarly, Figure 11.11 shows an object bouncing on a spring as it leaves a wavelike “trace” of its position on a moving strip of paper. Both waves are sine functions. All simple harmonic motion is intimately related to sine and cosine waves.

**Figure 11.10** The bouncing car makes a wavelike motion. If the restoring force in the suspension system can be described only by Hooke’s law, then the wave is a sine function. (The wave is the trace produced by the headlight as the car moves to the right.)
The displacement as a function of time \( t \) in any simple harmonic motion—that is, one in which the net restoring force can be described by Hooke’s law, is given by

\[
x(t) = X \cos \frac{2\pi t}{T},
\]  

where \( X \) is amplitude. At \( t = 0 \), the initial position is \( x_0 = X \), and the displacement oscillates back and forth with a period \( T \). (When \( t = T \), we get \( x = X \) again because \( \cos 2\pi = 1 \).) Furthermore, from this expression for \( x \), the velocity \( v \) as a function of time is given by:

\[
v(t) = -v_{\text{max}} \sin \left( \frac{2\pi t}{T} \right),
\]

where \( v_{\text{max}} = \frac{2\pi X}{T} = X \sqrt{k/m} \). The object has zero velocity at maximum displacement—for example, \( v = 0 \) when \( t = 0 \), and at that time \( x = X \). The minus sign in the first equation for \( v(t) \) gives the correct direction for the velocity. Just after the start of the motion, for instance, the velocity is negative because the system is moving back toward the equilibrium point. Finally, we can get an expression for acceleration using Newton’s second law. [Then we have \( x(t), v(t), t, \) and \( a(t) \), the quantities needed for kinematics and a description of simple harmonic motion.] According to Newton’s second law, the acceleration is \( a = F/m = kx/m \). So, \( a(t) \) is also a cosine function:

\[
a(t) = -\frac{kX}{m} \cos \frac{2\pi t}{T}.
\]

Hence, \( a(t) \) is directly proportional to and in the opposite direction to \( x(t) \).

Figure 11.12 shows the simple harmonic motion of an object on a spring and presents graphs of \( x(t), v(t), \) and \( a(t) \) versus time.
Figure 11.12 Graphs of $x(t)$, $v(t)$, and $a(t)$ versus $t$ for the motion of an object on a spring. The net force on the object can be described by Hooke’s law, and so the object undergoes simple harmonic motion. Note that the initial position has the vertical displacement at its maximum value $X$; $v$ is initially zero and then negative as the object moves down; and the initial acceleration is negative, back toward the equilibrium position and becomes zero at that point.

The most important point here is that these equations are mathematically straightforward and are valid for all simple harmonic motion. They are very useful in visualizing waves associated with simple harmonic motion, including visualizing how waves add with one another.

Check Your Understanding

Suppose you pluck a banjo string. You hear a single note that starts out loud and slowly quiets over time. Describe what happens to the sound waves in terms of period, frequency and amplitude as the sound decreases in volume.

Solution
Frequency and period remain essentially unchanged. Only amplitude decreases as volume decreases.

Check Your Understanding

A babysitter is pushing a child on a swing. At the point where the swing reaches $x$, where would the corresponding point on a wave of this motion be located?

Solution
\( x \) is the maximum deformation, which corresponds to the amplitude of the wave. The point on the wave would either be at the very top or the very bottom of the curve.

**Masses and Springs**

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energy for each spring.

(This media type is not supported in this reader. Click to open media in browser.)

Figure 11.13

### 11.4 The Simple Pendulum

![Figure 11.14 A simple pendulum has a small-diameter bob and a string that has a very small mass but is strong enough not to stretch appreciably. The linear displacement from equilibrium is \( s \), the length of the arc. Also shown are the forces on the bob, which result in a net force of \(-mg \sin \theta\) toward the equilibrium position—that is, a restoring force. Pendulums are in common usage. Some have crucial uses, such as in clocks; some are for fun, such as a child’s swing; and some are just there, such as the sinker on a fishing line. For small displacements, a pendulum is a simple harmonic oscillator. A simple pendulum is defined to have an object that has a small mass, also known as the pendulum bob, which is suspended from a light wire or string, such as shown in Figure 11.14. Exploring the simple pendulum a bit further, we can discover the conditions under which it performs simple harmonic motion, and we can derive an interesting expression for its period.

We begin by defining the displacement to be the arc length \( s \). We see from Figure 11.14 that the net force on the bob is tangent to the arc and equals \(-mg \sin \theta\). (The weight \( mg \) has components \( mg \cos \theta \) along the string and \( mg \sin \theta \) tangent to the arc.) Tension in the string exactly cancels the component \( mg \cos \theta \) parallel to the string. This leaves a net restoring force back toward the equilibrium position at \( \theta = 0 \).

Now, if we can show that the restoring force is directly proportional to the displacement, then we have a simple harmonic oscillator. In trying to determine if we have a simple harmonic oscillator, we should note that for small angles (less than about 15º), \( \sin \theta \approx \theta \) (\( \sin \theta \) and \( \theta \) differ by about 1% or less at smaller angles). Thus, for angles less than about 15º, the restoring force \( F \) is

\[
F \approx -mg\theta.
\]

The displacement \( s \) is directly proportional to \( \theta \). When \( \theta \) is expressed in radians, the arc length in a circle is related to its radius (\( L \) in this instance) by:

\[
s = L\theta,
\]

so that

\[
\theta = \frac{s}{L}.
\]

For small angles, then, the expression for the restoring force is:

\[
F \approx -\frac{mg}{L}s
\]

This expression is of the form:

\[
F = -kx,
\]
where the force constant is given by \( k = \frac{mg}{L} \) and the displacement is given by \( x = s \). For angles less than about 15°, the restoring force is directly proportional to the displacement, and the simple pendulum is a simple harmonic oscillator.

Using this equation, we can find the period of a pendulum for amplitudes less than about 15°. For the simple pendulum:

\[
T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{m}{mg/L}}.
\]  
(11.28)

Thus,

\[
T = 2\pi \sqrt{\frac{T}{g}}
\]  
(11.29)

for the period of a simple pendulum. This result is interesting because of its simplicity. The only things that affect the period of a simple pendulum are its length and the acceleration due to gravity. The period is completely independent of other factors, such as mass. As with simple harmonic oscillators, the period \( T \) for a pendulum is nearly independent of amplitude, especially if \( \theta \) is less than about 15°. Even simple pendulum clocks can be finely adjusted and accurate.

Note the dependence of \( T \) on \( g \). If the length of a pendulum is precisely known, it can actually be used to measure the acceleration due to gravity. Consider the following example.

**Example 11.5 Measuring Acceleration due to Gravity: The Period of a Pendulum**

What is the acceleration due to gravity in a region where a simple pendulum having a length 75.000 cm has a period of 1.7357 s?

**Strategy**

We are asked to find \( g \) given the period \( T \) and the length \( L \) of a pendulum. We can solve \( T = 2\pi \sqrt{\frac{T}{g}} \) for \( g \), assuming only that the angle of deflection is less than 15°.

**Solution**

1. Square \( T = 2\pi \sqrt{\frac{T}{g}} \) and solve for \( g \):

\[
g = 4\pi^2 \frac{L}{T^2}.
\]  
(11.30)

2. Substitute known values into the new equation:

\[
g = 4\pi^2 \frac{0.75000 \text{ m}}{(1.7357 \text{ s})^2}.
\]  
(11.31)

3. Calculate to find \( g \):

\[
g = 9.8281 \text{ m/s}^2.
\]  
(11.32)

**Discussion**

This method for determining \( g \) can be very accurate. This is why length and period are given to five digits in this example. For the precision of the approximation \( \sin \theta \approx \theta \) to be better than the precision of the pendulum length and period, the maximum displacement angle should be kept below about 0.5°.

**Making Career Connections**

Knowing \( g \) can be important in geological exploration; for example, a map of \( g \) over large geographical regions aids the study of plate tectonics and helps in the search for oil fields and large mineral deposits.

**Take Home Experiment: Determining \( g \)**

Use a simple pendulum to determine the acceleration due to gravity \( g \) in your own locale. Cut a piece of a string or dental floss so that it is about 1 m long. Attach a small object of high density to the end of the string (for example, a metal nut or a car key). Starting at an angle of less than 10°, allow the pendulum to swing and measure the pendulum’s period for 10 oscillations using a stopwatch. Calculate \( g \). How accurate is this measurement? How might it be improved?
Check Your Understanding

An engineer builds two simple pendula. Both are suspended from small wires secured to the ceiling of a room. Each pendulum hovers 2 cm above the floor. Pendulum 1 has a bob with a mass of 10 kg. Pendulum 2 has a bob with a mass of 100 kg. Describe how the motion of the pendula will differ if the bobs are both displaced by 12°.

Solution
The movement of the pendula will not differ at all because the mass of the bob has no effect on the motion of a simple pendulum. The pendula are only affected by the period (which is related to the pendulum’s length) and by the acceleration due to gravity.

Pendulum Lab
Play with one or two pendulums and discover how the period of a simple pendulum depends on the length of the string, the mass of the pendulum bob, and the amplitude of the swing. It’s easy to measure the period using the photogate timer. You can vary friction and the strength of gravity. Use the pendulum to find the value of $g$ on planet X. Notice the anharmonic behavior at large amplitude.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42243/1.16/#eip-idm504883360)

Figure 11.15

11.5 Energy and the Simple Harmonic Oscillator

To study the energy of a simple harmonic oscillator, we first consider all the forms of energy it can have. We know from Hooke’s Law: Stress and Strain Revisited that the energy stored in the deformation of a simple harmonic oscillator is a form of potential energy given by:

$$PE_{el} = \frac{1}{2}kx^2.$$  \hspace{1cm} (11.33)

Because a simple harmonic oscillator has no dissipative forces, the other important form of energy is kinetic energy $KE$.
Conservation of energy for these two forms is:

$$KE + PE_{el} = \text{constant}$$  \hspace{1cm} (11.34)

or

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}.$$  \hspace{1cm} (11.35)

This statement of conservation of energy is valid for all simple harmonic oscillators, including ones where the gravitational force plays a role.
Namely, for a simple pendulum we replace the velocity with $v = L\omega$, the spring constant with $k = mg/L$, and the displacement term with $x = L\theta$. Thus

$$\frac{1}{2}mL^2\omega^2 + \frac{1}{2}mgL\theta^2 = \text{constant}.$$  \hspace{1cm} (11.36)

In the case of undamped simple harmonic motion, the energy oscillates back and forth between kinetic and potential, going completely from one to the other as the system oscillates. So for the simple example of an object on a frictionless surface attached to a spring, as shown again in Figure 11.16, the motion starts with all of the energy stored in the spring. As the object starts to move, the elastic potential energy is converted to kinetic energy, becoming entirely kinetic energy at the equilibrium position. It is then converted back into elastic potential energy by the spring, the velocity becomes zero when the kinetic energy is completely converted, and so on. This concept provides extra insight here and in later applications of simple harmonic motion, such as alternating current circuits.
Figure 11.16 The transformation of energy in simple harmonic motion is illustrated for an object attached to a spring on a frictionless surface.

The conservation of energy principle can be used to derive an expression for velocity \( v \). If we start our simple harmonic motion with zero velocity and maximum displacement \( x = X \), then the total energy is

\[
\frac{1}{2} kX^2.
\] (11.37)

This total energy is constant and is shifted back and forth between kinetic energy and potential energy, at most times being shared by each. The conservation of energy for this system in equation form is thus:

\[
\frac{1}{2} m v^2 + \frac{1}{2} kx^2 = \frac{1}{2} kX^2.
\] (11.38)

Solving this equation for \( v \) yields:

\[
v = \pm \sqrt{\frac{k}{m}(X^2 - x^2))}.
\] (11.39)

Manipulating this expression algebraically gives:

\[
v = \pm \sqrt{\frac{k}{m}} \sqrt{1 - \frac{x^2}{X^2}}
\] (11.40)

and so

\[
v = \pm v_{\text{max}} \sqrt{1 - \frac{x^2}{X^2}},
\] (11.41)

where

\[
v_{\text{max}} = \sqrt{\frac{k}{m} X}.
\] (11.42)

From this expression, we see that the velocity is a maximum \( v_{\text{max}} \) at \( x = 0 \), as stated earlier in \( v(t) = - v_{\text{max}} \sin \frac{2\pi t}{T} \).

Notice that the maximum velocity depends on three factors. Maximum velocity is directly proportional to amplitude. As you might guess, the greater the maximum displacement the greater the maximum velocity. Maximum velocity is also greater for stiffer systems, because they exert greater force for the same displacement. This observation is seen in the expression for \( v_{\text{max}} \); it is proportional to the square root of the force constant \( k \). Finally, the maximum velocity is smaller for objects that have larger masses, because the maximum velocity is inversely proportional to the square root of \( m \). For a given force, objects that have large masses accelerate more slowly.

A similar calculation for the simple pendulum produces a similar result, namely:
\[ \omega_{\text{max}} = \sqrt{\frac{F}{m}} \theta_{\text{max}}. \]  

Example 11.6 Determine the Maximum Speed of an Oscillating System: A Bumpy Road

Suppose that a car is 900 kg and has a suspension system that has a force constant \( k = 6.53 \times 10^4 \) N/m. The car hits a bump and bounces with an amplitude of 0.100 m. What is its maximum vertical velocity if you assume no damping occurs?

**Strategy**

We can use the expression for \( v_{\text{max}} \) given in \( v_{\text{max}} = \sqrt{\frac{F}{m}} X \) to determine the maximum vertical velocity. The variables \( m \) and \( k \) are given in the problem statement, and the maximum displacement \( X \) is 0.100 m.

**Solution**

1. Identify known.

2. Substitute known values into \( v_{\text{max}} = \sqrt{\frac{F}{m}} X \):

   \[ v_{\text{max}} = \frac{6.53 \times 10^4 \text{ N/m}}{900 \text{ kg}} (0.100 \text{ m}). \]  

3. Calculate to find \( v_{\text{max}} = 0.852 \text{ m/s} \).

**Discussion**

This answer seems reasonable for a bouncing car. There are other ways to use conservation of energy to find \( v_{\text{max}} \). We could use it directly, as was done in the example featured in *Hooke's Law: Stress and Strain Revisited*.

The small vertical displacement \( y \) of an oscillating simple pendulum, starting from its equilibrium position, is given as

\[ y(t) = a \sin \omega t, \]  

(11.45)

where \( a \) is the amplitude, \( \omega \) is the angular velocity and \( t \) is the time taken. Substituting \( \omega = \frac{2\pi}{T} \), we have

\[ y(t) = a \sin \left( \frac{2\pi t}{T} \right). \]  

(11.46)

Thus, the displacement of pendulum is a function of time as shown above.

Also the velocity of the pendulum is given by

\[ v(t) = \frac{2a\pi}{T} \cos \left( \frac{2\pi t}{T} \right). \]  

(11.47)

so the motion of the pendulum is a function of time.

**Check Your Understanding**

Why does it hurt more if your hand is snapped with a ruler than with a loose spring, even if the displacement of each system is equal?

**Solution**

The ruler is a stiffer system, which carries greater force for the same amount of displacement. The ruler snaps your hand with greater force, which hurts more.

**Check Your Understanding**

You are observing a simple harmonic oscillator. Identify one way you could decrease the maximum velocity of the system.

**Solution**

You could increase the mass of the object that is oscillating.

### 11.6 Uniform Circular Motion and Simple Harmonic Motion
There is an easy way to produce simple harmonic motion by using uniform circular motion. Figure 11.18 shows one way of using this method. A ball is attached to a uniformly rotating vertical turntable, and its shadow is projected on the floor as shown. The shadow undergoes simple harmonic motion. Hooke’s law usually describes uniform circular motions (ω constant) rather than systems that have large visible displacements. So observing the projection of uniform circular motion, as in Figure 11.18, is often easier than observing a precise large-scale simple harmonic oscillator. If studied in sufficient depth, simple harmonic motion produced in this manner can give considerable insight into many aspects of oscillations and waves and is very useful mathematically. In our brief treatment, we shall indicate some of the major features of this relationship and how they might be useful.

Figure 11.17 The horses on this merry-go-round exhibit uniform circular motion. (credit: Wonderlane, Flickr)

Figure 11.18 The shadow of a ball rotating at constant angular velocity ω on a turntable goes back and forth in precise simple harmonic motion.

Figure 11.19 shows the basic relationship between uniform circular motion and simple harmonic motion. The point P travels around the circle at constant angular velocity ω. The point P is analogous to an object on the merry-go-round. The projection of the position of P onto a fixed axis undergoes simple harmonic motion and is analogous to the shadow of the object. At the time shown in the figure, the projection has position x and moves to the left with velocity v. The velocity of the point P around the circle equals v_max. The projection of v_max on the x-axis is the velocity v of the simple harmonic motion along the x-axis.
A point $P$ moving on a circular path with a constant angular velocity $\omega$ is undergoing uniform circular motion. Its projection on the x-axis undergoes simple harmonic motion. Also shown is the velocity of this point around the circle, $\vec{v}_{\text{max}}$, and its projection, which is $\vec{v}$. Note that these velocities form a similar triangle to the displacement triangle.

To see that the projection undergoes simple harmonic motion, note that its position $x$ is given by

$$x = X \cos \theta,$$

where $\theta = \omega t$, $\omega$ is the constant angular velocity, and $X$ is the radius of the circular path. Thus,

$$x = X \cos \omega t. \tag{11.49}$$

The angular velocity $\omega$ is in radians per unit time; in this case $2\pi$ radians is the time for one revolution $T$. That is, $\omega = 2\pi / T$. Substituting this expression for $\omega$, we see that the position $x$ is given by:

$$x(t) = \cos \left( \frac{2\pi t}{T} \right). \tag{11.50}$$

This expression is the same one we had for the position of a simple harmonic oscillator in Simple Harmonic Motion: A Special Periodic Motion. If we make a graph of position versus time as in Figure 11.20, we see again the wavelike character (typical of simple harmonic motion) of the projection of uniform circular motion onto the $x$-axis.

Now let us use Figure 11.19 to do some further analysis of uniform circular motion as it relates to simple harmonic motion. The triangle formed by the velocities in the figure and the triangle formed by the displacements ($X$, $x$, and $\sqrt{X^2 - x^2}$) are similar right triangles. Taking ratios of similar sides, we see that
\[ \frac{v}{v_{\text{max}}} = \sqrt{1 - \frac{x^2}{X^2}} = \sqrt{1 - \frac{x^2}{X^2}} \quad (11.51) \]

We can solve this equation for the speed \( v \) or

\[ v = v_{\text{max}} \sqrt{1 - \frac{x^2}{X^2}} \quad (11.52) \]

This expression for the speed of a simple harmonic oscillator is exactly the same as the equation obtained from conservation of energy considerations in **Energy and the Simple Harmonic Oscillator**. You can begin to see that it is possible to get all of the characteristics of simple harmonic motion from an analysis of the projection of uniform circular motion.

Finally, let us consider the period \( T \) of the motion of the projection. This period is the time it takes the point \( P \) to complete one revolution. That time is the circumference of the circle \( 2\pi X \) divided by the velocity around the circle, \( v_{\text{max}} \). Thus, the period \( T \) is

\[ T = \frac{2\pi X}{v_{\text{max}}} \quad (11.53) \]

We know from conservation of energy considerations that

\[ v_{\text{max}} = \sqrt{\frac{k}{m} X} \quad (11.54) \]

Solving this equation for \( X/v_{\text{max}} \) gives

\[ \frac{X}{v_{\text{max}}} = \sqrt{\frac{m}{k}} \quad (11.55) \]

Substituting this expression into the equation for \( T \) yields

\[ T = 2\pi \sqrt{\frac{m}{k}} \quad (11.56) \]

Thus, the period of the motion is the same as for a simple harmonic oscillator. We have determined the period for any simple harmonic oscillator using the relationship between uniform circular motion and simple harmonic motion.

Some modules occasionally refer to the connection between uniform circular motion and simple harmonic motion. Moreover, if you carry your study of physics and its applications to greater depths, you will find this relationship useful. It can, for example, help to analyze how waves add when they are superimposed.

### Check Your Understanding

Identify an object that undergoes uniform circular motion. Describe how you could trace the simple harmonic motion of this object as a wave.

**Solution**

A record player undergoes uniform circular motion. You could attach dowel rod to one point on the outside edge of the turntable and attach a pen to the other end of the dowel. As the record player turns, the pen will move. You can drag a long piece of paper under the pen, capturing its motion as a wave.

#### 11.7 Damped Harmonic Motion

A guitar string stops oscillating a few seconds after being plucked. To keep a child happy on a swing, you must keep pushing. Although we can often make friction and other non-conservative forces negligibly small, completely undamped motion is rare. In fact, we may even want to damp oscillations, such as with car shock absorbers.

For a system that has a small amount of damping, the period and frequency are nearly the same as for simple harmonic motion, but the amplitude gradually decreases as shown in **Figure 11.22**. This occurs because the non-conservative damping force...
removes energy from the system, usually in the form of thermal energy. In general, energy removal by non-conservative forces is described as

\[ W_{nc} = \Delta (KE + PE), \]  

(11.57)

where \( W_{nc} \) is work done by a non-conservative force (here the damping force). For a damped harmonic oscillator, \( W_{nc} \) is negative because it removes mechanical energy (\( KE + PE \)) from the system.

Figure 11.22 In this graph of displacement versus time for a harmonic oscillator with a small amount of damping, the amplitude slowly decreases, but the period and frequency are nearly the same as if the system were completely undamped.

If you gradually increase the amount of damping in a system, the period and frequency begin to be affected, because damping opposes and hence slows the back and forth motion. (The net force is smaller in both directions.) If there is very large damping, the system does not even oscillate—it slowly moves toward equilibrium. Figure 11.23 shows the displacement of a harmonic oscillator for different amounts of damping. When we want to damp out oscillations, such as in the suspension of a car, we may want the system to return to equilibrium as quickly as possible Critical damping is defined as the condition in which the damping of an oscillator results in it returning as quickly as possible to its equilibrium position The critically damped system may overshoot the equilibrium position, but if it does, it will do so only once. Critical damping is represented by Curve A in Figure 11.23. With less-than critical damping, the system will return to equilibrium faster but will overshoot and cross over one or more times. Such a system is underdamped; its displacement is represented by the curve in Figure 11.22. Curve B in Figure 11.23 represents an overdamped system. As with critical damping, it too may overshoot the equilibrium position, but will reach equilibrium over a longer period of time.

Figure 11.23 Displacement versus time for a critically damped harmonic oscillator (A) and an overdamped harmonic oscillator (B). The critically damped oscillator returns to equilibrium at \( X = 0 \) in the smallest time possible without overshooting.

Critical damping is often desired, because such a system returns to equilibrium rapidly and remains at equilibrium as well. In addition, a constant force applied to a critically damped system moves the system to a new equilibrium position in the shortest time possible without overshooting or oscillating about the new position. For example, when you stand on bathroom scales that have a needle gauge, the needle moves to its equilibrium position without oscillating. It would be quite inconvenient if the needle oscillated about the new equilibrium position for a long time before settling. Damping forces can vary greatly in character. Friction, for example, is sometimes independent of velocity (as assumed in most places in this text). But many damping forces depend on velocity—sometimes in complex ways, sometimes simply being proportional to velocity.

Example 11.7 Damping an Oscillatory Motion: Friction on an Object Connected to a Spring

Damping oscillatory motion is important in many systems, and the ability to control the damping is even more so. This is generally attained using non-conservative forces such as the friction between surfaces, and viscosity for objects moving through fluids. The following example considers friction. Suppose a 0.200-kg object is connected to a spring as shown in Figure 11.24, but there is simple friction between the object and the surface, and the coefficient of friction \( \mu_k \) is equal to 0.0800. (a) What is the frictional force between the surfaces? (b) What total distance does the object travel if it is released 0.100 m from equilibrium, starting at \( v = 0 \)? The force constant of the spring is \( k = 50.0 \, \text{N/m} \).
Strategy

This problem requires you to integrate your knowledge of various concepts regarding waves, oscillations, and damping. To solve an integrated concept problem, you must first identify the physical principles involved. Part (a) is about the frictional force. This is a topic involving the application of Newton’s Laws. Part (b) requires an understanding of work and conservation of energy, as well as some understanding of horizontal oscillatory systems.

Now that we have identified the principles we must apply in order to solve the problems, we need to identify the knowns and unknowns for each part of the question, as well as the quantity that is constant in Part (a) and Part (b) of the question.

Solution a

1. Choose the proper equation: Friction is \( f = \mu_k mg \).
2. Identify the known values.
3. Enter the known values into the equation:
   \[
   f = (0.0800)(0.200 \text{ kg})(9.80 \text{ m/s}^2).
   \]
   \[(11.58)\]
4. Calculate and convert units: \( f = 0.157 \text{ N} \).

Discussion a

The force here is small because the system and the coefficients are small.

Solution b

1. Choose the proper equation: Friction is \( f = \mu_k mg \).
2. Identify the known values.
3. Enter the known values into the equation:
   \[
   f = (0.0800)(0.200 \text{ kg})(9.80 \text{ m/s}^2).
   \]
   \[(11.58)\]
4. Calculate and convert units: \( f = 0.157 \text{ N} \).

Discussion a

The force here is small because the system and the coefficients are small.

Solution b

1. Choose the proper equation: Friction is \( f = \mu_k mg \).
2. Identify the known values.
3. Enter the known values into the equation:
   \[
   f = (0.0800)(0.200 \text{ kg})(9.80 \text{ m/s}^2).
   \]
   \[(11.58)\]
4. Calculate and convert units: \( f = 0.157 \text{ N} \).

Discussion a

The force here is small because the system and the coefficients are small.
equation to use:

\[ W_{nc} = \Delta (KE + PE) = PE_{el,f} - PE_{el,i} = \frac{1}{2}k\left(\frac{\mu k mg}{k} \right)^2 - X^2 \].

(11.60)

3. Recall that \( W_{nc} = -fd \).

4. Enter the friction as \( f = \mu k mg \) into \( W_{nc} = -fd \), thus

\[ W_{nc} = -\mu k mgd. \]

(11.61)

5. Combine these two equations to find

\[ \frac{1}{2}k\left(\frac{\mu k mg}{k} \right)^2 - X^2 = -\mu k mgd. \]

(11.62)

6. Solve the equation for \( d \):

\[ d = \frac{k}{2\mu k mg}\left(X^2 - \left(\frac{\mu k mg}{k} \right)^2\right) \]

(11.63)

7. Enter the known values into the resulting equation:

\[ \frac{50.0 \text{ N/m}}{2(0.0800)(0.200 \text{ kg})(9.80 \text{ m/s}^2)} \left( \frac{d}{(0.100 \text{ m})} \right) = \left( \frac{0.0800(0.200 \text{ kg})(9.80 \text{ m/s}^2)}{50.0 \text{ N/m}} \right) \]

(11.64)

8. Calculate \( d \) and convert units:

\[ d = 1.59 \text{ m}. \]

(11.65)

Discussion b

This is the total distance traveled back and forth across \( x = 0 \), which is the undamped equilibrium position. The number of oscillations about the equilibrium position will be more than \( d/X = (1.59 \text{ m})/(0.100 \text{ m}) = 15.9 \) because the amplitude of the oscillations is decreasing with time. At the end of the motion, this system will not return to \( x = 0 \) for this type of damping force, because static friction will exceed the restoring force. This system is underdamped. In contrast, an overdamped system with a simple constant damping force would not cross the equilibrium position \( x = 0 \) a single time. For example, if this system had a damping force 20 times greater, it would only move 0.0484 m toward the equilibrium position from its original 0.100-m position.

This worked example illustrates how to apply problem-solving strategies to situations that integrate the different concepts you have learned. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknowns using familiar problem-solving strategies. These are found throughout the text, and many worked examples show how to use them for single topics. In this integrated concepts example, you can see how to apply them across several topics. You will find these techniques useful in applications of physics outside a physics course, such as in your profession, in other science disciplines, and in everyday life.

Check Your Understanding

Why are completely undamped harmonic oscillators so rare?

Solution

Friction often comes into play whenever an object is moving. Friction causes damping in a harmonic oscillator.

Check Your Understanding

Describe the difference between overdamping, underdamping, and critical damping.

Solution

An overdamped system moves slowly toward equilibrium. An underdamped system moves quickly to equilibrium, but will oscillate about the equilibrium point as it does so. A critically damped system moves as quickly as possible toward equilibrium without oscillating about the equilibrium.
11.8 Forced Oscillations and Resonance

Sit in front of a piano sometime and sing a loud brief note at it with the dampers off its strings. It will sing the same note back at you—the strings, having the same frequencies as your voice, are resonating in response to the forces from the sound waves that you sent to them. Your voice and a piano’s strings is a good example of the fact that objects—in this case, piano strings—can be forced to oscillate but oscillate best at their natural frequency. In this section, we shall briefly explore applying a periodic driving force acting on a simple harmonic oscillator. The driving force puts energy into the system at a certain frequency, not necessarily the same as the natural frequency of the system. The natural frequency is the frequency at which a system would oscillate if there were no driving and no damping force.

Most of us have played with toys involving an object supported on an elastic band, something like the paddle ball suspended from a finger in Figure 11.26. Imagine the finger in the figure is your finger. At first you hold your finger steady, and the ball bounces up and down with a small amount of damping. If you move your finger up and down slowly, the ball will follow along without bouncing much on its own. As you increase the frequency at which you move your finger up and down, the ball will respond by oscillating with increasing amplitude. When you drive the ball at its natural frequency, the ball’s oscillations increase in amplitude with each oscillation for as long as you drive it. The phenomenon of driving a system with a frequency equal to its natural frequency is called resonance. A system being driven at its natural frequency is said to resonate. As the driving frequency gets progressively higher than the resonant or natural frequency, the amplitude of the oscillations becomes smaller, until the oscillations nearly disappear and your finger simply moves up and down with little effect on the ball.

Figure 11.26 The paddle ball on its rubber band moves in response to the finger supporting it. If the finger moves with the natural frequency \( f_0 \) of the ball on the rubber band, then a resonance is achieved, and the amplitude of the ball’s oscillations increases dramatically. At higher and lower driving frequencies, energy is transferred to the ball less efficiently, and it responds with lower-amplitude oscillations.

Figure 11.27 shows a graph of the amplitude of a damped harmonic oscillator as a function of the frequency of the periodic force driving it. There are three curves on the graph, each representing a different amount of damping. All three curves peak at the point where the frequency of the driving force equals the natural frequency of the harmonic oscillator. The highest peak, or greatest response, is for the least amount of damping, because less energy is removed by the damping force.
Figure 11.27 Amplitude of a harmonic oscillator as a function of the frequency of the driving force. The curves represent the same oscillator with the same natural frequency but with different amounts of damping. Resonance occurs when the driving frequency equals the natural frequency, and the greatest response is for the least amount of damping. The narrowest response is also for the least damping.

It is interesting that the widths of the resonance curves shown in Figure 11.27 depend on damping: the less the damping, the narrower the resonance. The message is that if you want a driven oscillator to resonate at a very specific frequency, you need as little damping as possible. Little damping is the case for piano strings and many other musical instruments. Conversely, if you want small-amplitude oscillations, such as in a car’s suspension system, then you want heavy damping. Heavy damping reduces the amplitude, but the tradeoff is that the system responds at more frequencies.

These features of driven harmonic oscillators apply to a huge variety of systems. When you tune a radio, for example, you are adjusting its resonant frequency so that it only oscillates to the desired station’s broadcast (driving) frequency. The more selective the radio is in discriminating between stations, the smaller its damping. Magnetic resonance imaging (MRI) is a widely used medical diagnostic tool in which atomic nuclei (mostly hydrogen nuclei) are made to resonate by incoming radio waves (on the order of 100 MHz). A child on a swing is driven by a parent at the swing’s natural frequency to achieve maximum amplitude. In all of these cases, the efficiency of energy transfer from the driving force into the oscillator is best at resonance. Speed bumps and gravel roads prove that even a car’s suspension system is not immune to resonance. In spite of finely engineered shock absorbers, which ordinarily convert mechanical energy to thermal energy almost as fast as it comes in, speed bumps still cause a large-amplitude oscillation. On gravel roads that are corrugated, you may have noticed that if you travel at the “wrong” speed, the bumps are very noticeable whereas at other speeds you may hardly feel the bumps at all. Figure 11.28 shows a photograph of a famous example (the Tacoma Narrows Bridge) of the destructive effects of a driven harmonic oscillation. The Millennium Bridge in London was closed for a short period of time for the same reason while inspections were carried out.

In our bodies, the chest cavity is a clear example of a system at resonance. The diaphragm and chest wall drive the oscillations of the chest cavity which result in the lungs inflating and deflating. The system is critically damped and the muscular diaphragm oscillates at the resonant value for the system, making it highly efficient.

Figure 11.28 In 1940, the Tacoma Narrows Bridge in Washington state collapsed. Heavy cross winds drove the bridge into oscillations at its resonant frequency. Damping decreased when support cables broke loose and started to slip over the towers, allowing increasingly greater amplitudes until the structure failed (credit: PRI’s Studio 360, via Flickr)

Check Your Understanding

A famous magic trick involves a performer singing a note toward a crystal glass until the glass shatters. Explain why the trick
works in terms of resonance and natural frequency.

Solution
The performer must be singing a note that corresponds to the natural frequency of the glass. As the sound wave is directed at the glass, the glass responds by resonating at the same frequency as the sound wave. With enough energy introduced into the system, the glass begins to vibrate and eventually shatters.

11.9 Waves

![Figure 11.29 Waves in the ocean behave similarly to all other types of waves. (credit: Steve Jurveston, Flickr)](image)

What do we mean when we say something is a wave? The most intuitive and easiest wave to imagine is the familiar water wave. More precisely, a wave is a disturbance that propagates, or moves from the place it was created. For water waves, the disturbance is in the surface of the water, perhaps created by a rock thrown into a pond or by a swimmer splashing the surface repeatedly. For sound waves, the disturbance is a change in air pressure, perhaps created by the oscillating cone inside a speaker. For earthquakes, there are several types of disturbances, including disturbance of Earth’s surface and pressure disturbances under the surface. Even radio waves are most easily understood using an analogy with water waves. Visualizing water waves is useful because there is more to it than just a mental image. Water waves exhibit characteristics common to all waves, such as amplitude, period, frequency and energy. All wave characteristics can be described by a small set of underlying principles.

A wave is a disturbance that propagates, or moves from the place it was created. The simplest waves repeat themselves for several cycles and are associated with simple harmonic motion. Let us start by considering the simplified water wave in Figure 11.30. The wave is an up and down disturbance of the water surface. It causes a sea gull to move up and down in simple harmonic motion as the wave crests and troughs (peaks and valleys) pass under the bird. The time for one complete up and down motion is the wave’s period $T$. The wave’s frequency is $f = 1 / T$, as usual. The wave itself moves to the right in the figure. This movement of the wave is actually the disturbance moving to the right, not the water itself (or the bird would move to the right). We define wave velocity $v_w$ to be the speed at which the disturbance moves. Wave velocity is sometimes also called the propagation velocity or propagation speed, because the disturbance propagates from one location to another.

Misconception Alert

Many people think that water waves push water from one direction to another. In fact, the particles of water tend to stay in one location, save for moving up and down due to the energy in the wave. The energy moves forward through the water, but the water stays in one place. If you feel yourself pushed in an ocean, what you feel is the energy of the wave, not a rush of water.
An idealized ocean wave passes under a sea gull that bobs up and down in simple harmonic motion. The wave has a wavelength \( \lambda \), which is the distance between adjacent identical parts of the wave. The up and down disturbance of the surface propagates parallel to the surface at a speed \( v_w \).

The water wave in the figure also has a length associated with it, called its wavelength \( \lambda \), the distance between adjacent identical parts of a wave. (\( \lambda \) is the distance parallel to the direction of propagation.) The speed of propagation \( v_w \) is the distance the wave travels in a given time, which is one wavelength in the time of one period. In equation form, that is

\[
v_w = \frac{\lambda}{T}
\]  

or

\[
v_w = f\lambda.
\]

This fundamental relationship holds for all types of waves. For water waves, \( v_w \) is the speed of a surface wave; for sound, \( v_w \) is the speed of sound; and for visible light, \( v_w \) is the speed of light, for example.

**Take-Home Experiment: Waves in a Bowl**

Fill a large bowl or basin with water and wait for the water to settle so there are no ripples. Gently drop a cork into the middle of the bowl. Estimate the wavelength and period of oscillation of the water wave that propagates away from the cork. Remove the cork from the bowl and wait for the water to settle again. Gently drop the cork at a height that is different from the first drop. Does the wavelength depend upon how high above the water the cork is dropped?

**Example 11.8 Calculate the Velocity of Wave Propagation: Gull in the Ocean**

Calculate the wave velocity of the ocean wave in Figure 11.30 if the distance between wave crests is 10.0 m and the time for a sea gull to bob up and down is 5.00 s.

**Strategy**

We are asked to find \( v_w \). The given information tells us that \( \lambda = 10.0 \) m and \( T = 5.00 \) s. Therefore, we can use

\[
v_w = \frac{\lambda}{T}
\]

to find the wave velocity.

**Solution**

1. Enter the known values into \( v_w = \frac{\lambda}{T} \):

\[
v_w = \frac{10.0 \text{ m}}{5.00 \text{ s}}.
\]

2. Solve for \( v_w \) to find \( v_w = 2.00 \text{ m/s} \).

**Discussion**

This slow speed seems reasonable for an ocean wave. Note that the wave moves to the right in the figure at this speed, not the varying speed at which the sea gull moves up and down.
Transverse and Longitudinal Waves

A simple wave consists of a periodic disturbance that propagates from one place to another. The wave in Figure 11.31 propagates in the horizontal direction while the surface is disturbed in the vertical direction. Such a wave is called a transverse wave or shear wave; in such a wave, the disturbance is perpendicular to the direction of propagation. In contrast, in a longitudinal wave or compressional wave, the disturbance is parallel to the direction of propagation. Figure 11.32 shows an example of a longitudinal wave. The size of the disturbance is its amplitude $X$ and is completely independent of the speed of propagation $V_w$.

![Figure 11.31](image1.png) In this example of a transverse wave, the wave propagates horizontally, and the disturbance in the cord is in the vertical direction.

![Figure 11.32](image2.png) In this example of a longitudinal wave, the wave propagates horizontally, and the disturbance in the cord is also in the horizontal direction.

Waves may be transverse, longitudinal, or a combination of the two. (Water waves are actually a combination of transverse and longitudinal. The simplified water wave illustrated in Figure 11.30 shows no longitudinal motion of the bird.) The waves on the strings of musical instruments are transverse—so are electromagnetic waves, such as visible light.

Sound waves in air and water are longitudinal. Their disturbances are periodic variations in pressure that are transmitted in fluids. Fluids do not have appreciable shear strength, and thus the sound waves in them must be longitudinal or compressional. Sound in solids can be both longitudinal and transverse.

![Figure 11.33](image3.png) The wave on a guitar string is transverse. The sound wave rattles a sheet of paper in a direction that shows the sound wave is longitudinal.

Earthquake waves under Earth's surface also have both longitudinal and transverse components (called compressional or P-waves and shear or S-waves, respectively). These components have important individual characteristics—they propagate at different speeds, for example. Earthquakes also have surface waves that are similar to surface waves on water.

### Check Your Understanding

**Why is it important to differentiate between longitudinal and transverse waves?**

**Solution**
In the different types of waves, energy can propagate in a different direction relative to the motion of the wave. This is important to understand how different types of waves affect the materials around them.

### Wave on a String

Watch a string vibrate in slow motion. Wiggle the end of the string and make waves, or adjust the frequency and amplitude of an oscillator. Adjust the damping and tension. The end can be fixed, loose, or open.

(This media type is not supported in this reader. Click to open media in browser.)(http://legacy.cnx.org/content/m42248/1.17/#fs-id1167065947839)

Figure 11.34

#### 11.10 Superposition and Interference

Figure 11.35 These waves result from the superposition of several waves from different sources, producing a complex pattern. (credit: waterborough, Wikimedia Commons)

Most waves do not look very simple. They look more like the waves in Figure 11.35 than like the simple water wave considered in Waves. (Simple waves may be created by a simple harmonic oscillation, and thus have a sinusoidal shape). Complex waves are more interesting, even beautiful, but they look formidable. Most waves appear complex because they result from several simple waves adding together. Luckily, the rules for adding waves are quite simple.

When two or more waves arrive at the same point, they superimpose themselves on one another. More specifically, the disturbances of waves are superimposed when they come together—a phenomenon called superposition. Each disturbance corresponds to a force, and forces add. If the disturbances are along the same line, then the resulting wave is a simple addition of the disturbances of the individual waves—that is, their amplitudes add. Figure 11.36 and Figure 11.37 illustrate superposition in two special cases, both of which produce simple results.

Figure 11.36 shows two identical waves that arrive at the same point exactly in phase. The crests of the two waves are precisely aligned, as are the troughs. This superposition produces pure constructive interference. Because the disturbances add, pure constructive interference produces a wave that has twice the amplitude of the individual waves, but has the same wavelength.

Figure 11.37 shows two identical waves that arrive exactly out of phase—that is, precisely aligned crest to trough—producing pure destructive interference. Because the disturbances are in the opposite direction for this superposition, the resulting amplitude is zero for pure destructive interference—the waves completely cancel.

Figure 11.36 Pure constructive interference of two identical waves produces one with twice the amplitude, but the same wavelength.
While pure constructive and pure destructive interference do occur, they require precisely aligned identical waves. The superposition of most waves produces a combination of constructive and destructive interference and can vary from place to place and time to time. Sound from a stereo, for example, can be loud in one spot and quiet in another. Varying loudness means the sound waves add partially constructively and partially destructively at different locations. A stereo has at least two speakers creating sound waves, and waves can reflect from walls. All these waves superimpose. An example of sounds that vary over time from constructive to destructive is found in the combined whine of airplane jets heard by a stationary passenger. The combined sound can fluctuate up and down in volume as the sound from the two engines varies in time from constructive to destructive. These examples are of waves that are similar.

An example of the superposition of two dissimilar waves is shown in Figure 11.38. Here again, the disturbances add and subtract, producing a more complicated looking wave.

Standing Waves

Sometimes waves do not seem to move; rather, they just vibrate in place. Unmoving waves can be seen on the surface of a glass of milk in a refrigerator, for example. Vibrations from the refrigerator motor create waves on the milk that oscillate up and down but do not seem to move across the surface. These waves are formed by the superposition of two or more moving waves, such as illustrated in Figure 11.39 for two identical waves moving in opposite directions. The waves move through each other with their disturbances adding as they go by. If the two waves have the same amplitude and wavelength, then they alternate between constructive and destructive interference. The resultant looks like a wave standing in place and, thus, is called a standing wave. Waves on the glass of milk are one example of standing waves. There are other standing waves, such as on guitar strings and in organ pipes. With the glass of milk, the two waves that produce standing waves may come from reflections from the side of the glass.

A closer look at earthquakes provides evidence for conditions appropriate for resonance, standing waves, and constructive and destructive interference. A building may be vibrated for several seconds with a driving frequency matching that of the natural frequency of vibration of the building—producing a resonance resulting in one building collapsing while neighboring buildings do not. Often buildings of a certain height are devastated while other taller buildings remain intact. The building height matches the condition for setting up a standing wave for that particular height. As the earthquake waves travel along the surface of Earth and reflect off denser rocks, constructive interference occurs at certain points. Often areas closer to the epicenter are not damaged while areas farther away are damaged.
Standing waves are also found on the strings of musical instruments and are due to reflections of waves from the ends of the string. Figure 11.40 and Figure 11.41 show three standing waves that can be created on a string that is fixed at both ends. **Nodes** are the points where the string does not move; more generally, nodes are where the wave disturbance is zero in a standing wave. The fixed ends of strings must be nodes, too, because the string cannot move there. The word **antinode** is used to denote the location of maximum amplitude in standing waves. Standing waves on strings have a frequency that is related to the propagation speed \( v_w \) of the disturbance on the string. The wavelength \( \lambda \) is determined by the distance between the points where the string is fixed in place.

The lowest frequency, called the **fundamental frequency**, is thus for the longest wavelength, which is seen to be \( \lambda_1 = 2L \). Therefore, the fundamental frequency is \( f_1 = v_w / \lambda_1 = v_w / 2L \). In this case, the **overtones** or harmonics are multiples of the fundamental frequency. As seen in Figure 11.41, the first harmonic can easily be calculated since \( \lambda_2 = L \). Thus, \( f_2 = v_w / \lambda_2 = v_w / 2L = 2f_1 \). Similarly, \( f_3 = 3f_1 \), and so on. All of these frequencies can be changed by adjusting the tension in the string. The greater the tension, the greater \( v_w \) is and the higher the frequencies. This observation is familiar to anyone who has ever observed a string instrument being tuned. We will see in later chapters that standing waves are crucial to many resonance phenomena, such as in sounding boxes on string instruments.

![Figure 11.39](image1.png) **Standing wave created by the superposition of two identical waves moving in opposite directions. The oscillations are at fixed locations in space and result from alternately constructive and destructive interference.**

![Figure 11.40](image2.png) **The figure shows a string oscillating at its fundamental frequency.**
Beats

Striking two adjacent keys on a piano produces a warbling combination usually considered to be unpleasant. The superposition of two waves of similar but not identical frequencies is the culprit. Another example is often noticeable in jet aircraft, particularly the two-engine variety, while taxiing. The combined sound of the engines goes up and down in loudness. This varying loudness happens because the sound waves have similar but not identical frequencies. The discordant warbling of the piano and the fluctuating loudness of the jet engine noise are both due to alternately constructive and destructive interference as the two waves go in and out of phase. Figure 11.42 illustrates this graphically.

The wave resulting from the superposition of two similar-frequency waves has a frequency that is the average of the two. This wave fluctuates in amplitude, or beats, with a frequency called the beat frequency. We can determine the beat frequency by adding two waves together mathematically. Note that a wave can be represented at one point in space as

\[ x = X \cos \left( \frac{2\pi f t}{T} \right) = X \cos(2\pi f t), \]  

where \( f = 1 / T \) is the frequency of the wave. Adding two waves that have different frequencies but identical amplitudes produces a resultant

\[ x = x_1 + x_2, \]  

More specifically,

\[ x = X \cos(2\pi f_1 t) + X \cos(2\pi f_2 t). \]  

Using a trigonometric identity, it can be shown that

\[ x = 2X \cos(\pi f_B t) \cos(2\pi f_{\text{ave}} t) \]  

where

\[ f_B = | f_1 - f_2 | \]  

is the beat frequency, and \( f_{\text{ave}} \) is the average of \( f_1 \) and \( f_2 \). These results mean that the resultant wave has twice the amplitude and the average frequency of the two superimposed waves, but it also fluctuates in overall amplitude at the beat frequency \( f_B \). The first cosine term in the expression effectively causes the amplitude to go up and down. The second cosine term is the wave with frequency \( f_{\text{ave}} \). This result is valid for all types of waves. However, if it is a sound wave, providing the two frequencies are similar, then what we hear is an average frequency that gets louder and softer (or warbles) at the beat frequency.

Making Career Connections

Piano tuners use beats routinely in their work. When comparing a note with a tuning fork, they listen for beats and adjust the
string until the beats go away (to zero frequency). For example, if the tuning fork has a 256 Hz frequency and two beats per second are heard, then the other frequency is either 254 or 258 Hz. Most keys hit multiple strings, and these strings are actually adjusted until they have nearly the same frequency and give a slow beat for richness. Twelve-string guitars and mandolins are also tuned using beats.

While beats may sometimes be annoying in audible sounds, we will find that beats have many applications. Observing beats is a very useful way to compare similar frequencies. There are applications of beats as apparently disparate as in ultrasonic imaging and radar speed traps.

Check Your Understanding

Imagine you are holding one end of a jump rope, and your friend holds the other. If your friend holds her end still, you can move your end up and down, creating a transverse wave. If your friend then begins to move her end up and down, generating a wave in the opposite direction, what resultant wave forms would you expect to see in the jump rope?

Solution

The rope would alternate between having waves with amplitudes two times the original amplitude and reaching equilibrium with no amplitude at all. The wavelengths will result in both constructive and destructive interference.

Check Your Understanding

Define nodes and antinodes.

Solution

Nodes are areas of wave interference where there is no motion. Antinodes are areas of wave interference where the motion is at its maximum point.

Check Your Understanding

You hook up a stereo system. When you test the system, you notice that in one corner of the room, the sounds seem dull. In another area, the sounds seem excessively loud. Describe how the sound moving about the room could result in these effects.

Solution

With multiple speakers putting out sounds into the room, and these sounds bouncing off walls, there is bound to be some wave interference. In the dull areas, the interference is probably mostly destructive. In the louder areas, the interference is probably mostly constructive.

Wave Interference

Make waves with a dripping faucet, audio speaker, or laser! Add a second source or a pair of slits to create an interference pattern.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42249/1.14/#fs-id1167067067106)
11.11 Energy in Waves: Intensity

All waves carry energy. The energy of some waves can be directly observed. Earthquakes can shake whole cities to the ground, performing the work of thousands of wrecking balls.

Loud sounds pulverize nerve cells in the inner ear, causing permanent hearing loss. Ultrasound is used for deep-heat treatment of muscle strains. A laser beam can burn away a malignancy. Water waves chew up beaches.

The amount of energy in a wave is related to its amplitude. Large-amplitude earthquakes produce large ground displacements. Loud sounds have higher pressure amplitudes and come from larger-amplitude source vibrations than soft sounds. Large ocean breakers churn up the shore more than small ones. More quantitatively, a wave is a displacement that is resisted by a restoring force. The larger the displacement $x$, the larger the force $F = kx$ needed to create it. Because work $W$ is related to force multiplied by distance ($Fx$) and energy is put into the wave by the work done to create it, the energy in a wave is related to amplitude. In fact, a wave’s energy is directly proportional to its amplitude squared because

$$W \propto Fx = kx^2.$$  \hspace{1cm} (11.74)

The energy effects of a wave depend on time as well as amplitude. For example, the longer deep-heat ultrasound is applied, the more energy it transfers. Waves can also be concentrated or spread out. Sunlight, for example, can be focused to burn wood. Earthquakes spread out, so they do less damage the farther they get from the source. In both cases, changing the area the waves cover has important effects. All these pertinent factors are included in the definition of intensity $I$ as power per unit area:

$$I = \frac{P}{A}$$  \hspace{1cm} (11.75)

where $P$ is the power carried by the wave through area $A$. The definition of intensity is valid for any energy in transit, including that carried by waves. The SI unit for intensity is watts per square meter ($\text{W/m}^2$). For example, infrared and visible energy from the Sun impinge on Earth at an intensity of 1300 $\text{W/m}^2$ just above the atmosphere. There are other intensity-related units in use, too. The most common is the decibel. For example, a 90 decibel sound level corresponds to an intensity of $10^{-3}$ $\text{W/m}^2$. (This quantity is not much power per unit area considering that 90 decibels is a relatively high sound level. Decibels will be discussed in some detail in a later chapter.

**Example 11.9 Calculating intensity and power: How much energy is in a ray of sunlight?**

The average intensity of sunlight on Earth’s surface is about 700 $\text{W/m}^2$.

(a) Calculate the amount of energy that falls on a solar collector having an area of 0.500 $\text{m}^2$ in 4.00 h.

(b) What intensity would such sunlight have if concentrated by a magnifying glass onto an area 200 times smaller than its own?

**Strategy a**

Because power is energy per unit time or $P = \frac{E}{t}$, the definition of intensity can be written as $I = \frac{P}{A} = \frac{E}{At}$, and this equation can be solved for $E$ with the given information.

**Solution a**

1. Begin with the equation that states the definition of intensity:
\[ I = \frac{P}{A}. \]  

(11.76)

2. Replace \( P \) with its equivalent \( E / t \):

\[ I = \frac{E}{t}. \]  

(11.77)

3. Solve for \( E \):

\[ E = IAt. \]  

(11.78)

4. Substitute known values into the equation:

\[ E = (700 \text{ W/m}^2)(0.500 \text{ m}^2)(4.00 \text{ h})(3600 \text{ s/h})]. \]  

(11.79)

5. Calculate to find \( E \) and convert units:

\[ 5.04 \times 10^6 \text{ J}, \]  

(11.80)

**Discussion a**
The energy falling on the solar collector in 4 h in part is enough to be useful—for example, for heating a significant amount of water.

**Strategy b**
Taking a ratio of new intensity to old intensity and using primes for the new quantities, we will find that it depends on the ratio of the areas. All other quantities will cancel.

**Solution b**
1. Take the ratio of intensities, which yields:

\[ \frac{I'}{I} = \frac{P'/A'}{P/A} = \frac{A}{A'} \text{ (The powers cancel because } P' = P). \]  

(11.81)

2. Identify the knowns:

\[ A = 200A', \]  

(11.82)

\[ \frac{I'}{I} = 200. \]  

(11.83)

3. Substitute known quantities:

\[ I' = 200I = 200(700 \text{ W/m}^2). \]  

(11.84)

4. Calculate to find \( I' \):

\[ I' = 1.40 \times 10^5 \text{ W/m}^2. \]  

(11.85)

**Discussion b**
Decreasing the area increases the intensity considerably. The intensity of the concentrated sunlight could even start a fire.

**Example 11.10** Determine the combined intensity of two waves: Perfect constructive interference

If two identical waves are spatially separated, each having an intensity of 1.00 W/m², interfere perfectly constructively in a particular location, what is the intensity of the wave at this particular location?

**Strategy**
We know from **Superposition and Interference** that when two identical waves, which have equal amplitudes \( X \), interfere perfectly constructively, the resulting wave has an amplitude of \( 2X \). Because a wave’s intensity is proportional to amplitude squared, the intensity of the resulting wave is four times as great as in the individual waves.

**Solution**
1. Recall that intensity is proportional to amplitude squared.

2. Calculate the new amplitude:

\[ I' \propto (X)^2 = (2X)^2 = 4X^2. \]  

(11.86)

3. Recall that the intensity of the old amplitude was:

\[ I \propto X^2. \]  

(11.87)
4. Take the ratio of new intensity to the old intensity. This gives:

\[ \frac{I'}{I} = 4. \]  

(11.88)

5. Calculate to find \( I' \):

\[ I' = 4I = 4.00 \text{ W/m}^2. \]  

(11.89)

Discussion

The intensity goes up by a factor of 4 when the amplitude doubles. This answer is a little disquieting. The two individual waves each have intensities of 1.00 W/m\(^2\), yet their sum has an intensity of 4.00 W/m\(^2\), which may appear to violate conservation of energy. This violation, of course, cannot happen. What does happen is intriguing. The area over which the intensity is 4.00 W/m\(^2\) is much less than the area covered by the two waves before they interfered. There are other areas where the intensity is zero. The addition of waves is not as simple as our first look in Superposition and Interference suggested. We actually get a pattern of both constructive interference and destructive interference whenever two waves are added. For example, if we have two stereo speakers putting out 1.00 W/m\(^2\) each, there will be places in the room where the intensity is 4.00 W/m\(^2\), other places where the intensity is zero, and others in between. Figure 11.45 shows what this interference might look like. We will pursue interference patterns elsewhere in this text.

Figure 11.45 These stereo speakers produce both constructive interference and destructive interference in the room, a property common to the superposition of all types of waves. The shading is proportional to intensity.

Check Your Understanding

Which measurement of a wave is most important when determining the wave’s intensity?

Solution

Amplitude, because a wave’s energy is directly proportional to its amplitude squared.

Glossary

amplitude: the maximum displacement from the equilibrium position of an object oscillating around the equilibrium position

antinode: the location of maximum amplitude in standing waves

beat frequency: the frequency of the amplitude fluctuations of a wave

constructive interference: when two waves arrive at the same point exactly in phase; that is, the crests of the two waves are precisely aligned, as are the troughs

critical damping: the condition in which the damping of an oscillator causes it to return as quickly as possible to its equilibrium position without oscillating back and forth about this position

defformation: displacement from equilibrium

destructive interference: when two identical waves arrive at the same point exactly out of phase; that is, precisely aligned crest to trough

elastic potential energy: potential energy stored as a result of deformation of an elastic object, such as the stretching of a spring
force constant: a constant related to the rigidity of a system; the larger the force constant, the more rigid the system; the force constant is represented by $k$

frequency: number of events per unit of time

fundamental frequency: the lowest frequency of a periodic waveform

intensity: power per unit area

longitudinal wave: a wave in which the disturbance is parallel to the direction of propagation

natural frequency: the frequency at which a system would oscillate if there were no driving and no damping forces

nodes: the points where the string does not move; more generally, nodes are where the wave disturbance is zero in a standing wave

oscillate: moving back and forth regularly between two points

over damping: the condition in which damping of an oscillator causes it to return to equilibrium without oscillating; oscillator moves more slowly toward equilibrium than in the critically damped system

overtones: multiples of the fundamental frequency of a sound

period: time it takes to complete one oscillation

periodic motion: motion that repeats itself at regular time intervals

resonance: the phenomenon of driving a system with a frequency equal to the system's natural frequency

resonate: a system being driven at its natural frequency

restoring force: force acting in opposition to the force caused by a deformation

simple harmonic motion: the oscillatory motion in a system where the net force can be described by Hooke's law

simple harmonic oscillator: a device that implements Hooke's law, such as a mass that is attached to a spring, with the other end of the spring being connected to a rigid support such as a wall

simple pendulum: an object with a small mass suspended from a light wire or string

superposition: the phenomenon that occurs when two or more waves arrive at the same point

transverse wave: a wave in which the disturbance is perpendicular to the direction of propagation

under damping: the condition in which damping of an oscillator causes it to return to equilibrium with the amplitude gradually decreasing to zero; system returns to equilibrium faster but overshoots and crosses the equilibrium position one or more times

wave: a disturbance that moves from its source and carries energy

wave velocity: the speed at which the disturbance moves. Also called the propagation velocity or propagation speed

wavelength: the distance between adjacent identical parts of a wave

**Section Summary**

11.1 Hooke’s Law: Stress and Strain Revisited

- An oscillation is a back and forth motion of an object between two points of deformation.
- An oscillation may create a wave, which is a disturbance that propagates from where it was created.
- The simplest type of oscillations and waves are related to systems that can be described by Hooke's law:

  $$F = -kx,$$

  where $F$ is the restoring force, $x$ is the displacement from equilibrium or deformation, and $k$ is the force constant of the system.

- Elastic potential energy $PE_{el}$ stored in the deformation of a system that can be described by Hooke's law is given by

  $$PE_{el} = (1/2)kx^2.$$

11.2 Period and Frequency in Oscillations

- Periodic motion is a repetitious oscillation.
• The time for one oscillation is the period $T$.
• The number of oscillations per unit time is the frequency $f$.
• These quantities are related by
  $$ f = \frac{1}{T}. $$

### 11.3 Simple Harmonic Motion: A Special Periodic Motion

• Simple harmonic motion is oscillatory motion for a system that can be described only by Hooke’s law. Such a system is also called a simple harmonic oscillator.
• Maximum displacement is the amplitude $X$. The period $T$ and frequency $f$ of a simple harmonic oscillator are given by
  $$ T = 2\pi\sqrt{\frac{m}{k}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, $$
  where $m$ is the mass of the system.
• Displacement in simple harmonic motion as a function of time is given by $x(t) = X \cos \frac{2\pi}{T}t$.
• The velocity is given by $v(t) = -v_{\text{max}} \sin \frac{2\pi}{T}t$, where $v_{\text{max}} = \sqrt{k/m}X$.
• The acceleration is found to be $a(t) = -\frac{kX}{m} \cos \frac{2\pi}{T}t$.

### 11.4 The Simple Pendulum

• A mass $m$ suspended by a wire of length $L$ is a simple pendulum and undergoes simple harmonic motion for amplitudes less than about $15^\circ$.
  The period of a simple pendulum is
  $$ T = 2\pi\sqrt{\frac{L}{g}}, $$
  where $L$ is the length of the string and $g$ is the acceleration due to gravity.

### 11.5 Energy and the Simple Harmonic Oscillator

• Energy in the simple harmonic oscillator is shared between elastic potential energy and kinetic energy, with the total being constant:
  $$ \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}. $$
• Maximum velocity depends on three factors: it is directly proportional to amplitude, it is greater for stiffer systems, and it is smaller for objects that have larger masses:
  $$ v_{\text{max}} = \sqrt{\frac{k}{m}}X. $$

### 11.6 Uniform Circular Motion and Simple Harmonic Motion

A projection of uniform circular motion undergoes simple harmonic oscillation.

### 11.7 Damped Harmonic Motion

• Damped harmonic oscillators have non-conservative forces that dissipate their energy.
• Critical damping returns the system to equilibrium as fast as possible without overshooting.
• An underdamped system will oscillate through the equilibrium position.
• An overdamped system moves more slowly toward equilibrium than one that is critically damped.

### 11.8 Forced Oscillations and Resonance

• A system’s natural frequency is the frequency at which the system will oscillate if not affected by driving or damping forces.
• A periodic force driving a harmonic oscillator at its natural frequency produces resonance. The system is said to resonate.
• The less damping a system has, the higher the amplitude of the forced oscillations near resonance. The more damping a system has, the broader response it has to varying driving frequencies.

### 11.9 Waves

• A wave is a disturbance that moves from the point of creation with a wave velocity $v_w$.
• A wave has a wavelength $\lambda$, which is the distance between adjacent identical parts of the wave.
• Wave velocity and wavelength are related to the wave’s frequency and period by $v_w = \frac{\lambda}{T}$ or $v_w = f\lambda$. 
A transverse wave has a disturbance perpendicular to its direction of propagation, whereas a longitudinal wave has a disturbance parallel to its direction of propagation.

11.10 Superposition and Interference

- Superposition is the combination of two waves at the same location.
- Constructive interference occurs when two identical waves are superimposed in phase.
- Destructive interference occurs when two identical waves are superimposed exactly out of phase.
- A standing wave is one in which two waves superimpose to produce a wave that varies in amplitude but does not propagate.
- Nodes are points of no motion in standing waves.
- An antinode is the location of maximum amplitude of a standing wave.
- Waves on a string are resonant standing waves with a fundamental frequency and can occur at higher multiples of the fundamental, called overtones or harmonics.
- Beats occur when waves of similar frequencies \( f_1 \) and \( f_2 \) are superimposed. The resulting amplitude oscillates with a beat frequency given by

\[
f_B = \left| f_1 - f_2 \right|
\]

11.11 Energy in Waves: Intensity

Intensity is defined to be the power per unit area:

\[
I = \frac{P}{A}
\]

Conceptual Questions

11.1 Hooke’s Law: Stress and Strain Revisited

1. Describe a system in which elastic potential energy is stored.

11.3 Simple Harmonic Motion: A Special Periodic Motion

2. What conditions must be met to produce simple harmonic motion?

3. (a) If frequency is not constant for some oscillation, can the oscillation be simple harmonic motion?

(b) Can you think of any examples of harmonic motion where the frequency may depend on the amplitude?

4. Give an example of a simple harmonic oscillator, specifically noting how its frequency is independent of amplitude.

5. Explain why you expect an object made of a stiff material to vibrate at a higher frequency than a similar object made of a spongy material.

6. As you pass a freight truck with a trailer on a highway, you notice that its trailer is bouncing up and down slowly. Is it more likely that the trailer is heavily loaded or nearly empty? Explain your answer.

7. Some people modify cars to be much closer to the ground than when manufactured. Should they install stiffer springs? Explain your answer.

11.4 The Simple Pendulum

8. Pendulum clocks are made to run at the correct rate by adjusting the pendulum’s length. Suppose you move from one city to another where the acceleration due to gravity is slightly greater, taking your pendulum clock with you, will you have to lengthen or shorten the pendulum to keep the correct time, other factors remaining constant? Explain your answer.

11.5 Energy and the Simple Harmonic Oscillator

9. Explain in terms of energy how dissipative forces such as friction reduce the amplitude of a harmonic oscillator. Also explain how a driving mechanism can compensate. (A pendulum clock is such a system.)

11.7 Damped Harmonic Motion

10. Give an example of a damped harmonic oscillator. (They are more common than undamped or simple harmonic oscillators.)

11. How would a car bounce after a bump under each of these conditions?

- overdamping
- underdamping
- critical damping

12. Most harmonic oscillators are damped and, if undriven, eventually come to a stop. How is this observation related to the second law of thermodynamics?
11.8 Forced Oscillations and Resonance

13. Why are soldiers in general ordered to “route step” (walk out of step) across a bridge?

11.9 Waves

14. Give one example of a transverse wave and another of a longitudinal wave, being careful to note the relative directions of the disturbance and wave propagation in each.

15. What is the difference between propagation speed and the frequency of a wave? Does one or both affect wavelength? If so, how?

11.10 Superposition and Interference

16. Speakers in stereo systems have two color-coded terminals to indicate how to hook up the wires. If the wires are reversed, the speaker moves in a direction opposite that of a properly connected speaker. Explain why it is important to have both speakers connected the same way.

11.11 Energy in Waves: Intensity

17. Two identical waves undergo pure constructive interference. Is the resultant intensity twice that of the individual waves? Explain your answer.

18. Circular water waves decrease in amplitude as they move away from where a rock is dropped. Explain why.
11.1 Hooke’s Law: Stress and Strain Revisited

1. Fish are hung on a spring scale to determine their mass (most fishermen feel no obligation to truthfully report the mass).
   (a) What is the force constant of the spring in such a scale if it stretches 8.00 cm for a 10.0 kg load?
   (b) What is the mass of a fish that stretches the spring 5.50 cm?
   (c) How far apart are the half-kilogram marks on the scale?

2. It is weigh-in time for the local under-85-kg rugby team. The bathroom scale used to assess eligibility can be described by Hooke’s law and is depressed 0.75 cm by its maximum load of 120 kg. (a) What is the spring’s effective spring constant? (b) A player stands on the scales and depresses it by 0.48 cm. Is he eligible to play on this under-85 kg team?

3. One type of BB gun uses a spring-driven plunger to blow the BB from its barrel. (a) Calculate the force constant of its plunger’s spring if you must compress it 0.150 m to drive the 0.0500-kg plunger to a top speed of 20.0 m/s. (b) What force must be exerted to compress the spring?

4. (a) The springs of a pickup truck act like a single spring with a force constant of $1.30 \times 10^5$ N/m. By how much will the truck be depressed by its maximum load of 1000 kg? (b) If the pickup truck has four identical springs, what is the force constant of each?

5. When an 80.0-kg man stands on a pogo stick, the spring is compressed 0.120 m.
   (a) What is the force constant of the spring? (b) Will the spring be compressed more when he hops down the road?

6. A spring has a length of 0.200 m when a 0.300-kg mass hangs from it, and a length of 0.750 m when a 1.95-kg mass hangs from it. (a) What is the force constant of the spring? (b) What is the unloaded length of the spring?

11.2 Period and Frequency in Oscillations

7. What is the period of 60.0 Hz electrical power?

8. If your heart rate is 150 beats per minute during strenuous exercise, what is the time per beat in units of seconds?

9. Find the frequency of a tuning fork that takes $2.50 \times 10^{-3}$ s to complete one oscillation.

10. A stroboscope is set to flash every 8.00 $\times 10^{-5}$ s. What is the frequency of the flashes?

11. A tire has a tread pattern with a crevice every 2.00 cm. Each crevice makes a single vibration as the tire moves. What is the frequency of these vibrations if the car moves at 30.0 m/s?

12. Engineering Application

Each piston of an engine makes a sharp sound every other revolution of the engine. (a) How fast is a race car going if its eight-cylinder engine emits a sound of frequency 750 Hz, given that the engine makes 2000 revolutions per kilometer? (b) At how many revolutions per minute is the engine rotating?
20. 

Figure 11.46 This child’s toy relies on springs to keep infants entertained. (credit: By Humboldthead, Flickr)

The device pictured in Figure 11.46 entertains infants while keeping them from wandering. The child bounces in a harness suspended from a door frame by a spring constant.

(a) If the spring stretches 0.250 m while supporting an 8.0-kg child, what is its spring constant?

(b) What is the time for one complete bounce of this child? (c) What is the child’s maximum velocity if the amplitude of her bounce is 0.200 m?

21. 

A 90.0-kg skydiver hanging from a parachute bounces up and down with a period of 1.50 s. What is the new period of oscillation when a second skydiver, whose mass is 60.0 kg, hangs from the legs of the first, as seen in Figure 11.47.

Figure 11.47 The oscillations of one skydiver are about to be affected by a second skydiver. (credit: U.S. Army, www.army.mil)

11.4 The Simple Pendulum

As usual, the acceleration due to gravity in these problems is taken to be $g = 9.80 \text{ m/s}^2$, unless otherwise specified.

22. What is the length of a pendulum that has a period of 0.500 s?

23. Some people think a pendulum with a period of 1.00 s can be driven with “mental energy” or psycho kinetically, because its period is the same as an average heartbeat. True or not, what is the length of such a pendulum?

24. What is the period of a 1.00-m-long pendulum?

25. How long does it take a child on a swing to complete one swing if her center of gravity is 4.00 m below the pivot?

26. The pendulum on a cuckoo clock is 5.00 cm long. What is its frequency?

27. Two parakeets sit on a swing with their combined center of mass 10.0 cm below the pivot. At what frequency do they swing?

28. (a) A pendulum that has a period of 3.00000 s and that is located where the acceleration due to gravity is $9.79 \text{ m/s}^2$ is moved to a location where it the acceleration due to gravity is $9.82 \text{ m/s}^2$. What is its new period? (b) Explain why so many digits are needed in the value for the period, based on the relation between the period and the acceleration due to gravity.

29. A pendulum with a period of 2.00000 s in one location is moved to a new location where the period is now 1.99796 s. What is the acceleration due to gravity at its new location?

30. (a) What is the effect on the period of a pendulum if you double its length? (b) What is the effect on the period of a pendulum if you decrease its length by 5.00%?

31. Find the ratio of the new/old periods of a pendulum if the pendulum were transported from Earth to the Moon, where the acceleration due to gravity is $1.63 \text{ m/s}^2$.

32. At what rate will a pendulum clock run on the Moon, where the acceleration due to gravity is $1.63 \text{ m/s}^2$, if it keeps time accurately on Earth? That is, find the time (in hours) it takes the clock’s hour hand to make one revolution on the Moon.

33. Suppose the length of a clock’s pendulum is changed by 1.000%, exactly at noon one day. What time will it read 24.00 hours later, assuming it the pendulum has kept perfect time before the change? Note that there are two answers, and perform the calculation to four-digit precision.

34. If a pendulum-driven clock gains 5.00 s/day, what fractional change in pendulum length must be made for it to keep perfect time?

11.5 Energy and the Simple Harmonic Oscillator

35. #exercise/nylon
36. Engineering Application
Near the top of the Citigroup Center building in New York City, there is an object with mass of 4.00 × 10^5 kg on springs that have adjustable force constants. Its function is to dampen wind-driven oscillations of the building by oscillating at the same frequency as the building is being driven—the driving force is transferred to the object, which oscillates instead of the entire building. (a) What effective force constant should the springs have to make the object oscillate with a period of 2.00 s? (b) What energy is stored in the springs for a 2.00-m displacement from equilibrium?

11.6 Uniform Circular Motion and Simple Harmonic Motion
37. (a) What is the maximum velocity of an 85.0-kg person bouncing on a bathroom scale having a force constant of 1.50 × 10^6 N/m, if the amplitude of the bounce is 0.200 cm? (b) What is the maximum energy stored in the spring?
38. A novelty clock has a 0.0100-kg mass object bouncing on a spring that has a force constant of 1.25 N/m. What is the maximum velocity of the object if the object bounces 3.00 cm above and below its equilibrium position? (b) How many joules of kinetic energy does the object have at its maximum velocity?
39. At what positions is the speed of a simple harmonic oscillator half its maximum? That is, what values of x/X gives \( v = \pm \frac{v_{\text{max}}}{2} \), where X is the amplitude of the motion?
40. A ladybug sits 12.0 cm from the center of a Beatles music album spinning at 33.33 rpm. What is the maximum velocity of its shadow on the wall behind the turntable, if illuminated parallel to the record by the parallel rays of the setting Sun?

11.7 Damped Harmonic Motion
41. The amplitude of a lightly damped oscillator decreases by 3.0% during each cycle. What percentage of the mechanical energy of the oscillator is lost in each cycle?

11.8 Forced Oscillations and Resonance
42. How much energy must the shock absorbers of a 1200-kg car dissipate in order to damp a bounce that initially has a velocity of 0.800 m/s at the equilibrium position? Assume the car returns to its original vertical position.
43. If a car has a suspension system with a force constant of 5.00 × 10^4 N/m, how much energy must the car’s shocks remove to dampen an oscillation starting with a maximum displacement of 0.0750 m?
44. (a) How much will a spring that has a force constant of 40.0 N/m be stretched by an object with a mass of 0.500 kg when hung motionless from the spring? (b) Calculate the decrease in gravitational potential energy of the 0.500-kg object when it descends this distance. (c) Part of this gravitational energy goes into the spring. Calculate the energy stored in the spring by this stretch, and compare it with the gravitational potential energy. Explain where the rest of the energy might go.
45. Suppose you have a 0.750-kg object on a horizontal surface connected to a spring that has a force constant of 150 N/m. There is simple friction between the object and surface with a static friction coefficient of \( \mu_s = 0.100 \). (a) How far can the spring be stretched without moving the mass? (b) If the object is set into oscillation with an amplitude twice the distance found in part (a), and the kinetic coefficient of friction is \( \mu_k = 0.0850 \), what total distance does it travel before stopping? Assume it starts at the maximum amplitude.
46. Engineering Application: A suspension bridge oscillates with an effective force constant of 1.00 × 10^8 N/m. (a) How much energy is needed to make it oscillate with an amplitude of 0.100 m? (b) If soldiers march across the bridge with a cadence equal to the bridge’s natural frequency and impart 1.00 × 10^4 J of energy each second, how long does it take for the bridge’s oscillations to go from 0.100 m to 0.500 m amplitude?

11.9 Waves
47. Storms in the South Pacific can create waves that travel all the way to the California coast, which are 12,000 km away. How long does it take them if they travel at 15.0 m/s?
48. Waves on a swimming pool propagate at 0.750 m/s. You splash the water at one end of the pool and observe the wave go to the opposite end, reflect, and return in 30.0 s. How far away is the other end of the pool?
49. Wind gusts create ripples on the ocean that have a wavelength of 5.00 cm and propagate at 2.00 m/s. What is their frequency?
50. How many times a minute does a boat bob up and down on ocean waves that have a wavelength of 40.0 m and a propagation speed of 5.00 m/s?
51. Scouts at a camp shake the rope bridge they have just crossed and observe the wave crests to be 8.00 m apart. If they shake the bridge twice per second, what is the propagation speed of the waves?
52. What is the wavelength of the waves you create in a swimming pool if you splash your hand at a rate of 2.00 Hz and the waves propagate at 0.800 m/s?
53. What is the wavelength of an earthquake that shakes you with a frequency of 10.0 Hz and gets to another city 84.0 km away in 12.0 s?
54. Radio waves transmitted through space at 3.00 × 10^8 m/s by the Voyager spacecraft have a wavelength of 0.120 m. What is their frequency?
55. Your ear is capable of differentiating sounds that arrive at the ear just 1.00 ms apart. What is the minimum distance between two speakers that produce sounds that arrive at noticeably different times on a day when the speed of sound is 340 m/s?
56. (a) Seismographs measure the arrival times of earthquakes with a precision of 0.100 s. To get the distance to the epicenter of the quake, they compare the arrival times of S- and P-waves, which travel at different speeds. Figure 11.48 If S- and P-waves travel at 4.00 and 7.20 km/s, respectively, in the region considered, how precisely can the distance to the source of the earthquake be determined? (b) Seismic waves from underground detonations of nuclear bombs can be used to locate the test site and detect violations of test bans. Discuss whether your answer to (a) implies a serious limit to such detection. (Note also that the uncertainty is greater if there is an uncertainty in the propagation speeds of the S- and P-waves.)

11.11 Energy in Waves: Intensity

63. Medical Application

Ultrasound of intensity 1.50×10^{-2} W/m^2 is produced by the rectangular head of a medical imaging device measuring 3.00 cm. What is its power output?

64. The low-frequency speaker of a stereo set has a surface area of 0.05 m^2 and produces 1W of acoustical power. What is the intensity at the speaker? If the speaker projects sound uniformly in all directions, at what distance from the speaker is the intensity 0.1 W/m^2?

65. To increase intensity of a wave by a factor of 50, by what factor should the amplitude be increased?

66. Engineering Application

A device called an insolation meter is used to measure the intensity of sunlight has an area of 100 cm^2 and registers 6.50 W. What is the intensity in W/m^2?

67. Astronomy Application

Energy from the Sun arrives at the top of the Earth’s atmosphere with an intensity of 1.30 kW/m^2. How long does it take for 1.8×10^9 J to arrive on an area of 1.00 m^2?

68. Suppose you have a device that extracts energy from ocean breakers in direct proportion to their intensity. If the device produces 10.0 kW of power on a day when the breakers are 1.20 m high, how much will it produce when they are 0.600 m high?

69. Engineering Application

(a) A photovoltaic array of (solar cells) is 10.0% efficient in gathering solar energy and converting it to electricity. If the average intensity of sunlight on one day is 700 W/m^2, what area should your array have to gather energy at the rate of 100 W? (b) What is the maximum cost of the array if it must pay for itself in two years of operation averaging 10.0 hours per day? Assume that it earns money at the rate of 9.00 ¢ per kilowatt-hour.

70. A microphone receiving a pure sound tone feeds an oscilloscope, producing a wave on its screen. If the sound intensity is originally 2.00×10^{-5} W/m^2, but is turned up until the amplitude increases by 30.0%, what is the new intensity?

71. Medical Application

(a) What is the intensity in W/m^2 of a laser beam used to burn away cancerous tissue that, when 90.0% absorbed, puts 500 J of energy into a circular spot 2.00 mm in diameter in 4.00 s? (b) Discuss how this intensity compares to the average intensity of sunlight (about 700 W/m^2) and the implications that would have if the laser beam entered your eye. Note how the amount of damage depends on the time duration of the exposure.
Chapter 12 | Electric Charge and Electric Field

12. ELECTRIC CHARGE AND ELECTRIC FIELD

Figure 12.1 Static electricity from this plastic slide causes the child’s hair to stand on end. The sliding motion stripped electrons away from the child’s body, leaving an excess of positive charges, which repel each other along each strand of hair. (credit: Ken Bosma/Wikimedia Commons)

Chapter Outline

12.1. Static Electricity and Charge: Conservation of Charge
• Define electric charge, and describe how the two types of charge interact.
• Describe three common situations that generate static electricity.
• State the law of conservation of charge.

12.2. Conductors and Insulators
• Define conductor and insulator, explain the difference, and give examples of each.
• Describe three methods for charging an object.
• Explain what happens to an electric force as you move farther from the source.
• Define polarization.

12.3. Coulomb’s Law
• State Coulomb’s law in terms of how the electrostatic force changes with the distance between two objects.
• Calculate the electrostatic force between two charged point forces, such as electrons or protons.
• Compare the electrostatic force to the gravitational attraction for a proton and an electron; for a human and the Earth.

12.4. Electric Field: Concept of a Field Revisited
• Describe a force field and calculate the strength of an electric field due to a point charge.
• Calculate the force exerted on a test charge by an electric field.
• Explain the relationship between electrical force (F) on a test charge and electrical field strength (E).

12.5. Electric Field Lines: Multiple Charges
• Calculate the total force (magnitude and direction) exerted on a test charge from more than one charge.
• Describe an electric field diagram of a positive point charge; of a negative point charge with twice the magnitude of positive charge.
• Draw the electric field lines between two points of the same charge; between two points of opposite charge.

12.6. Electric Forces in Biology
• Describe how a water molecule is polar.
• Explain electrostatic screening by a water molecule within a living cell.

12.7. Conductors and Electric Fields in Static Equilibrium
• List the three properties of a conductor in electrostatic equilibrium.
• Explain the effect of an electric field on free charges in a conductor.
• Explain why no electric field may exist inside a conductor.
• Describe the electric field surrounding Earth.
• Explain what happens to an electric field applied to an irregular conductor.
Describe how a lightning rod works.

Explain how a metal car may protect passengers inside from the dangerous electric fields caused by a downed line touching the car.

12.8. Applications of Electrostatics

Name several real-world applications of the study of electrostatics.

Introduction to Electric Charge and Electric Field

The image of American politician and scientist Benjamin Franklin (1706–1790) flying a kite in a thunderstorm is familiar to every schoolchild. (See Figure 12.2.) In this experiment, Franklin demonstrated a connection between lightning and static electricity. Sparks were drawn from a key hung on a kite string during an electrical storm. These sparks were like those produced by static electricity, such as the spark that jumps from your finger to a metal doorknob after you walk across a wool carpet. What Franklin demonstrated in his dangerous experiment was a connection between phenomena on two different scales: one the grand power of an electrical storm, the other an effect of more human proportions. Connections like this one reveal the underlying unity of the laws of nature, an aspect we humans find particularly appealing.

Figure 12.2 When Benjamin Franklin demonstrated that lightning was related to static electricity, he made a connection that is now part of the evidence that all directly experienced forces except the gravitational force are manifestations of the electromagnetic force.

Much has been written about Franklin. His experiments were only part of the life of a man who was a scientist, inventor, revolutionary, statesman, and writer. Franklin’s experiments were not performed in isolation, nor were they the only ones to reveal connections.

For example, the Italian scientist Luigi Galvani (1737–1798) performed a series of experiments in which static electricity was used to stimulate contractions of leg muscles of dead frogs, an effect already known in humans subjected to static discharges. But Galvani also found that if he joined two metal wires (say copper and zinc) end to end and touched the other ends to muscles, he produced the same effect in frogs as static discharge. Alessandro Volta (1745–1827), partly inspired by Galvani’s work, experimented with various combinations of metals and developed the battery.

During the same era, other scientists made progress in discovering fundamental connections. The periodic table was developed as the systematic properties of the elements were discovered. This influenced the development and refinement of the concept of atoms as the basis of matter. Such submicroscopic descriptions of matter also help explain a great deal more.

Atomic and molecular interactions, such as the forces of friction, cohesion, and adhesion, are now known to be manifestations of the electromagnetic force. Static electricity is just one aspect of the electromagnetic force, which also includes moving electricity and magnetism.

All the macroscopic forces that we experience directly, such as the sensations of touch and the tension in a rope, are due to the electromagnetic force, one of the four fundamental forces in nature. The gravitational force, another fundamental force, is actually sensed through the electromagnetic interaction of molecules, such as between those in our feet and those on the top of a bathroom scale. (The other two fundamental forces, the strong nuclear force and the weak nuclear force, cannot be sensed on the human scale.)

This chapter begins the study of electromagnetic phenomena at a fundamental level. The next several chapters will cover static electricity, moving electricity, and magnetism—collectively known as electromagnetism. In this chapter, we begin with the study of electric phenomena due to charges that are at least temporarily stationary, called electrostatics, or static electricity.
12.1 Static Electricity and Charge: Conservation of Charge

What makes plastic wrap cling? Static electricity. Not only are applications of static electricity common these days, its existence has been known since ancient times. The first record of its effects dates to ancient Greeks who noted more than 500 years B.C. that polishing amber temporarily enabled it to attract bits of straw (see Figure 12.3). The very word electric derives from the Greek word for amber (electron).

Many of the characteristics of static electricity can be explored by rubbing things together. Rubbing creates the spark you get from walking across a wool carpet, for example. Static cling generated in a clothes dryer and the attraction of straw to recently polished amber also result from rubbing. Similarly, lightning results from air movements under certain weather conditions. You can also rub a balloon on your hair, and the static electricity created can then make the balloon cling to a wall. We also have to be cautious of static electricity, especially in dry climates. When we pump gasoline, we are warned to discharge ourselves (after sliding across the seat) on a metal surface before grabbing the gas nozzle. Attendants in hospital operating rooms must wear booties with a conductive strip of aluminum foil on the bottoms to avoid creating sparks which may ignite flammable anesthesia gases combined with the oxygen being used.

Some of the most basic characteristics of static electricity include:

- The effects of static electricity are explained by a physical quantity not previously introduced, called electric charge.
- There are only two types of charge, one called positive and the other called negative.
- Like charges repel, whereas unlike charges attract.
- The force between charges decreases with distance.

How do we know there are two types of electric charge? When various materials are rubbed together in controlled ways, certain combinations of materials always produce one type of charge on one material and the opposite type on the other. By convention, we call one type of charge “positive”, and the other type “negative.” For example, when glass is rubbed with silk, the glass becomes positively charged and the silk negatively charged. Since the glass and silk have opposite charges, they attract one another like clothes that have rubbed together in a dryer. Two glass rods rubbed with silk in this manner will repel one another, since each rod has positive charge on it. Similarly, two silk cloths so rubbed will repel, since both cloths have negative charge. Figure 12.4 shows how these simple materials can be used to explore the nature of the force between charges.

Figure 12.4 A glass rod becomes positively charged when rubbed with silk, while the silk becomes negatively charged. (a) The glass rod is attracted to the silk because their charges are opposite. (b) Two similarly charged glass rods repel. (c) Two similarly charged silk cloths repel.

More sophisticated questions arise. Where do these charges come from? Can you create or destroy charge? Is there a smallest unit of charge? Exactly how does the force depend on the amount of charge and the distance between charges? Such questions obviously occurred to Benjamin Franklin and other early researchers, and they interest us even today.
Charge Carried by Electrons and Protons

Franklin wrote in his letters and books that he could see the effects of electric charge but did not understand what caused the phenomenon. Today we have the advantage of knowing that normal matter is made of atoms, and that atoms contain positive and negative charges, usually in equal amounts.

Figure 12.5 shows a simple model of an atom with negative electrons orbiting its positive nucleus. The nucleus is positive due to the presence of positively charged protons. Nearly all charge in nature is due to electrons and protons, which are two of the three building blocks of most matter. (The third is the neutron, which is neutral, carrying no charge.) Other charge-carrying particles are observed in cosmic rays and nuclear decay, and are created in particle accelerators. All but the electron and proton survive only a short time and are quite rare by comparison.

The charges of electrons and protons are identical in magnitude but opposite in sign. Furthermore, all charged objects in nature are integral multiples of this basic quantity of charge, meaning that all charges are made of combinations of a basic unit of charge. Usually, charges are formed by combinations of electrons and protons. The magnitude of this basic charge is

$$|q_e| = 1.60 \times 10^{-19} \text{ C}. \tag{12.1}$$

The symbol $q$ is commonly used for charge and the subscript $e$ indicates the charge of a single electron (or proton).

The SI unit of charge is the coulomb (C). The number of protons needed to make a charge of 1.00 C is

$$\frac{1 \text{ proton}}{1.60 \times 10^{-19} \text{ C}} = 6.25 \times 10^{18} \text{ protons} \tag{12.2}.$$ 

Similarly, $6.25 \times 10^{18}$ electrons have a combined charge of $-1.00$ coulomb. Just as there is a smallest bit of an element (an atom), there is a smallest bit of charge. There is no directly observed charge smaller than $|q_e|$ (see Things Great and Small: The Submicroscopic Origin of Charge), and all observed charges are integral multiples of $|q_e|$.

Things Great and Small: The Submicroscopic Origin of Charge

With the exception of exotic, short-lived particles, all charge in nature is carried by electrons and protons. Electrons carry the charge we have named negative. Protons carry an equal-magnitude charge that we call positive. (See Figure 12.6.) Electron and proton charges are considered fundamental building blocks, since all other charges are integral multiples of those carried by electrons and protons. Electrons and protons are also two of the three fundamental building blocks of ordinary matter. The neutron is the third and has zero total charge.

Figure 12.6 shows a person touching a Van de Graaff generator and receiving excess positive charge. The expanded view of a hair shows the existence of both types of charges but an excess of positive. The repulsion of these positive like charges causes the strands of hair to repel other strands of hair and to stand up. The further blowup shows an artist’s conception of an electron and a proton perhaps found in an atom in a strand of hair.
When this person touches a Van de Graaff generator, some electrons are attracted to the generator, resulting in an excess of positive charge, causing her hair to stand on end. The charges in one hair are shown. An artist’s conception of an electron and a proton illustrate the particles carrying the negative and positive charges. We cannot really see these particles with visible light because they are so small (the electron seems to be an infinitesimal point), but we know a great deal about their measurable properties, such as the charges they carry.

The electron seems to have no substructure; in contrast, when the substructure of protons is explored by scattering extremely energetic electrons from them, it appears that there are point-like particles inside the proton. These sub-particles, named quarks, have never been directly observed, but they are believed to carry fractional charges as seen in Figure 12.7. Charges on electrons and protons and all other directly observable particles are unitary, but these quark substructures carry charges of either $-\frac{1}{3}$ or $+\frac{2}{3}$. There are continuing attempts to observe fractional charge directly and to learn of the properties of quarks, which are perhaps the ultimate substructure of matter.

Separation of Charge in Atoms

Charges in atoms and molecules can be separated—for example, by rubbing materials together. Some atoms and molecules have a greater affinity for electrons than others and will become negatively charged by close contact in rubbing, leaving the other material positively charged. (See Figure 12.8.) Positive charge can similarly be induced by rubbing. Methods other than rubbing can also separate charges. Batteries, for example, use combinations of substances that interact in such a way as to separate charges. Chemical interactions may transfer negative charge from one substance to the other, making one battery terminal negative and leaving the first one positive.
When materials are rubbed together, charges can be separated, particularly if one material has a greater affinity for electrons than another.

(a) Both the amber and cloth are originally neutral, with equal positive and negative charges. Only a tiny fraction of the charges are involved, and only a few of them are shown here. (b) When rubbed together, some negative charge is transferred to the amber, leaving the cloth with a net positive charge. (c) When separated, the amber and cloth now have net charges, but the absolute value of the net positive and negative charges will be equal.

No charge is actually created or destroyed when charges are separated as we have been discussing. Rather, existing charges are moved about. In fact, in all situations the total amount of charge is always constant. This universally obeyed law of nature is called the **law of conservation of charge**.

### Law of Conservation of Charge

Total charge is constant in any process.

In more exotic situations, such as in particle accelerators, mass, \( \Delta m \), can be created from energy in the amount \( \Delta m = \frac{E}{c^2} \).

Sometimes, the created mass is charged, such as when an electron is created. Whenever a charged particle is created, another having an opposite charge is always created along with it, so that the total charge created is zero. Usually, the two particles are “matter-antimatter” counterparts. For example, an antielectron would usually be created at the same time as an electron. The antielectron has a positive charge (it is called a positron), and so the total charge created is zero. (See Figure 12.9.) All particles have antimatter counterparts with opposite signs. When matter and antimatter counterparts are brought together, they completely annihilate one another. By annihilate, we mean that the mass of the two particles is converted to energy \( E \), again obeying the relationship \( \Delta m = \frac{E}{c^2} \). Since the two particles have equal and opposite charge, the total charge is zero before and after the annihilation; thus, total charge is conserved.

### Making Connections: Conservation Laws

Only a limited number of physical quantities are universally conserved. Charge is one—energy, momentum, and angular momentum are others. Because they are conserved, these physical quantities are used to explain more phenomena and form more connections than other, less basic quantities. We find that conserved quantities give us great insight into the rules followed by nature and hints to the organization of nature. Discoveries of conservation laws have led to further discoveries, such as the weak nuclear force and the quark substructure of protons and other particles.
Figure 12.9 (a) When enough energy is present, it can be converted into matter. Here the matter created is an electron–antielectron pair. \( m_e \) is the electron’s mass.) The total charge before and after this event is zero. (b) When matter and antimatter collide, they annihilate each other; the total charge is conserved at zero before and after the annihilation.

The law of conservation of charge is absolute—it has never been observed to be violated. Charge, then, is a special physical quantity, joining a very short list of other quantities in nature that are always conserved. Other conserved quantities include energy, momentum, and angular momentum.

Balloons and Static Electricity

Why does a balloon stick to your sweater? Rub a balloon on a sweater, then let go of the balloon and it flies over and sticks to the sweater. View the charges in the sweater, balloons, and the wall.

(Figure 12.10 Figure 12.10 This media type is not supported in this reader. Click to open media in browser.)

12.2 Conductors and Insulators

Figure 12.11 This power adapter uses metal wires and connectors to conduct electricity from the wall socket to a laptop computer. The conducting wires allow electrons to move freely through the cables, which are shielded by rubber and plastic. These materials act as insulators that don’t allow electric charge to escape outward. (credit: Evan-Amos, Wikimedia Commons)

Some substances, such as metals and salty water, allow charges to move through them with relative ease. Some of the electrons in metals and similar conductors are not bound to individual atoms or sites in the material. These free electrons can move through the material much as air moves through loose sand. Any substance that has free electrons and allows charge to move relatively freely through it is called a conductor. The moving electrons may collide with fixed atoms and molecules, losing some energy, but they can move in a conductor. Superconductors allow the movement of charge without any loss of energy. Salty water and other similar conducting materials contain free ions that can move through them. An ion is an atom or molecule having a positive or negative (nonzero) total charge. In other words, the total number of electrons is not equal to the total number
of protons. Other substances, such as glass, do not allow charges to move through them. These are called insulators. Electrons and ions in insulators are bound in the structure and cannot move easily—as much as $10^{23}$ times more slowly than in conductors. Pure water and dry table salt are insulators, for example, whereas molten salt and salty water are conductors.

Figure 12.12 An electroscope is a favorite instrument in physics demonstrations and student laboratories. It is typically made with gold foil leaves hung from a (conducting) metal stem and is insulated from the room air in a glass-walled container. (a) A positively charged glass rod is brought near the tip of the electroscope, attracting electrons to the top and leaving a net positive charge on the leaves. Like charges in the light flexible gold leaves repel, separating them. (b) When the rod is touched against the ball, electrons are attracted and transferred, reducing the net charge on the glass rod but leaving the electroscope positively charged. (c) The excess charges are evenly distributed in the stem and leaves of the electroscope once the glass rod is removed.

**Charging by Contact**

Figure 12.12 shows an electroscope being charged by touching it with a positively charged glass rod. Because the glass rod is an insulator, it must actually touch the electroscope to transfer charge to or from it. (Note that the extra positive charges reside on the surface of the glass rod as a result of rubbing it with silk before starting the experiment.) Since only electrons move in metals, we see that they are attracted to the top of the electroscope. There, some are transferred to the positive rod by touch, leaving the electroscope with a net positive charge.

Electrostatic repulsion in the leaves of the charged electroscope separates them. The electrostatic force has a horizontal component that results in the leaves moving apart as well as a vertical component that is balanced by the gravitational force. Similarly, the electroscope can be negatively charged by contact with a negatively charged object.

**Charging by Induction**

It is not necessary to transfer excess charge directly to an object in order to charge it. Figure 12.13 shows a method of induction wherein a charge is created in a nearby object, without direct contact. Here we see two neutral metal spheres in contact with one another but insulated from the rest of the world. A positively charged rod is brought near one of them, attracting negative charge to that side, leaving the other sphere positively charged. This is an example of induced polarization of neutral objects. Polarization is the separation of charges in an object that remains neutral. If the spheres are now separated (before the rod is pulled away), each sphere will have a net charge. Note that the object closest to the charged rod receives an opposite charge when charged by induction. Note also that no charge is removed from the charged rod, so that this process can be repeated without depleting the supply of excess charge.

Another method of charging by induction is shown in Figure 12.14. The neutral metal sphere is polarized when a charged rod is brought near it. The sphere is then grounded, meaning that a conducting wire is run from the sphere to the ground. Since the earth is large and most ground is a good conductor, it can supply or accept excess charge easily. In this case, electrons are attracted to the sphere through a wire called the ground wire, because it supplies a conducting path to the ground. The ground connection is broken before the charged rod is removed, leaving the sphere with an excess charge opposite to that of the rod. Again, an opposite charge is achieved when charging by induction and the charged rod loses none of its excess charge.
Figure 12.13 Charging by induction. (a) Two uncharged or neutral metal spheres are in contact with each other but insulated from the rest of the world. (b) A positively charged glass rod is brought near the sphere on the left, attracting negative charge and leaving the other sphere positively charged. (c) The spheres are separated before the rod is removed, thus separating negative and positive charge. (d) The spheres retain net charges after the inducing rod is removed—without ever having been touched by a charged object.

Figure 12.14 Charging by induction, using a ground connection. (a) A positively charged rod is brought near a neutral metal sphere, polarizing it. (b) The sphere is grounded, allowing electrons to be attracted from the earth’s ample supply. (c) The ground connection is broken. (d) The positive rod is removed, leaving the sphere with an induced negative charge.
Both positive and negative objects attract a neutral object by polarizing its molecules. (a) A positive object brought near a neutral insulator polarizes its molecules. There is a slight shift in the distribution of the electrons orbiting the molecule, with unlike charges being brought nearer and like charges moved away. Since the electrostatic force decreases with distance, there is a net attraction. (b) A negative object produces the opposite polarization, but again attracts the neutral object. (c) The same effect occurs for a conductor; since the unlike charges are closer, there is a net attraction.

Neutral objects can be attracted to any charged object. The pieces of straw attracted to polished amber are neutral, for example. If you run a plastic comb through your hair, the charged comb can pick up neutral pieces of paper. Figure 12.15 shows how the polarization of atoms and molecules in neutral objects results in their attraction to a charged object.

When a charged rod is brought near a neutral substance, an insulator in this case, the distribution of charge in atoms and molecules is shifted slightly. Opposite charge is attracted nearer the external charged rod, while like charge is repelled. Since the electrostatic force decreases with distance, the repulsion of like charges is weaker than the attraction of unlike charges, and so there is a net attraction. Thus a positively charged glass rod attracts neutral pieces of paper, as will a negatively charged rubber rod. Some molecules, like water, are polar molecules. Polar molecules have a natural or inherent separation of charge, although they are neutral overall. Polar molecules are particularly affected by other charged objects and show greater polarization effects than molecules with naturally uniform charge distributions.

Check Your Understanding

Can you explain the attraction of water to the charged rod in the figure below?

**Solution**

Water molecules are polarized, giving them slightly positive and slightly negative sides. This makes water even more...
susceptible to a charged rod’s attraction. In addition, tap water contains dissolved ions (positive and negative charges). As the water flows downward, due to the force of gravity, the charged conductor exerts a net attraction to the opposite charges in the stream of water, pulling it closer.

**John Travoltage**

Make sparks fly with John Travoltage. Wiggle Johnnie’s foot and he picks up charges from the carpet. Bring his hand close to the door knob and get rid of the excess charge.

(This media type is not supported in this reader. Click to open media in browser.)

Through the work of scientists in the late 18th century, the main features of the electrostatic force—the existence of two types of charge, the observation that like charges repel, unlike charges attract, and the decrease of force with distance—were eventually refined, and expressed as a mathematical formula. The mathematical formula for the electrostatic force is called **Coulomb’s law** after the French physicist Charles Coulomb (1736–1806), who performed experiments and first proposed a formula to calculate it.

Coulomb’s law calculates the magnitude of the force between two point charges, \( q_1 \) and \( q_2 \), separated by a distance \( r \). In SI units, the constant \( k \) is equal to

\[
k = 8.988 \times 10^9 \text{N} \cdot \text{m}^2/\text{C}^2 \approx 8.99 \times 10^9 \text{N} \cdot \text{m}^2/\text{C}^2.
\]

The electrostatic force is a vector quantity and is expressed in units of newtons. The force is understood to be along the line joining the two charges. (See Figure 12.19.)

Although the formula for Coulomb’s law is simple, it was no mean task to prove it. The experiments Coulomb did, with the primitive equipment then available, were difficult. Modern experiments have verified Coulomb’s law to great precision. For example, it has been shown that the force is inversely proportional to distance between two objects squared \( (F \propto 1/r^2) \) to an accuracy of 1 part in \( 10^{16} \). No exceptions have ever been found, even at the small distances within the atom.
Figure 12.19 The magnitude of the electrostatic force $F$ between point charges $q_1$ and $q_2$ separated by a distance $r$ is given by Coulomb’s law. Note that Newton’s third law (every force exerted creates an equal and opposite force) applies as usual—the force on $q_1$ is equal in magnitude and opposite in direction to the force it exerts on $q_2$. (a) Like charges. (b) Unlike charges.

Example 12.1 How Strong is the Coulomb Force Relative to the Gravitational Force?

Compare the electrostatic force between an electron and proton separated by $0.530 \times 10^{-10}$ m with the gravitational force between them. This distance is their average separation in a hydrogen atom.

Strategy

To compare the two forces, we first compute the electrostatic force using Coulomb’s law, $F = k \frac{|q_1 q_2|}{r^2}$. We then calculate the gravitational force using Newton’s universal law of gravitation. Finally, we take a ratio to see how the forces compare in magnitude.

Solution

Entering the given and known information about the charges and separation of the electron and proton into the expression of Coulomb’s law yields

$$F = k \frac{|q_1 q_2|}{r^2}$$

$$= \left(8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2\right) \frac{(1.60 \times 10^{-19} \text{ C})(1.60 \times 10^{-19} \text{ C})}{(0.530 \times 10^{-10} \text{ m})^2}$$

Thus the Coulomb force is

$$F = 8.19 \times 10^{-8} \text{ N}.$$

The charges are opposite in sign, so this is an attractive force. This is a very large force for an electron—it would cause an acceleration of $8.99 \times 10^{22}$ m/s² (verification is left as an end-of-section problem). The gravitational force is given by Newton’s law of gravitation as:

$$F_G = G \frac{mM}{r^2},$$

where $G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$. Here $m$ and $M$ represent the electron and proton masses, which can be found in the appendices. Entering values for the knowns yields

$$F_G = (6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2) \frac{(9.11 \times 10^{-31} \text{ kg})(1.67 \times 10^{-27} \text{ kg})}{(0.530 \times 10^{-10} \text{ m})^2} = 3.61 \times 10^{-47} \text{ N}$$

This is also an attractive force, although it is traditionally shown as positive since gravitational force is always attractive. The ratio of the magnitude of the electrostatic force to gravitational force in this case is, thus,

$$\frac{F}{F_G} = 2.27 \times 10^{39}. $$

Discussion

This is a remarkably large ratio! Note that this will be the ratio of electrostatic force to gravitational force for an electron and a proton at any distance (taking the ratio before entering numerical values shows that the distance cancels). This ratio gives some indication of just how much larger the Coulomb force is than the gravitational force between two of the most common particles in nature.

As the example implies, gravitational force is completely negligible on a small scale, where the interactions of individual charged particles are important. On a large scale, such as between the Earth and a person, the reverse is true. Most objects are nearly electrically neutral, and so attractive and repulsive Coulomb forces nearly cancel. Gravitational force on a large scale dominates interactions between large objects because it is always attractive, while Coulomb forces tend to cancel.
12.4 Electric Field: Concept of a Field Revisited

Contact forces, such as between a baseball and a bat, are explained on the small scale by the interaction of the charges in atoms and molecules in close proximity. They interact through forces that include the Coulomb force. Action at a distance is a force between objects that are not close enough for their atoms to “touch.” That is, they are separated by more than a few atomic diameters.

For example, a charged rubber comb attracts neutral bits of paper from a distance via the Coulomb force. It is very useful to think of an object being surrounded in space by a force field. The force field carries the force to another object (called a test object) some distance away.

Concept of a Field

A field is a way of conceptualizing and mapping the force that surrounds any object and acts on another object at a distance without apparent physical connection. For example, the gravitational field surrounding the earth (and all other masses) represents the gravitational force that would be experienced if another mass were placed at a given point within the field.

In the same way, the Coulomb force field surrounding any charge extends throughout space. Using Coulomb’s law, $F = k|q_1q_2|/r^2$, its magnitude is given by the equation $F = k|qQ|/r^2$, for a point charge (a particle having a charge $Q$) acting on a test charge $q$ at a distance $r$ (see Figure 12.20). Both the magnitude and direction of the Coulomb force field depend on $Q$ and the test charge $q$.

Figure 12.20 The Coulomb force field due to a positive charge $Q$ is shown acting on two different charges. Both charges are the same distance from $Q$. (a) Since $q_1$ is positive, the force $F_1$ acting on it is repulsive. (b) The charge $q_2$ is negative and greater in magnitude than $q_1$, and so the force $F_2$ acting on it is attractive and stronger than $F_1$. The Coulomb force field is thus not unique at any point in space, because it depends on the test charges $q_1$ and $q_2$ as well as the charge $Q$.

To simplify things, we would prefer to have a field that depends only on $Q$ and not on the test charge $q$. The electric field is defined in such a manner that it represents only the charge creating it and is unique at every point in space. Specifically, the electric field $E$ is defined to be the ratio of the Coulomb force to the test charge:

$$E = \frac{F}{q},$$

(12.11)

where $F$ is the electrostatic force (or Coulomb force) exerted on a positive test charge $q$. It is understood that $E$ is in the same direction as $F$. It is also assumed that $q$ is so small that it does not alter the charge distribution creating the electric field.

The units of electric field are newtons per coulomb (N/C). If the electric field is known, then the electrostatic force on any charge $q$ is simply obtained by multiplying charge times electric field, or $F = qE$. Consider the electric field due to a point charge $Q$.

According to Coulomb’s law, the force it exerts on a test charge $q$ is $F = k|qQ|/r^2$. Thus the magnitude of the electric field, $E$, for a point charge is

$$E = \frac{|F|}{q} = \frac{|qQ|}{qr^2} = k\frac{|Q|}{r^2}.$$  

(12.12)

Since the test charge cancels, we see that

$$E = k\frac{|Q|}{r^2}.$$  

(12.13)
The electric field is thus seen to depend only on the charge \( Q \) and the distance \( r \); it is completely independent of the test charge \( q \).

**Example 12.2 Calculating the Electric Field of a Point Charge**

Calculate the strength and direction of the electric field \( E \) due to a point charge of 2.00 nC (nano-Coulombs) at a distance of 5.00 mm from the charge.

**Strategy**

We can find the electric field created by a point charge by using the equation \( E = \frac{kQ}{r^2} \).

**Solution**

Here \( Q = 2.00 \times 10^{-9} \) C and \( r = 5.00 \times 10^{-3} \) m. Entering those values into the above equation gives

\[
E = \frac{kQ}{r^2} = \frac{(8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)(2.00 \times 10^{-9} \text{ C})}{(5.00 \times 10^{-3} \text{ m})^2} = 7.19 \times 10^5 \text{ N/C}.
\]

**Discussion**

This electric field strength is the same at any point 5.00 mm away from the charge \( Q \) that creates the field. It is positive, meaning that it has a direction pointing away from the charge \( Q \).

**Example 12.3 Calculating the Force Exerted on a Point Charge by an Electric Field**

What force does the electric field found in the previous example exert on a point charge of \(-0.250 \mu\text{C}\) ?

**Strategy**

Since we know the electric field strength and the charge in the field, the force on that charge can be calculated using the definition of electric field \( E = \frac{F}{q} \) rearranged to \( F = qE \).

**Solution**

The magnitude of the force on a charge \( q = -0.250 \mu\text{C} \) exerted by a field of strength \( E = 7.20 \times 10^5 \) N/C is thus,

\[
F = -qE = (0.250 \times 10^{-6} \text{ C})(7.20 \times 10^5 \text{ N/C}) = 0.180 \text{ N}.
\]

Because \( q \) is negative, the force is directed opposite to the direction of the field.

**Discussion**

The force is attractive, as expected for unlike charges. (The field was created by a positive charge and here acts on a negative charge.) The charges in this example are typical of common static electricity, and the modest attractive force obtained is similar to forces experienced in static cling and similar situations.

**Electric Field of Dreams**

Play ball! Add charges to the Field of Dreams and see how they react to the electric field. Turn on a background electric field and adjust the direction and magnitude.

(This media type is not supported in this reader. Click to open media in browser.)

Figure 12.21

### 12.5 Electric Field Lines: Multiple Charges

Drawings using lines to represent electric fields around charged objects are very useful in visualizing field strength and
direction. Since the electric field has both magnitude and direction, it is a vector. Like all vectors, the electric field can be represented by an arrow that has length proportional to its magnitude and that points in the correct direction. (We have used arrows extensively to represent force vectors, for example.)

**Figure 12.22** shows two pictorial representations of the same electric field created by a positive point charge $Q$. **Figure 12.22** (b) shows the standard representation using continuous lines. **Figure 12.22** (a) shows numerous individual arrows with each arrow representing the force on a test charge $q$. Field lines are essentially a map of infinitesimal force vectors.

![Figure 12.22](image)

**Figure 12.22** Two equivalent representations of the electric field due to a positive charge $Q$. (a) Arrows representing the electric field's magnitude and direction. (b) In the standard representation, the arrows are replaced by continuous field lines having the same direction at any point as the electric field. The closeness of the lines is directly related to the strength of the electric field. A test charge placed anywhere will feel a force in the direction of the field line; this force will have a strength proportional to the density of the lines (being greater near the charge, for example).

Note that the electric field is defined for a positive test charge $q$, so that the field lines point away from a positive charge and toward a negative charge. (See **Figure 12.23**.) The electric field strength is exactly proportional to the number of field lines per unit area, since the magnitude of the electric field for a point charge is $E = kQ/r^2$ and area is proportional to $r^2$. This pictorial representation, in which field lines represent the direction and their closeness (that is, their areal density or the number of lines crossing a unit area) represents strength, is used for all fields: electrostatic, gravitational, magnetic, and others.

![Figure 12.23](image)

**Figure 12.23** The electric field surrounding three different point charges. (a) A positive charge. (b) A negative charge of equal magnitude. (c) A larger negative charge.

In many situations, there are multiple charges. The total electric field created by multiple charges is the vector sum of the individual fields created by each charge. The following example shows how to add electric field vectors.

**Example 12.4 Adding Electric Fields**

Find the magnitude and direction of the total electric field due to the two point charges, $q_1$ and $q_2$, at the origin of the coordinate system as shown in **Figure 12.24**.
The electric fields $E_1$ and $E_2$ at the origin O add to $E_{\text{tot}}$.

Strategy
Since the electric field is a vector (having magnitude and direction), we add electric fields with the same vector techniques used for other types of vectors. We first must find the electric field due to each charge at the point of interest, which is the origin of the coordinate system (O) in this instance. We pretend that there is a positive test charge, $q$, at point O, which allows us to determine the direction of the fields $E_1$ and $E_2$. Once those fields are found, the total field can be determined using vector addition.

Solution
The electric field strength at the origin due to $q_1$ is labeled $E_1$ and is calculated:

$$E_1 = k \frac{q_1}{r_1^2} = \left(8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2\right) \frac{\left(5.00 \times 10^{-9} \text{ C}\right)}{(2.00 \times 10^{-2} \text{ m})^2}$$

$$E_1 = 1.124 \times 10^5 \text{ N/C}.$$ 

Similarly, $E_2$ is

$$E_2 = k \frac{q_2}{r_2^2} = \left(8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2\right) \frac{\left(10.0 \times 10^{-9} \text{ C}\right)}{(4.00 \times 10^{-2} \text{ m})^2}$$

$$E_2 = 0.5619 \times 10^5 \text{ N/C}.$$ 

Four digits have been retained in this solution to illustrate that $E_1$ is exactly twice the magnitude of $E_2$. Now arrows are drawn to represent the magnitudes and directions of $E_1$ and $E_2$. (See Figure 12.24.) The direction of the electric field is that of the force on a positive charge so both arrows point directly away from the positive charges that create them. The arrow for $E_1$ is exactly twice the length of that for $E_2$. The arrows form a right triangle in this case and can be added using the Pythagorean theorem. The magnitude of the total field $E_{\text{tot}}$ is

$$E_{\text{tot}} = \left(E_1^2 + E_2^2\right)^{1/2}$$

$$= \left\{(1.124 \times 10^5 \text{ N/C})^2 + (0.5619 \times 10^5 \text{ N/C})^2\right\}^{1/2}$$

$$= 1.26 \times 10^5 \text{ N/C}.$$ 

The direction is

$$\theta = \tan^{-1}\left(\frac{E_1}{E_2}\right)$$

$$= \tan^{-1}\left(\frac{1.124 \times 10^5 \text{ N/C}}{0.5619 \times 10^5 \text{ N/C}}\right)$$

$$= 63.4^\circ,$$ 

or $63.4^\circ$ above the x-axis.
Discussion

In cases where the electric field vectors to be added are not perpendicular, vector components or graphical techniques can be used. The total electric field found in this example is the total electric field at only one point in space. To find the total electric field due to these two charges over an entire region, the same technique must be repeated for each point in the region. This impossibly lengthy task (there are an infinite number of points in space) can be avoided by calculating the total field at representative points and using some of the unifying features noted next.

Figure 12.25 shows how the electric field from two point charges can be drawn by finding the total field at representative points and drawing electric field lines consistent with those points. While the electric fields from multiple charges are more complex than those of single charges, some simple features are easily noticed.

For example, the field is weaker between like charges, as shown by the lines being farther apart in that region. (This is because the fields from each charge exert opposing forces on any charge placed between them.) (See Figure 12.25 and Figure 12.26(a.) Furthermore, at a great distance from two like charges, the field becomes identical to the field from a single, larger charge.

Figure 12.26(b) shows the electric field of two unlike charges. The field is stronger between the charges. In that region, the fields from each charge are in the same direction, and so their strengths add. The field of two unlike charges is weak at large distances, because the fields of the individual charges are in opposite directions and so their strengths subtract. At very large distances, the field of two unlike charges looks like that of a smaller single charge.

Figure 12.25 Two positive point charges $q_1$ and $q_2$ produce the resultant electric field shown. The field is calculated at representative points and then smooth field lines drawn following the rules outlined in the text.
We use electric field lines to visualize and analyze electric fields (the lines are a pictorial tool, not a physical entity in themselves). The properties of electric field lines for any charge distribution can be summarized as follows:

1. Field lines must begin on positive charges and terminate on negative charges, or at infinity in the hypothetical case of isolated charges.
2. The number of field lines leaving a positive charge or entering a negative charge is proportional to the magnitude of the charge.
3. The strength of the field is proportional to the closeness of the field lines—more precisely, it is proportional to the number of lines per unit area perpendicular to the lines.
4. The direction of the electric field is tangent to the field line at any point in space.
5. Field lines can never cross.

The last property means that the field is unique at any point. The field line represents the direction of the field; so if they crossed, the field would have two directions at that location (an impossibility if the field is unique).

### Charges and Fields

Move point charges around on the playing field and then view the electric field, voltages, equipotential lines, and more. It's colorful, it's dynamic, it's free.

(This media type is not supported in this reader. Click to open media in browser.)

Figure 12.27

---

**12.6 Electric Forces in Biology**

Classical electrostatics has an important role to play in modern molecular biology. Large molecules such as proteins, nucleic acids, and so on—so important to life—are usually electrically charged. DNA itself is highly charged; it is the electrostatic force that not only holds the molecule together but gives the molecule structure and strength. Figure 12.28 is a schematic of the DNA double helix.
The four nucleotide bases are given the symbols A (adenine), C (cytosine), G (guanine), and T (thymine). The order of the four bases varies in each strand, but the pairing between bases is always the same. C and G are always paired and A and T are always paired, which helps to preserve the order of bases in cell division (mitosis) so as to pass on the correct genetic information. Since the Coulomb force drops with distance \( F \propto \frac{1}{r^2} \), the distances between the base pairs must be small enough that the electrostatic force is sufficient to hold them together.

DNA is a highly charged molecule, with about \( 2q_e \) (fundamental charge) per \( 0.3 \times 10^{-9} \text{ m} \). The distance separating the two strands that make up the DNA structure is about 1 nm, while the distance separating the individual atoms within each base is about 0.3 nm.

One might wonder why electrostatic forces do not play a larger role in biology than they do if we have so many charged molecules. The reason is that the electrostatic force is “diluted” due to screening between molecules. This is due to the presence of other charges in the cell.

**Polarity of Water Molecules**

The best example of this charge screening is the water molecule, represented as \( \text{H}_2\text{O} \). Water is a strongly polar molecule. Its 10 electrons (8 from the oxygen atom and 2 from the two hydrogen atoms) tend to remain closer to the oxygen nucleus than the hydrogen nuclei. This creates two centers of equal and opposite charges—what is called a dipole, as illustrated in Figure 12.29. The magnitude of the dipole is called the dipole moment.

These two centers of charge will terminate some of the electric field lines coming from a free charge, as on a DNA molecule. This results in a reduction in the strength of the Coulomb interaction. One might say that screening makes the Coulomb force a short range force rather than long range.

Other ions of importance in biology that can reduce or screen Coulomb interactions are \( \text{Na}^+ \), \( \text{K}^+ \), and \( \text{Cl}^- \). These ions are located both inside and outside of living cells. The movement of these ions through cell membranes is crucial to the motion of nerve impulses through nerve axons.

Recent studies of electrostatics in biology seem to show that electric fields in cells can be extended over larger distances, in spite of screening, by “microtubules” within the cell. These microtubules are hollow tubes composed of proteins that guide the movement of chromosomes when cells divide, the motion of other organisms within the cell, and provide mechanisms for motion of some cells (as motors).

**12.7 Conductors and Electric Fields in Static Equilibrium**

Conductors contain free charges that move easily. When excess charge is placed on a conductor or the conductor is put into a static electric field, charges in the conductor quickly respond to reach a steady state called electrostatic equilibrium.

Figure 12.30 shows the effect of an electric field on free charges in a conductor. The free charges move until the field is perpendicular to the conductor’s surface. There can be no component of the field parallel to the surface in electrostatic equilibrium, since, if there were, it would produce further movement of charge. A positive free charge is shown, but free charges can be either positive or negative and are, in fact, negative in metals. The motion of a positive charge is equivalent to the motion of a negative charge in the opposite direction.
When an electric field $\mathbf{E}$ is applied to a conductor, free charges inside the conductor move until the field is perpendicular to the surface.

(a) The electric field is a vector quantity, with both parallel and perpendicular components. The parallel component ($E_\parallel$) exerts a force ($F_\parallel$) on the free charge $q$, which moves the charge until $F_\parallel = 0$. (b) The resulting field is perpendicular to the surface. The free charge has been brought to the conductor’s surface, leaving electrostatic forces in equilibrium.

A conductor placed in an electric field will be polarized. Figure 12.31 shows the result of placing a neutral conductor in an originally uniform electric field. The field becomes stronger near the conductor but entirely disappears inside it.

Figure 12.30 When an electric field $\mathbf{E}$ is applied to a conductor, free charges inside the conductor move until the field is perpendicular to the surface. (a) The electric field is a vector quantity, with both parallel and perpendicular components. The parallel component ($E_\parallel$) exerts a force ($F_\parallel$) on the free charge $q$, which moves the charge until $F_\parallel = 0$. (b) The resulting field is perpendicular to the surface. The free charge has been brought to the conductor’s surface, leaving electrostatic forces in equilibrium.

A conductor placed in an electric field will be polarized. Figure 12.31 shows the result of placing a neutral conductor in an originally uniform electric field. The field becomes stronger near the conductor but entirely disappears inside it.

Figure 12.31 This illustration shows a spherical conductor in static equilibrium with an originally uniform electric field. Free charges move within the conductor, polarizing it, until the electric field lines are perpendicular to the surface. The field lines end on excess negative charge on one section of the surface and begin again on excess positive charge on the opposite side. No electric field exists inside the conductor, since free charges in the conductor would continue moving in response to any field until it was neutralized.

**Misconception Alert: Electric Field inside a Conductor**

Excess charges placed on a spherical conductor repel and move until they are evenly distributed, as shown in Figure 12.32. Excess charge is forced to the surface until the field inside the conductor is zero. Outside the conductor, the field is exactly the same as if the conductor were replaced by a point charge at its center equal to the excess charge.
The mutual repulsion of excess positive charges on a spherical conductor distributes them uniformly on its surface. The resulting electric field is perpendicular to the surface and zero inside. Outside the conductor, the field is identical to that of a point charge at the center equal to the excess charge.

Properties of a Conductor in Electrostatic Equilibrium

1. The electric field is zero inside a conductor.
2. Just outside a conductor, the electric field lines are perpendicular to its surface, ending or beginning on charges on the surface.
3. Any excess charge resides entirely on the surface or surfaces of a conductor.

The properties of a conductor are consistent with the situations already discussed and can be used to analyze any conductor in electrostatic equilibrium. This can lead to some interesting new insights, such as described below.

How can a very uniform electric field be created? Consider a system of two metal plates with opposite charges on them, as shown in Figure 12.33. The properties of conductors in electrostatic equilibrium indicate that the electric field between the plates will be uniform in strength and direction. Except near the edges, the excess charges distribute themselves uniformly, producing field lines that are uniformly spaced (hence uniform in strength) and perpendicular to the surfaces (hence uniform in direction, since the plates are flat). The edge effects are less important when the plates are close together.

A near uniform electric field of approximately 150 N/C, directed downward, surrounds Earth, with the magnitude increasing slightly as we get closer to the surface. What causes the electric field? At around 100 km above the surface of Earth we have a layer of charged particles, called the ionosphere. The ionosphere is responsible for a range of phenomena including the electric field surrounding Earth. In fair weather the ionosphere is positive and the Earth largely negative, maintaining the electric field (Figure 12.34(a)).

In storm conditions clouds form and localized electric fields can be larger and reversed in direction (Figure 12.34(b)). The exact charge distributions depend on the local conditions, and variations of Figure 12.34(b) are possible.

If the electric field is sufficiently large, the insulating properties of the surrounding material break down and it becomes conducting. For air this occurs at around $3 \times 10^6$ N/C. Air ionizes ions and electrons recombine, and we get discharge in the form of lightning sparks and corona discharge.
Electric Fields on Uneven Surfaces

So far we have considered excess charges on a smooth, symmetrical conductor surface. What happens if a conductor has sharp corners or is pointed? Excess charges on a nonuniform conductor become concentrated at the sharpest points. Additionally, excess charge may move on or off the conductor at the sharpest points.

To see how and why this happens, consider the charged conductor in Figure 12.35. The electrostatic repulsion of like charges is most effective in moving them apart on the flattest surface, and so they become least concentrated there. This is because the forces between identical pairs of charges at either end of the conductor are identical, but the components of the forces parallel to the surfaces are different. The component parallel to the surface is greatest on the flattest surface and, hence, more effective in moving the charge.

The same effect is produced on a conductor by an externally applied electric field, as seen in Figure 12.35 (c). Since the field lines must be perpendicular to the surface, more of them are concentrated on the most curved parts.

Applications of Conductors

On a very sharply curved surface, such as shown in Figure 12.36, the charges are so concentrated at the point that the resulting electric field can be great enough to remove them from the surface. This can be useful.

Lightning rods work best when they are most pointed. The large charges created in storm clouds induce an opposite charge on a building that can result in a lightning bolt hitting the building. The induced charge is bled away continually by a lightning rod, preventing the more dramatic lightning strike.

Of course, we sometimes wish to prevent the transfer of charge rather than to facilitate it. In that case, the conductor should be very smooth and have as large a radius of curvature as possible. (See Figure 12.37.) Smooth surfaces are used on high-voltage transmission lines, for example, to avoid leakage of charge into the air.

Another device that makes use of some of these principles is a Faraday cage. This is a metal shield that encloses a volume. All electrical charges will reside on the outside surface of this shield, and there will be no electrical field inside. A Faraday cage is used to prohibit stray electrical fields in the environment from interfering with sensitive measurements, such as the electrical signals inside a nerve cell.

During electrical storms if you are driving a car, it is best to stay inside the car as its metal body acts as a Faraday cage with zero electrical field inside. If in the vicinity of a lightning strike, its effect is felt on the outside of the car and the inside is unaffected, provided you remain totally inside. This is also true if an active (“hot”) electrical wire was broken (in a storm or an accident) and fell on your car.
A very pointed conductor has a large charge concentration at the point. The electric field is very strong at the point and can exert a force large enough to transfer charge on or off the conductor. Lightning rods are used to prevent the buildup of large excess charges on structures and, thus, are pointed.

Figure 12.36

(a) A lightning rod is pointed to facilitate the transfer of charge. (credit: Romaine, Wikimedia Commons) (b) This Van de Graaff generator has a smooth surface with a large radius of curvature to prevent the transfer of charge and allow a large voltage to be generated. The mutual repulsion of like charges is evident in the person’s hair while touching the metal sphere. (credit: Jon ‘ShakataGaNai’ Davis/Wikimedia Commons).

12.8 Applications of Electrostatics

The study of electrostatics has proven useful in many areas. This module covers just a few of the many applications of electrostatics.

The Van de Graaff Generator

Van de Graaff generators (or Van de Graaffs) are not only spectacular devices used to demonstrate high voltage due to static electricity—they are also used for serious research. The first was built by Robert Van de Graaff in 1931 (based on original suggestions by Lord Kelvin) for use in nuclear physics research. Figure 12.38 shows a schematic of a large research version. Van de Graaffs utilize both smooth and pointed surfaces, and conductors and insulators to generate large static charges and, hence, large voltages.

A very large excess charge can be deposited on the sphere, because it moves quickly to the outer surface. Practical limits arise because the large electric fields polarize and eventually ionize surrounding materials, creating free charges that neutralize excess charge or allow it to escape. Nevertheless, voltages of 15 million volts are well within practical limits.
Figure 12.38 Schematic of Van de Graaff generator. A battery (A) supplies excess positive charge to a pointed conductor, the points of which spray the charge onto a moving insulating belt near the bottom. The pointed conductor (B) on top in the large sphere picks up the charge. (The induced electric field at the points is so large that it removes the charge from the belt.) This can be done because the charge does not remain inside the conducting sphere but moves to its outside surface. An ion source inside the sphere produces positive ions, which are accelerated away from the positive sphere to high velocities.

Take-Home Experiment: Electrostatics and Humidity
Rub a comb through your hair and use it to lift pieces of paper. It may help to tear the pieces of paper rather than cut them neatly. Repeat the exercise in your bathroom after you have had a long shower and the air in the bathroom is moist. Is it easier to get electrostatic effects in dry or moist air? Why would torn paper be more attractive to the comb than cut paper? Explain your observations.

Xerography
Most copy machines use an electrostatic process called xerography—a word coined from the Greek words xeros for dry and graphos for writing. The heart of the process is shown in simplified form in Figure 12.39.

A selenium-coated aluminum drum is sprayed with positive charge from points on a device called a corotron. Selenium is a substance with an interesting property—it is a photoconductor. That is, selenium is an insulator when in the dark and a conductor when exposed to light.

In the first stage of the xerography process, the conducting aluminum drum is grounded so that a negative charge is induced under the thin layer of uniformly positively charged selenium. In the second stage, the surface of the drum is exposed to the image of whatever is to be copied. Where the image is light, the selenium becomes conducting, and the positive charge is neutralized. In dark areas, the positive charge remains, and so the image has been transferred to the drum.

The third stage takes a dry black powder, called toner, and sprays it with a negative charge so that it will be attracted to the positive regions of the drum. Next, a blank piece of paper is given a greater positive charge than on the drum so that it will pull the toner from the drum. Finally, the paper and electrostatically held toner are passed through heated pressure rollers, which melt and permanently adhere the toner within the fibers of the paper.
Figure 12.39 Xerography is a dry copying process based on electrostatics. The major steps in the process are the charging of the photoconducting drum, transfer of an image creating a positive charge duplicate, attraction of toner to the charged parts of the drum, and transfer of toner to the paper. Not shown are heat treatment of the paper and cleansing of the drum for the next copy.

Laser Printers

Laser printers use the xerographic process to make high-quality images on paper, employing a laser to produce an image on the photoconducting drum as shown in Figure 12.40. In its most common application, the laser printer receives output from a computer, and it can achieve high-quality output because of the precision with which laser light can be controlled. Many laser printers do significant information processing, such as making sophisticated letters or fonts, and may contain a computer more powerful than the one giving them the raw data to be printed.

Figure 12.40 In a laser printer, a laser beam is scanned across a photoconducting drum, leaving a positive charge image. The other steps for charging the drum and transferring the image to paper are the same as in xerography. Laser light can be very precisely controlled, enabling laser printers to produce high-quality images.

Ink Jet Printers and Electrostatic Painting

The ink jet printer, commonly used to print computer-generated text and graphics, also employs electrostatics. A nozzle makes a fine spray of tiny ink droplets, which are then given an electrostatic charge. (See Figure 12.41.)

Once charged, the droplets can be directed, using pairs of charged plates, with great precision to form letters and images on paper. Ink jet printers can produce color images by using a black jet and three other jets with primary colors, usually cyan, magenta, and yellow, much as a color television produces color. (This is more difficult with xerography, requiring multiple drums and toners.)
Figure 12.41 The nozzle of an ink-jet printer produces small ink droplets, which are sprayed with electrostatic charge. Various computer-driven devices are then used to direct the droplets to the correct positions on a page.

Electrostatic painting employs electrostatic charge to spray paint onto odd-shaped surfaces. Mutual repulsion of like charges causes the paint to fly away from its source. Surface tension forms drops, which are then attracted by unlike charges to the surface to be painted. Electrostatic painting can reach those hard-to-get at places, applying an even coat in a controlled manner. If the object is a conductor, the electric field is perpendicular to the surface, tending to bring the drops in perpendicularly. Corners and points on conductors will receive extra paint. Felt can similarly be applied.

Smoke Precipitators and Electrostatic Air Cleaning

Another important application of electrostics is found in air cleaners, both large and small. The electrostatic part of the process places excess (usually positive) charge on smoke, dust, pollen, and other particles in the air and then passes the air through an oppositely charged grid that attracts and retains the charged particles. (See Figure 12.42.)

Large electrostatic precipitators are used industrially to remove over 99% of the particles from stack gas emissions associated with the burning of coal and oil. Home precipitators, often in conjunction with the home heating and air conditioning system, are very effective in removing polluting particles, irritants, and allergens.

Figure 12.42 (a) Schematic of an electrostatic precipitator. Air is passed through grids of opposite charge. The first grid charges airborne particles, while the second attracts and collects them. (b) The dramatic effect of electrostatic precipitators is seen by the absence of smoke from this power plant. (credit: Cmdalgleish, Wikimedia Commons)

Problem-Solving Strategies for Electrostatics

1. Examine the situation to determine if static electricity is involved. This may concern separated stationary charges, the forces among them, and the electric fields they create.
2. Identify the system of interest. This includes noting the number, locations, and types of charges involved.
3. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful. Determine
whether the Coulomb force is to be considered directly—if so, it may be useful to draw a free-body diagram, using electric field lines.

4. Make a list of what is given or can be inferred from the problem as stated (identify the knowns). It is important to distinguish the Coulomb force $F$ from the electric field $E$, for example.

5. Solve the appropriate equation for the quantity to be determined (the unknown) or draw the field lines as requested.

6. Examine the answer to see if it is reasonable: Does it make sense? Are units correct and the numbers involved reasonable?

Integrated Concepts

The Integrated Concepts exercises for this module involve concepts such as electric charges, electric fields, and several other topics. Physics is most interesting when applied to general situations involving more than a narrow set of physical principles. The electric field exerts force on charges, for example, and hence the relevance of Dynamics: Force and Newton's Laws of Motion. The following topics are involved in some or all of the problems labeled “Integrated Concepts”:

- Kinematics
- Two-Dimensional Kinematics
- Dynamics: Force and Newton's Laws of Motion
- Uniform Circular Motion and Gravitation
- Statics and Torque
- Fluid Statics (https://legacy.cnx.org/content/m42185/latest)

The following worked example illustrates how this strategy is applied to an Integrated Concept problem:

### Example 12.5 Acceleration of a Charged Drop of Gasoline

If steps are not taken to ground a gasoline pump, static electricity can be placed on gasoline when filling your car’s tank. Suppose a tiny drop of gasoline has a mass of $4.00 \times 10^{-15}$ kg and is given a positive charge of $3.20 \times 10^{-19}$ C. (a) Find the weight of the drop. (b) Calculate the electric force on the drop if there is an upward electric field of strength $3.00 \times 10^5$ N/C due to other static electricity in the vicinity. (c) Calculate the drop's acceleration.

**Strategy**

To solve an integrated concept problem, we must first identify the physical principles involved and identify the chapters in which they are found. Part (a) of this example asks for weight. This is a topic of dynamics and is defined in Dynamics: Force and Newton's Laws of Motion. Part (b) deals with electric force on a charge, a topic of Electric Charge and Electric Field. Part (c) asks for acceleration, knowing forces and mass. These are part of Newton's laws, also found in Dynamics: Force and Newton's Laws of Motion.

The following solutions to each part of the example illustrate how the specific problem-solving strategies are applied. These involve identifying knowns and unknowns, checking to see if the answer is reasonable, and so on.

**Solution for (a)**

Weight is mass times the acceleration due to gravity, as first expressed in

$$w = mg.$$  \hspace{1cm} (12.20)

Entering the given mass and the average acceleration due to gravity yields

$$w = (4.00 \times 10^{-15} \text{ kg})(9.80 \text{ m/s}^2) = 3.92 \times 10^{-14} \text{ N}. \hspace{1cm} (12.21)$$

**Discussion for (a)**

This is a small weight, consistent with the small mass of the drop.

**Solution for (b)**

The force an electric field exerts on a charge is given by rearranging the following equation:

$$F = qE.$$  \hspace{1cm} (12.22)

Here we are given the charge ($3.20 \times 10^{-19}$ C is twice the fundamental unit of charge) and the electric field strength, and so the electric force is found to be

$$F = (3.20 \times 10^{-19} \text{ C})(3.00 \times 10^5 \text{ N/C}) = 9.60 \times 10^{-14} \text{ N}. \hspace{1cm} (12.23)$$

**Discussion for (b)**

While this is a small force, it is greater than the weight of the drop.

**Solution for (c)**
The acceleration can be found using Newton’s second law, provided we can identify all of the external forces acting on the drop. We assume only the drop’s weight and the electric force are significant. Since the drop has a positive charge and the electric field is given to be upward, the electric force is upward. We thus have a one-dimensional (vertical direction) problem, and we can state Newton’s second law as

$$a = \frac{F_{\text{net}}}{m}. \quad (12.24)$$

where $F_{\text{net}} = F - w$. Entering this and the known values into the expression for Newton’s second law yields

$$a = \frac{F - w}{m} = \frac{9.60 \times 10^{-14} \text{ N} - 3.92 \times 10^{-14} \text{ N}}{4.00 \times 10^{-15} \text{ kg}} = 14.2 \text{ m/s}^2. \quad (12.25)$$

Discussion for (c)

This is an upward acceleration great enough to carry the drop to places where you might not wish to have gasoline.

This worked example illustrates how to apply problem-solving strategies to situations that include topics in different chapters. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknown using familiar problem-solving strategies. These are found throughout the text, and many worked examples show how to use them for single topics. In this integrated concepts example, you can see how to apply them across several topics. You will find these techniques useful in applications of physics outside a physics course, such as in your profession, in other science disciplines, and in everyday life. The following problems will build your skills in the broad application of physical principles.

Unreasonable Results

The Unreasonable Results exercises for this module have results that are unreasonable because some premise is unreasonable or because certain of the premises are inconsistent with one another. Physical principles applied correctly then produce unreasonable results. The purpose of these problems is to give practice in assessing whether nature is being accurately described, and if it is not to trace the source of difficulty.

Problem-Solving Strategy

To determine if an answer is reasonable, and to determine the cause if it is not, do the following.

1. Solve the problem using strategies as outlined above. Use the format followed in the worked examples in the text to solve the problem as usual.

2. Check to see if the answer is reasonable. Is it too large or too small, or does it have the wrong sign, improper units, and so on?

3. If the answer is unreasonable, look for what specifically could cause the identified difficulty. Usually, the manner in which the answer is unreasonable is an indication of the difficulty. For example, an extremely large Coulomb force could be due to the assumption of an excessively large separated charge.

Glossary

conductor: a material that allows electrons to move separately from their atomic orbits

conductor: an object with properties that allow charges to move about freely within it

Coulomb force: another term for the electrostatic force

Coulomb interaction: the interaction between two charged particles generated by the Coulomb forces they exert on one another

Coulomb’s law: the mathematical equation calculating the electrostatic force vector between two charged particles
dipole: a molecule’s lack of symmetrical charge distribution, causing one side to be more positive and another to be more negative
electric charge: a physical property of an object that causes it to be attracted toward or repelled from another charged object; each charged object generates and is influenced by a force called an electromagnetic force
electric field: a three-dimensional map of the electric force extended out into space from a point charge
electric field lines: a series of lines drawn from a point charge representing the magnitude and direction of force exerted by
that charge

**electromagnetic force**: one of the four fundamental forces of nature; the electromagnetic force consists of static electricity, moving electricity and magnetism

**electron**: a particle orbiting the nucleus of an atom and carrying the smallest unit of negative charge

**electrostatic equilibrium**: an electrostatically balanced state in which all free electrical charges have stopped moving about

**electrostatic force**: the amount and direction of attraction or repulsion between two charged bodies

**electrostatic precipitators**: filters that apply charges to particles in the air, then attract those charges to a filter, removing them from the airstream

**electrostatic repulsion**: the phenomenon of two objects with like charges repelling each other

**electrostatics**: the study of electric forces that are static or slow-moving

**Faraday cage**: a metal shield which prevents electric charge from penetrating its surface

**field**: a map of the amount and direction of a force acting on other objects, extending out into space

**free charge**: an electrical charge (either positive or negative) which can move about separately from its base molecule

**free electron**: an electron that is free to move away from its atomic orbit

**grounded**: when a conductor is connected to the Earth, allowing charge to freely flow to and from Earth’s unlimited reservoir

**grounded**: connected to the ground with a conductor, so that charge flows freely to and from the Earth to the grounded object

**induction**: the process by which an electrically charged object brought near a neutral object creates a charge in that object

**ink-jet printer**: small ink droplets sprayed with an electric charge are controlled by electrostatic plates to create images on paper

**insulator**: a material that holds electrons securely within their atomic orbits

**ionosphere**: a layer of charged particles located around 100 km above the surface of Earth, which is responsible for a range of phenomena including the electric field surrounding Earth

**laser printer**: uses a laser to create a photoconductive image on a drum, which attracts dry ink particles that are then rolled onto a sheet of paper to print a high-quality copy of the image

**law of conservation of charge**: states that whenever a charge is created, an equal amount of charge with the opposite sign is created simultaneously

**photoconductor**: a substance that is an insulator until it is exposed to light, when it becomes a conductor

**point charge**: A charged particle, designated \( q \), generating an electric field

**polar molecule**: a molecule with an asymmetrical distribution of positive and negative charge

**polarization**: slight shifting of positive and negative charges to opposite sides of an atom or molecule

**polarized**: a state in which the positive and negative charges within an object have collected in separate locations

**proton**: a particle in the nucleus of an atom and carrying a positive charge equal in magnitude and opposite in sign to the amount of negative charge carried by an electron

**screening**: the dilution or blocking of an electrostatic force on a charged object by the presence of other charges nearby

**static electricity**: a buildup of electric charge on the surface of an object

**test charge**: A particle (designated \( q \)) with either a positive or negative charge set down within an electric field generated by a point charge

**Van de Graaff generator**: a machine that produces a large amount of excess charge, used for experiments with high voltage

**vector**: a quantity with both magnitude and direction

**vector addition**: mathematical combination of two or more vectors, including their magnitudes, directions, and positions

**xerography**: a dry copying process based on electrostatics
12.1 Static Electricity and Charge: Conservation of Charge

- There are only two types of charge, which we call positive and negative.
- Like charges repel, unlike charges attract, and the force between charges decreases with the square of the distance.
- The vast majority of positive charge in nature is carried by protons, while the vast majority of negative charge is carried by electrons.
- The electric charge of one electron is equal in magnitude and opposite in sign to the charge of one proton.
- An ion is an atom or molecule that has nonzero total charge due to having unequal numbers of electrons and protons.
- The SI unit for charge is the coulomb (C), with protons and electrons having charges of opposite sign but equal magnitude; the magnitude of this basic charge $|q_e| = 1.60 \times 10^{-19}$ C.
- Whenever charge is created or destroyed, equal amounts of positive and negative are involved.
- Most often, existing charges are separated from neutral objects to obtain some net charge.
- Both positive and negative charges exist in neutral objects and can be separated by rubbing one object with another. For macroscopic objects, negatively charged means an excess of electrons and positively charged means a depletion of electrons.
- The law of conservation of charge ensures that whenever a charge is created, an equal charge of the opposite sign is created at the same time.

12.2 Conductors and Insulators

- Polarization is the separation of positive and negative charges in a neutral object.
- A conductor is a substance that allows charge to flow freely through its atomic structure.
- An insulator holds charge within its atomic structure.
- Objects with like charges repel each other, while those with unlike charges attract each other.
- A conducting object is said to be grounded if it is connected to the Earth through a conductor. Grounding allows transfer of charge to and from the Earth’s large reservoir.
- Objects can be charged by contact with another charged object and obtain the same sign charge.
- If an object is temporarily grounded, it can be charged by induction, and obtains the opposite sign charge.
- Polarized objects have their positive and negative charges concentrated in different areas, giving them a non-symmetrical charge.
- Polar molecules have an inherent separation of charge.

12.3 Coulomb’s Law

- Frenchman Charles Coulomb was the first to publish the mathematical equation that describes the electrostatic force between two objects.
- Coulomb’s law gives the magnitude of the force between point charges. It is

$$F = \frac{kq_1q_2}{r^2},$$

where $q_1$ and $q_2$ are two point charges separated by a distance $r$, and $k \approx 8.99 \times 10^9$ N·m$^2$/C$^2$

- This Coulomb force is extremely basic, since most charges are due to point-like particles. It is responsible for all electrostatic effects and underlies most macroscopic forces.
- The Coulomb force is extraordinarily strong compared with the gravitational force, another basic force—but unlike gravitational force it can cancel, since it can be either attractive or repulsive.
- The electrostatic force between two subatomic particles is far greater than the gravitational force between the same two particles.

12.4 Electric Field: Concept of a Field Revisited

- The electrostatic force field surrounding a charged object extends out into space in all directions.
- The electrostatic force exerted by a point charge on a test charge at a distance $r$ depends on the charge of both charges, as well as the distance between the two.
- The electric field $E$ is defined to be

$$E = \frac{F}{q},$$

where $F$ is the Coulomb or electrostatic force exerted on a small positive test charge $q$. $E$ has units of N/C.
- The magnitude of the electric field $E$ created by a point charge $Q$ is

$$E = \frac{kQ}{r^2}.$$
where \( r \) is the distance from \( Q \). The electric field \( \mathbf{E} \) is a vector and fields due to multiple charges add like vectors.

### 12.5 Electric Field Lines: Multiple Charges

- Drawings of electric field lines are useful visual tools. The properties of electric field lines for any charge distribution are that:
  - Field lines must begin on positive charges and terminate on negative charges, or at infinity in the hypothetical case of isolated charges.
  - The number of field lines leaving a positive charge or entering a negative charge is proportional to the magnitude of the charge.
  - The strength of the field is proportional to the closeness of the field lines—more precisely, it is proportional to the number of lines per unit area perpendicular to the lines.
  - The direction of the electric field is tangent to the field line at any point in space.
  - Field lines can never cross.

### 12.6 Electric Forces in Biology

- Many molecules in living organisms, such as DNA, carry a charge.
- An uneven distribution of the positive and negative charges within a polar molecule produces a dipole.
- The effect of a Coulomb field generated by a charged object may be reduced or blocked by other nearby charged objects.
- Biological systems contain water, and because water molecules are polar, they have a strong effect on other molecules in living systems.

### 12.7 Conductors and Electric Fields in Static Equilibrium

- A conductor allows free charges to move about within it.
- The electrical forces around a conductor will cause free charges to move around inside the conductor until static equilibrium is reached.
- Any excess charge will collect along the surface of a conductor.
- Conductors with sharp corners or points will collect more charge at those points.
- A lightning rod is a conductor with sharply pointed ends that collect excess charge on the building caused by an electrical storm and allow it to dissipate back into the air.
- Electrical storms result when the electrical field of Earth’s surface in certain locations becomes more strongly charged, due to changes in the insulating effect of the air.
- A Faraday cage acts like a shield around an object, preventing electric charge from penetrating inside.

### 12.8 Applications of Electrostatics

- Electrostatics is the study of electric fields in static equilibrium.
- In addition to research using equipment such as a Van de Graaff generator, many practical applications of electrostatics exist, including photocopiers, laser printers, ink-jet printers and electrostatic air filters.

### Conceptual Questions

1. **Static Electricity and Charge: Conservation of Charge**
   1. There are very large numbers of charged particles in most objects. Why, then, don’t most objects exhibit static electricity?
   2. Why do most objects tend to contain nearly equal numbers of positive and negative charges?

2. **Conductors and Insulators**
   3. An eccentric inventor attempts to levitate by first placing a large negative charge on himself and then putting a large positive charge on the ceiling of his workshop. Instead, while attempting to place a large negative charge on himself, his clothes fly off. Explain.
   4. If you have charged an electroscope by contact with a positively charged object, describe how you could use it to determine the charge of other objects. Specifically, what would the leaves of the electroscope do if other charged objects were brought near its knob?
   5. When a glass rod is rubbed with silk, it becomes positive and the silk becomes negative—yet both attract dust. Does the dust have a third type of charge that is attracted to both positive and negative? Explain.
   6. Why does a car always attract dust right after it is polished? (Note that car wax and car tires are insulators.)
   7. Describe how a positively charged object can be used to give another object a negative charge. What is the name of this process?
   8. What is grounding? What effect does it have on a charged conductor? On a charged insulator?
12.3 Coulomb’s Law

9. Figure 12.43 shows the charge distribution in a water molecule, which is called a polar molecule because it has an inherent separation of charge. Given water’s polar character, explain what effect humidity has on removing excess charge from objects.

Figure 12.43 Schematic representation of the outer electron cloud of a neutral water molecule. The electrons spend more time near the oxygen than the hydrogens, giving a permanent charge separation as shown. Water is thus a polar molecule. It is more easily affected by electrostatic forces than molecules with uniform charge distributions.

10. Using Figure 12.43, explain, in terms of Coulomb’s law, why a polar molecule (such as in Figure 12.43) is attracted by both positive and negative charges.

11. Given the polar character of water molecules, explain how ions in the air form nucleation centers for rain droplets.

12.4 Electric Field: Concept of a Field Revisited

12. Why must the test charge \( q \) in the definition of the electric field be vanishingly small?

13. Are the direction and magnitude of the Coulomb force unique at a given point in space? What about the electric field?

12.5 Electric Field Lines: Multiple Charges

14. Compare and contrast the Coulomb force field and the electric field. To do this, make a list of five properties for the Coulomb force field analogous to the five properties listed for electric field lines. Compare each item in your list of Coulomb force field properties with those of the electric field—are they the same or different? (For example, electric field lines cannot cross. Is the same true for Coulomb field lines?)

15. Figure 12.44 shows an electric field extending over three regions, labeled I, II, and III. Answer the following questions. (a) Are there any isolated charges? If so, in what region and what are their signs? (b) Where is the field strongest? (c) Where is it weakest? (d) Where is the field the most uniform?

Figure 12.44
12.6 Electric Forces in Biology

16. A cell membrane is a thin layer enveloping a cell. The thickness of the membrane is much less than the size of the cell. In a static situation the membrane has a charge distribution of \(-2.5 \times 10^{-6} \, \text{C/m}^2\) on its inner surface and \(+2.5 \times 10^{-6} \, \text{C/m}^2\) on its outer surface. Draw a diagram of the cell and the surrounding cell membrane. Include on this diagram the charge distribution and the corresponding electric field. Is there any electric field inside the cell? Is there any electric field outside the cell?

12.7 Conductors and Electric Fields in Static Equilibrium

17. Is the object in Figure 12.45 a conductor or an insulator? Justify your answer.

Figure 12.45

18. If the electric field lines in the figure above were perpendicular to the object, would it necessarily be a conductor? Explain.

19. The discussion of the electric field between two parallel conducting plates, in this module states that edge effects are less important if the plates are close together. What does close mean? That is, is the actual plate separation crucial, or is the ratio of plate separation to plate area crucial?

20. Would the self-created electric field at the end of a pointed conductor, such as a lightning rod, remove positive or negative charge from the conductor? Would the same sign charge be removed from a neutral pointed conductor by the application of a similar externally created electric field? (The answers to both questions have implications for charge transfer utilizing points.)

21. Why is a golfer with a metal club over her shoulder vulnerable to lightning in an open fairway? Would she be any safer under a tree?

22. Can the belt of a Van de Graaff accelerator be a conductor? Explain.

23. Are you relatively safe from lightning inside an automobile? Give two reasons.

24. Discuss pros and cons of a lightning rod being grounded versus simply being attached to a building.

25. Using the symmetry of the arrangement, show that the net Coulomb force on the charge \(q\) at the center of the square below (Figure 12.46) is zero if the charges on the four corners are exactly equal.

Figure 12.46 Four point charges \(q_a\), \(q_b\), \(q_c\), and \(q_d\) lie on the corners of a square and \(q\) is located at its center.

26. (a) Using the symmetry of the arrangement, show that the electric field at the center of the square in Figure 12.46 is zero if the charges on the four corners are exactly equal. (b) Show that this is also true for any combination of charges in which \(q_a = q_d\) and \(q_b = q_c\).

27. (a) What is the direction of the total Coulomb force on \(q\) in Figure 12.46 if \(q\) is negative, \(q_a = q_c\) and both are negative, and \(q_b = q_d\) and both are positive? (b) What is the direction of the electric field at the center of the square in this situation?
28. Considering Figure 12.46, suppose that \( q_a = q_d \) and \( q_b = q_c \). First show that \( q \) is in static equilibrium. (You may neglect the gravitational force.) Then discuss whether the equilibrium is stable or unstable, noting that this may depend on the signs of the charges and the direction of displacement of \( q \) from the center of the square.

29. If \( q_a = 0 \) in Figure 12.46, under what conditions will there be no net Coulomb force on \( q \)?

30. In regions of low humidity, one develops a special “grip” when opening car doors, or touching metal door knobs. This involves placing as much of the hand on the device as possible, not just the ends of one’s fingers. Discuss the induced charge and explain why this is done.

31. Tollbooth stations on roadways and bridges usually have a piece of wire stuck in the pavement before them that will touch a car as it approaches. Why is this done?

32. Suppose a woman carries an excess charge. To maintain her charged status can she be standing on ground wearing just any pair of shoes? How would you discharge her? What are the consequences if she simply walks away?
Problems & Exercises

12.1 Static Electricity and Charge: Conservation of Charge
1. Common static electricity involves charges ranging from nanocoulombs to microcoulombs. (a) How many electrons are needed to form a charge of $-2.00 \mu C$? (b) How many electrons must be removed from a neutral object to leave a net charge of 0.500 $\mu C$?
2. If $1.80 \times 10^{20}$ electrons move through a pocket calculator during a full day’s operation, how many coulombs of charge moved through it?
3. To start a car engine, the car battery moves $3.75 \times 10^{21}$ electrons through the starter motor. How many coulombs of charge were moved?
4. A certain lightning bolt moves 40.0 C of charge. How many fundamental units of charge $|q_e|$ is this?

12.2 Conductors and Insulators
5. Suppose a speck of dust in an electrostatic precipitator has $1.00 \times 10^{12}$ protons in it and has a net charge of $-5.00 \mu C$ (a very large charge for a small speck). How many electrons does it have?
6. An amoeba has $1.00 \times 10^{16}$ protons and a net charge of 0.300 pC. (a) How many fewer electrons are there than protons? (b) If you paired them up, what fraction of the protons would have no electrons?
7. A 50.0 g ball of copper has a net charge of 2.00 $\mu C$. What fraction of the copper’s electrons has been removed? (Each copper atom has 29 protons, and copper has an atomic mass of 63.5.)
8. What net charge would you place on a 100 g piece of sulfur if you put an extra electron on 1 in $10^{12}$ of its atoms? (Sulfur has an atomic mass of 32.1.)
9. How many coulombs of positive charge are there in 4.00 kg of plutonium, given its atomic mass is 244 and that each plutonium atom has 94 protons?

12.3 Coulomb’s Law
10. What is the repulsive force between two pith balls that are 8.00 cm apart and have equal charges of $-30.0 \mu C$?
11. (a) How strong is the attractive force between a glass rod with a 0.700 $\mu C$ charge and a silk cloth with a $-0.600 \mu C$ charge, which are 12.0 cm apart, using the approximation that they act like point charges? (b) Discuss how the answer to this problem might be affected if the charges are distributed over some area and do not act like point charges.
12. Two point charges exert a 5.00 N force on each other. What will the force become if the distance between them is increased by a factor of three?
13. Two point charges are brought closer together, increasing the force between them by a factor of 25. By what factor was their separation decreased?
14. How far apart must two point charges of 75.0 nC (typical of static electricity) be to have a force of 1.00 N between them?
15. If two equal charges each of 1 C each are separated in air by a distance of 1 km, what is the magnitude of the force acting between them? You will see that even at a distance as large as 1 km, the repulsive force is substantial because 1 C is a very significant amount of charge.
16. A test charge of $+2 \mu C$ is placed halfway between a charge of $+6 \mu C$ and another of $+4 \mu C$ separated by 10 cm. (a) What is the magnitude of the force on the test charge? (b) What is the direction of this force (away from or toward the $+6 \mu C$ charge)?
17. Bare free charges do not remain stationary when close together. To illustrate this, calculate the acceleration of two isolated protons separated by 2.00 nm (a typical distance between gas atoms). Explicitly show how you follow the steps in the Problem-Solving Strategy for electrostatics.
18. (a) By what factor must you change the distance between two point charges to change the force between them by a factor of 10? (b) Explain how the distance can either increase or decrease by this factor and still cause a factor of 10 change in the force.
19. Suppose you have a total charge $q_{\text{tot}}$ that you can split in any manner. Once split, the separation distance is fixed. How do you split the charge to achieve the greatest force?
20. (a) Common transparent tape becomes charged when pulled from a dispenser. If one piece is placed above another, the repulsive force can be great enough to support the top piece’s weight. Assuming equal point charges (only an approximation), calculate the magnitude of the charge if electrostatic force is great enough to support the weight of a 10.0 mg piece of tape held 1.00 cm above another. (b) Discuss whether the magnitude of this charge is consistent with what is typical of static electricity.
21. (a) Find the ratio of the electrostatic to gravitational force between two electrons. (b) What is this ratio for two protons? (c) Why is the ratio different for electrons and protons?
22. At what distance is the electrostatic force between two protons equal to the weight of one proton?
23. A certain five cent coin contains 5.00 g of nickel. What fraction of the nickel atoms’ electrons, removed and placed 1.00 m above it, would support the weight of this coin? The atomic mass of nickel is 58.7, and each nickel atom contains 28 electrons and 28 protons.
24. (a) Two point charges totaling 8.00 $\mu C$ exert a repulsive force of 0.150 N on one another when separated by 0.500 m. What is the charge on each? (b) What is the charge on each if the force is attractive?
25. Point charges of $5.00 \mu C$ and $-3.00 \mu C$ are placed 0.250 m apart. (a) Where can a third charge be placed so that the net force on it is zero? (b) What if both charges are positive?
26. Two point charges \( q_1 \) and \( q_2 \) are 3.00 m apart, and their total charge is 20 \( \mu C \). (a) If the force of repulsion between them is 0.075N, what are magnitudes of the two charges? (b) If one charge attracts the other with a force of 0.525N, what are the magnitudes of the two charges? Note that you may need to solve a quadratic equation to reach your answer.

**12.4 Electric Field: Concept of a Field Revisited**

27. What is the magnitude and direction of an electric field that exerts a \( 2.00 \times 10^{-5} \) N upward force on a \(-1.75 \mu C\) charge?

28. What is the magnitude and direction of the force exerted on a \( 3.50 \mu C\) charge by a 250 N/C electric field that points due east?

29. Calculate the magnitude of the electric field 2.00 m from a point charge of 5.00 mC (such as found on the terminal of a Van de Graaff).

30. (a) What magnitude point charge creates a 10,000 N/C electric field at a distance of 0.250 m? (b) How large is the field at 10.0 m?

31. Calculate the initial (from rest) acceleration of a proton in a \( 5.00 \times 10^{6} \) N/C electric field (such as created by a research Van de Graaff). Explicitly show how you follow the steps in the Problem-Solving Strategy for electrostatics.

32. (a) Find the magnitude and direction of an electric field that exerts a \( 4.80 \times 10^{-17} \) N westward force on an electron. (b) What magnitude and direction force does this field exert on a proton?

**12.5 Electric Field Lines: Multiple Charges**

33. (a) Sketch the electric field lines near a point charge \(+q\). (b) Do the same for a point charge \(-3.00q\).

34. Sketch the electric field lines a long distance from the charge distributions shown in Figure 12.26 (a) and (b).

35. Figure 12.47 shows the electric field lines near two charges \( q_1 \) and \( q_2 \). What is the ratio of their magnitudes? (b) Sketch the electric field lines a long distance from the charges shown in the figure.

36. Sketch the electric field lines in the vicinity of two opposite charges, where the negative charge is three times greater in magnitude than the positive. (See Figure 12.47 for a similar situation).

**12.7 Conductors and Electric Fields in Static Equilibrium**

37. Sketch the electric field lines in the vicinity of the conductor in Figure 12.48 given the field was originally uniform and parallel to the object’s long axis. Is the resulting field small near the long side of the object?

38. Sketch the electric field lines in the vicinity of the conductor in Figure 12.49 given the field was originally uniform and parallel to the object’s long axis. Is the resulting field small near the long side of the object?
39. Sketch the electric field between the two conducting plates shown in Figure 12.50, given the top plate is positive and an equal amount of negative charge is on the bottom plate. Be certain to indicate the distribution of charge on the plates.

40. Sketch the electric field lines in the vicinity of the charged insulator in Figure 12.51 noting its nonuniform charge distribution.

41. What is the force on the charge located at \( x = 8.00 \, \text{cm} \) in Figure 12.52(a) given that \( q = 1.00 \, \mu \text{C} \)?

42. (a) Find the total electric field at \( x = 1.00 \, \text{cm} \) in Figure 12.52(b) given that \( q = 5.00 \, \text{nC} \). (b) Find the total electric field at \( x = 11.00 \, \text{cm} \) in Figure 12.52(b). (c) If the charges are allowed to move and eventually be brought to rest by friction, what will the final charge configuration be? (That is, will there be a single charge, double charge, etc., and what will its value(s) be?)

43. (a) Find the electric field at \( x = 5.00 \, \text{cm} \) in Figure 12.52(a), given that \( q = 1.00 \, \mu \text{C} \). (b) At what position between 3.00 and 8.00 cm is the total electric field the same as that for \(-2q\) alone? (c) Can the electric field be zero anywhere between 0.00 and 8.00 cm? (d) At very large positive or negative values of \( x \), the electric field approaches zero in both (a) and (b). In which does it most rapidly approach zero and why? (e) At what position to the right of 11.0 cm is the total electric field zero, other than at infinity? (Hint: A graphing calculator can yield considerable insight in this problem.)

44. (a) Find the total Coulomb force on a charge of 2.00 nC located at \( x = 4.00 \, \text{cm} \) in Figure 12.52(b), given that \( q = 1.00 \, \mu \text{C} \). (b) Find the \( x \)-position at which the electric field is zero in Figure 12.52(b).

45. Using the symmetry of the arrangement, determine the direction of the force on \( q \) in the figure below, given that \( q_a = q_b = +7.50 \, \mu \text{C} \) and \( q_c = q_d = -7.50 \, \mu \text{C} \). (b) Calculate the magnitude of the force on the charge \( q \), given that the square is 10.0 cm on a side and \( q = 2.00 \, \mu \text{C} \).

46. (a) Using the symmetry of the arrangement, determine the direction of the electric field at the center of the square in Figure 12.53, given that \( q_a = q_b = -1.00 \, \mu \text{C} \) and \( q_c = q_d = +1.00 \, \mu \text{C} \). (b) Calculate the magnitude of the electric field at the location of \( q \), given that the square is 10.0 cm on a side.

47. Find the electric field at the location of \( q_a \) in Figure 12.53 given that \( q_b = q_c = q_d = +2.00 \, \text{nC} \), \( q = -1.00 \, \text{nC} \), and the square is 20.0 cm on a side.

48. Find the total Coulomb force on the charge \( q \) in Figure 12.53, given that \( q = 1.00 \, \mu \text{C} \), \( q_a = 2.00 \, \mu \text{C} \), \( q_b = -3.00 \, \mu \text{C} \), \( q_c = -4.00 \, \mu \text{C} \), and \( q_d = +1.00 \, \mu \text{C} \). The square is 50.0 cm on a side.
49. (a) Find the electric field at the location of \( q_a \) in Figure 12.54, given that \( q_b = +10.00 \, \mu \text{C} \) and \( q_c = -5.00 \, \mu \text{C} \). (b) What is the force on \( q_a \), given that \( q_a = +1.50 \, \text{nC} \) ?

![Figure 12.54](image)

Figure 12.54 Point charges located at the corners of an equilateral triangle 25.0 cm on a side.

50. (a) Find the electric field at the center of the triangular configuration of charges in Figure 12.54, given that \( q_a = +2.50 \, \text{nC} \), \( q_b = -8.00 \, \text{nC} \), and \( q_c = +1.50 \, \text{nC} \). (b) Is there any combination of charges, other than \( q_a = q_b = q_c \), that will produce a zero strength electric field at the center of the triangular configuration?

12.8 Applications of Electrostatics

51. (a) What is the electric field 5.00 m from the center of the terminal of a Van de Graaff with a 3.00 mC charge, noting that the field is equivalent to that of a point charge at the center of the terminal? (b) At this distance, what force does the field exert on a 2.00 \( \mu \text{C} \) charge on the Van de Graaff's belt?

52. (a) What is the direction and magnitude of an electric field that supports the weight of a free electron near the surface of Earth? (b) Discuss what the small value for this field implies regarding the relative strength of the gravitational and electrostatic forces.

53. A simple and common technique for accelerating electrons is shown in Figure 12.55, where there is a uniform electric field between two plates. Electrons are released, usually from a hot filament, near the negative plate, and there is a small hole in the positive plate that allows the electrons to continue moving. (a) Calculate the acceleration of the electron if the field strength is \( 2.50 \times 10^4 \, \text{N/C} \). (b) Explain why the electron will not be pulled back to the positive plate once it moves through the hole.

![Figure 12.55](image)

Figure 12.55 Parallel conducting plates with opposite charges on them create a relatively uniform electric field used to accelerate electrons to the right. Those that go through the hole can be used to make a TV or computer screen glow or to produce X-rays.

54. Earth has a net charge that produces an electric field of approximately 150 N/C downward at its surface. (a) What is the magnitude and sign of the excess charge, noting the electric field of a conducting sphere is equivalent to a point charge at its center? (b) What acceleration will the field produce on a free electron near Earth's surface? (c) What mass object with a single extra electron will have its weight supported by this field?

55. Point charges of 25.0 \( \mu \text{C} \) and 45.0 \( \mu \text{C} \) are placed 0.500 m apart. (a) At what point along the line between them is the electric field zero? (b) What is the electric field halfway between them?

56. What can you say about two charges \( q_1 \) and \( q_2 \), if the electric field one-fourth of the way from \( q_1 \) to \( q_2 \) is zero?

57. Integrated Concepts

Calculate the angular velocity \( \omega \) of an electron orbiting a proton in the hydrogen atom, given the radius of the orbit is \( 0.530 \times 10^{-10} \, \text{m} \). You may assume that the proton is stationary and the centripetal force is supplied by Coulomb attraction.
58. Integrated Concepts

An electron has an initial velocity of $5.00 \times 10^6 \text{ m/s}$ in a uniform $2.00 \times 10^5 \text{ N/C}$ strength electric field. The field accelerates the electron in the direction opposite to its initial velocity. (a) What is the direction of the electric field? (b) How far does the electron travel before coming to rest? (c) How long does it take the electron to come to rest? (d) What is the electron's velocity when it returns to its starting point?

59. Integrated Concepts

The practical limit to an electric field in air is about $3.00 \times 10^6 \text{ N/C}$. Above this strength, sparking takes place because air begins to ionize and charges flow, reducing the field. (a) Calculate the distance a free proton must travel in this field to reach 3.00% of the speed of light, starting from rest. (b) Is this practical in air, or must it occur in a vacuum?

60. Integrated Concepts

A 5.00 g charged insulating ball hangs on a 30.0 cm long string in a uniform horizontal electric field as shown in Figure 12.56. Given the charge on the ball is $1.00 \mu\text{C}$, find the strength of the field.

61. Integrated Concepts

Figure 12.57 shows an electron passing between two charged metal plates that create an 100 N/C vertical electric field perpendicular to the electron’s original horizontal velocity. (These can be used to change the electron's direction, such as in an oscilloscope.) The initial speed of the electron is $3.00 \times 10^6 \text{ m/s}$, and the horizontal distance it travels in the uniform field is 4.00 cm. (a) What is its vertical deflection? (b) What is the vertical component of its final velocity? (c) At what angle does it exit? Neglect any edge effects.

62. Integrated Concepts

The classic Millikan oil drop experiment was the first to obtain an accurate measurement of the charge on an electron. In it, oil drops were suspended against the gravitational force by a vertical electric field. (See Figure 12.58.) Given the oil drop to be $1.00 \mu\text{m}$ in radius and have a density of $920 \text{ kg/m}^3$:

(a) Find the weight of the drop. (b) If the drop has a single excess electron, find the electric field strength needed to balance its weight.

Figure 12.58 In the Millikan oil drop experiment, small drops can be suspended in an electric field by the force exerted on a single excess electron. Classically, this experiment was used to determine the electron charge $q_e$ by measuring the electric field and mass of the drop.
63. Integrated Concepts
(a) In Figure 12.59, four equal charges \( q \) lie on the corners of a square. A fifth charge \( Q \) is on a mass \( m \) directly above the center of the square, at a height equal to the length \( d \) of one side of the square. Determine the magnitude of \( q \) in terms of \( Q, m, \) and \( d \), if the Coulomb force is to equal the weight of \( m \). (b) Is this equilibrium stable or unstable? Discuss.

64. Unreasonable Results
(a) Calculate the electric field strength near a 10.0 cm diameter conducting sphere that has 1.00 C of excess charge on it. (b) What is unreasonable about this result? (c) Which assumptions are responsible?

65. Unreasonable Results
(a) Two 0.500 g raindrops in a thunderhead are 1.00 cm apart when they each acquire 1.00 mC charges. Find their acceleration. (b) What is unreasonable about this result? (c) Which premise or assumption is responsible?

66. Unreasonable Results
A wrecking yard inventor wants to pick up cars by charging a 0.400 m diameter ball and inducing an equal and opposite charge on the car. If a car has a 1000 kg mass and the ball is to be able to lift it from a distance of 1.00 m: (a) What minimum charge must be used? (b) What is the electric field near the surface of the ball? (c) Why are these results unreasonable? (d) Which premise or assumption is responsible?

67. Construct Your Own Problem
Consider two insulating balls with evenly distributed equal and opposite charges on their surfaces, held with a certain distance between the centers of the balls. Construct a problem in which you calculate the electric field (magnitude and direction) due to the balls at various points along a line running through the centers of the balls and extending to infinity on either side. Choose interesting points and comment on the meaning of the field at those points. For example, at what points might the field be just that due to one ball and where does the field become negligibly small? Among the things to be considered are the magnitudes of the charges and the distance between the centers of the balls. Your instructor may wish for you to consider the electric field off axis or for a more complex array of charges, such as those in a water molecule.

68. Construct Your Own Problem
Consider identical spherical conducting space ships in deep space where gravitational fields from other bodies are negligible compared to the gravitational attraction between the ships. Construct a problem in which you place identical excess charges on the space ships to exactly counter their gravitational attraction. Calculate the amount of excess charge needed. Examine whether that charge depends on the distance between the centers of the ships, the masses of the ships, or any other factors. Discuss whether this would be an easy, difficult, or even impossible thing to do in practice.
Chapter 13 | Electric Potential and Electric Field

13. ELECTRIC POTENTIAL AND ELECTRIC FIELD

Chapter Outline

- Define electric potential and electric potential energy.
- Describe the relationship between potential difference and electrical potential energy.
- Explain electron volt and its usage in submicroscopic process.
- Determine electric potential energy given potential difference and amount of charge.

13.2. Electric Potential in a Uniform Electric Field
- Describe the relationship between voltage and electric field.
- Derive an expression for the electric potential and electric field.
- Calculate electric field strength given distance and voltage.

13.3. Electrical Potential Due to a Point Charge
- Explain point charges and express the equation for electric potential of a point charge.
- Distinguish between electric potential and electric field.
- Determine the electric potential of a point charge given charge and distance.

13.4. Equipotential Lines
- Explain equipotential lines and equipotential surfaces.
- Describe the action of grounding an electrical appliance.
- Compare electric field and equipotential lines.

13.5. Capacitors and Dielectrics
- Describe the action of a capacitor and define capacitance.
- Explain parallel plate capacitors and their capacitances.
- Discuss the process of increasing the capacitance of a dielectric.
- Determine capacitance given charge and voltage.

13.6. Capacitors in Series and Parallel
- Derive expressions for total capacitance in series and in parallel.
- Identify series and parallel parts in the combination of connection of capacitors.
- Calculate the effective capacitance in series and parallel given individual capacitances.

13.7. Energy Stored in Capacitors
- List some uses of capacitors.
Introduction to Electric Potential and Electric Energy

In Electric Charge and Electric Field, we just scratched the surface (or at least rubbed it) of electrical phenomena. Two of the most familiar aspects of electricity are its energy and voltage. We know, for example, that great amounts of electrical energy can be stored in batteries, are transmitted cross-country through power lines, and may jump from clouds to explode the sap of trees. In a similar manner, at molecular levels, ions cross cell membranes and transfer information. We also know about voltages associated with electricity. Batteries are typically a few volts, the outlets in your home produce 120 volts, and power lines can be as high as hundreds of thousands of volts. But energy and voltage are not the same thing. A motorcycle battery, for example, is small and would not be very successful in replacing the much larger car battery, yet each has the same voltage. In this chapter, we shall examine the relationship between voltage and electrical energy and begin to explore some of the many applications of electricity.


When a free positive charge \( q \) is accelerated by an electric field, such as shown in Figure 13.2, it is given kinetic energy. The process is analogous to an object being accelerated by a gravitational field. It is as if the charge is going down an electrical hill where its electric potential energy is converted to kinetic energy. Let us explore the work done on a charge \( q \) by the electric field in this process, so that we may develop a definition of electric potential energy.

Figure 13.2 A charge accelerated by an electric field is analogous to a mass going down a hill. In both cases potential energy is converted to another form. Work is done by a force, but since this force is conservative, we can write \( W = -\Delta PE \).

The electrostatic or Coulomb force is conservative, which means that the work done on \( q \) is independent of the path taken. This is exactly analogous to the gravitational force in the absence of dissipative forces such as friction. When a force is conservative, it is possible to define a potential energy associated with the force, and it is usually easier to deal with the potential energy (because it depends only on position) than to calculate the work directly.

We use the letters PE to denote electric potential energy, which has units of joules (J). The change in potential energy, \( \Delta PE \), is crucial, since the work done by a conservative force is the negative of the change in potential energy; that is, \( W = -\Delta PE \). For example, work \( W \) done to accelerate a positive charge from rest is positive and results from a loss in PE, or a negative \( \Delta PE \). There must be a minus sign in front of \( \Delta PE \) to make \( W \) positive. PE can be found at any point by taking one point as a reference and calculating the work needed to move a charge to the other point.

Potential Energy

\[ W = -\Delta PE \]  For example, work \( W \) done to accelerate a positive charge from rest is positive and results from a loss in PE, or a negative \( \Delta PE \). There must be a minus sign in front of \( \Delta PE \) to make \( W \) positive. PE can be found at any point by taking one point as a reference and calculating the work needed to move a charge to the other point.

Gravitational potential energy and electric potential energy are quite analogous. Potential energy accounts for work done by a conservative force and gives added insight regarding energy and energy transformation without the necessity of dealing with the force directly. It is much more common, for example, to use the concept of voltage (related to electric potential energy) than to deal with the Coulomb force directly.
Calculating the work directly is generally difficult, since $W = Fd \cos \theta$ and the direction and magnitude of $F$ can be complex for multiple charges, for odd-shaped objects, and along arbitrary paths. But we do know that, since $F = qE$, the work, and hence $\Delta PE$, is proportional to the test charge $q$. To have a physical quantity that is independent of test charge, we define electric potential $V$ (or simply potential, since electric is understood) to be the potential energy per unit charge:

$$V = \frac{PE}{q}.$$  \hspace{1cm} (13.1)

### Electric Potential

This is the electric potential energy per unit charge.

$$V = \frac{PE}{q}.$$

(13.2)

Since $PE$ is proportional to $q$, the dependence on $q$ cancels. Thus $V$ does not depend on $q$. The change in potential energy $\Delta PE$ is crucial, and so we are concerned with the difference in potential or potential difference $\Delta V$ between two points, where

$$\Delta V = V_B - V_A = \frac{\Delta PE}{q}.$$  \hspace{1cm} (13.3)

The potential difference between points A and B, $V_B - V_A$, is thus defined to be the change in potential energy of a charge $q$ moved from A to B, divided by the charge. Units of potential difference are joules per coulomb, given the name volt (V) after Alessandro Volta.

$$1 \text{ V} = 1 \text{ J/C}.$$  \hspace{1cm} (13.4)

### Potential Difference

The potential difference between points A and B, $V_B - V_A$, is defined to be the change in potential energy of a charge $q$ moved from A to B, divided by the charge. Units of potential difference are joules per coulomb, given the name volt (V) after Alessandro Volta.

$$1 \text{ V} = 1 \text{ J/C}.$$  \hspace{1cm} (13.5)

The familiar term voltage is the common name for potential difference. Keep in mind that whenever a voltage is quoted, it is understood to be the potential difference between two points. For example, every battery has two terminals, and its voltage is the potential difference between them. More fundamentally, the point you choose to be zero volts is arbitrary. This is analogous to the fact that gravitational potential energy has an arbitrary zero, such as sea level or perhaps a lecture hall floor.

In summary, the relationship between potential difference (or voltage) and electrical potential energy is given by

$$\Delta V = \frac{\Delta PE}{q} \text{ and } \Delta PE = q\Delta V.$$  \hspace{1cm} (13.6)

### Potential Difference and Electrical Potential Energy

The relationship between potential difference (or voltage) and electrical potential energy is given by

$$\Delta V = \frac{\Delta PE}{q} \text{ and } \Delta PE = q\Delta V.$$  \hspace{1cm} (13.7)

The second equation is equivalent to the first.

Voltage is not the same as energy. Voltage is the energy per unit charge. Thus a motorcycle battery and a car battery can both have the same voltage (more precisely, the same potential difference between battery terminals), yet one stores much more energy than the other since $\Delta PE = q\Delta V$. The car battery can move more charge than the motorcycle battery, although both are 12 V batteries.

### Example 13.1 Calculating Energy

Suppose you have a 12.0 V motorcycle battery that can move 5000 C of charge, and a 12.0 V car battery that can move 60,000 C of charge. How much energy does each deliver? (Assume that the numerical value of each charge is accurate to three significant figures.)

**Strategy**

To say we have a 12.0 V battery means that its terminals have a 12.0 V potential difference. When such a battery moves
charge, it puts the charge through a potential difference of 12.0 V, and the charge is given a change in potential energy equal to \( \Delta PE = q \Delta V \).

So to find the energy output, we multiply the charge moved by the potential difference.

**Solution**

For the motorcycle battery, \( q = 5000 \text{ C} \) and \( \Delta V = 12.0 \text{ V} \). The total energy delivered by the motorcycle battery is

\[
\Delta PE_{\text{cycle}} = (5000 \text{ C})(12.0 \text{ V}) = 6.00 \times 10^4 \text{ J}.
\]

Similarly, for the car battery, \( q = 60,000 \text{ C} \) and

\[
\Delta PE_{\text{car}} = (60,000 \text{ C})(12.0 \text{ V}) = 7.20 \times 10^5 \text{ J}.
\]

**Discussion**

While voltage and energy are related, they are not the same thing. The voltages of the batteries are identical, but the energy supplied by each is quite different. Note also that as a battery is discharged, some of its energy is used internally and its terminal voltage drops, such as when headlights dim because of a low car battery. The energy supplied by the battery is still calculated as in this example, but not all of the energy is available for external use.

Note that the energies calculated in the previous example are absolute values. The change in potential energy for the battery is negative, since it loses energy. These batteries, like many electrical systems, actually move negative charge—electrons in particular. The batteries repel electrons from their negative terminals (A) through whatever circuitry is involved and attract them to their positive terminals (B) as shown in Figure 13.3. The change in potential is \( \Delta V = V_B - V_A = +12 \text{ V} \) and the charge \( q \) is negative, so that \( \Delta PE = q \Delta V \) is negative, meaning the potential energy of the battery has decreased when \( q \) has moved from A to B.

**Example 13.2 How Many Electrons Move through a Headlight Each Second?**

When a 12.0 V car battery runs a single 30.0 W headlight, how many electrons pass through it each second?

**Strategy**

To find the number of electrons, we must first find the charge that moved in 1.00 s. The charge moved is related to voltage and energy through the equation \( \Delta PE = q \Delta V \). A 30.0 W lamp uses 30.0 joules per second. Since the battery loses energy, we have \( \Delta PE = -30.0 \text{ J} \) and, since the electrons are going from the negative terminal to the positive, we see that \( \Delta V = +12.0 \text{ V} \).

**Solution**

To find the charge \( q \) moved, we solve the equation \( \Delta PE = q \Delta V \):
Entering the values for $\Delta PE$ and $\Delta V$, we get

$$q = \frac{\Delta PE}{\Delta V} = \frac{-30.0 \text{ J}}{+12.0 \text{ V}} = -2.50 \text{ C}.$$  

(13.11)

The number of electrons $n_e$ is the total charge divided by the charge per electron. That is,

$$n_e = \frac{-2.50 \text{ C}}{-1.60 \times 10^{-19} \text{ C/e}^-} = 1.56 \times 10^{19} \text{ electrons}.$$  

(13.12)

**Discussion**

This is a very large number. It is no wonder that we do not ordinarily observe individual electrons with so many being present in ordinary systems. In fact, electricity had been in use for many decades before it was determined that the moving charges in many circumstances were negative. Positive charge moving in the opposite direction of negative charge often produces identical effects; this makes it difficult to determine which is moving or whether both are moving.

**The Electron Volt**

The energy per electron is very small in macroscopic situations like that in the previous example—a tiny fraction of a joule. But on a submicroscopic scale, such energy per particle (electron, proton, or ion) can be of great importance. For example, even a tiny fraction of a joule can be great enough for these particles to destroy organic molecules and harm living tissue. The particle may do its damage by direct collision, or it may create harmful x rays, which can also inflict damage. It is useful to have an energy unit related to submicroscopic effects. Figure 13.4 shows a situation related to the definition of such an energy unit. An electron is accelerated between two charged metal plates as it might be in an old-model television tube or oscilloscope. The electron is given kinetic energy that is later converted to another form—light in the television tube, for example. (Note that downhill for the electron is uphill for a positive charge.) Since energy is related to voltage by $\Delta PE = q\Delta V$, we can think of the joule as a coulomb-volt.

**Figure 13.4**

A typical electron gun accelerates electrons using a potential difference between two metal plates. The energy of the electron in electron volts is numerically the same as the voltage between the plates. For example, a 5000 V potential difference produces 5000 eV electrons.

On the submicroscopic scale, it is more convenient to define an energy unit called the **electron volt** (eV), which is the energy given to a fundamental charge accelerated through a potential difference of 1 V. In equation form,

$$1 \text{ eV} = (1.60 \times 10^{-19} \text{ C})(1 \text{ V}) = (1.60 \times 10^{-19} \text{ C})(1 \text{ J/C}) = 1.60 \times 10^{-19} \text{ J}.$$  

(13.13)

**Electron Volt**

On the submicroscopic scale, it is more convenient to define an energy unit called the electron volt (eV), which is the energy given to a fundamental charge accelerated through a potential difference of 1 V. In equation form,
\[ 1 \text{ eV} = \left( 1.60 \times 10^{-19} \text{ C} \right) (1 \text{ V}) = \left( 1.60 \times 10^{-19} \text{ C} \right) (1 \text{ J/C}) \]

An electron accelerated through a potential difference of 1 V is given an energy of 1 eV. It follows that an electron accelerated through 50 V is given 50 eV. A potential difference of 100,000 V (100 kV) will give an electron an energy of 100,000 eV (100 keV), and so on. Similarly, an ion with a double positive charge accelerated through 100 V will be given 200 eV of energy. These simple relationships between accelerating voltage and particle charges make the electron volt a simple and convenient energy unit in such circumstances.

**Connections: Energy Units**

The electron volt (eV) is the most common energy unit for submicroscopic processes. This will be particularly noticeable in the chapters on modern physics. Energy is so important to so many subjects that there is a tendency to define a special energy unit for each major topic. There are, for example, calories for food energy, kilowatt-hours for electrical energy, and therms for natural gas energy.

The electron volt is commonly employed in submicroscopic processes—chemical valence energies and molecular and nuclear binding energies are among the quantities often expressed in electron volts. For example, about 5 eV of energy is required to break up certain organic molecules. If a proton is accelerated from rest through a potential difference of 30 kV, it is given an energy of 30 keV (30,000 eV) and it can break up as many as 6000 of these molecules (30,000 eV ÷ 5 eV per molecule = 6000 molecules). Nuclear decay energies are on the order of 1 MeV (1,000,000 eV) per event and can, thus, produce significant biological damage.

**Conservation of Energy**

The total energy of a system is conserved if there is no net addition (or subtraction) of work or heat transfer. For conservative forces, such as the electrostatic force, conservation of energy states that mechanical energy is a constant.

**Mechanical energy** is the sum of the kinetic energy and potential energy of a system; that is, \( KE + PE = \text{constant} \). A loss of PE of a charged particle becomes an increase in its KE. Here PE is the electric potential energy. Conservation of energy is stated in equation form as

\[ KE + PE = \text{constant} \tag{13.15} \]

or

\[ KE_i + PE_i = KE_f + PE_f \tag{13.16} \]

where \( i \) and \( f \) stand for initial and final conditions. As we have found many times before, considering energy can give us insights and facilitate problem solving.

**Example 13.3 Electrical Potential Energy Converted to Kinetic Energy**

Calculate the final speed of a free electron accelerated from rest through a potential difference of 100 V. (Assume that this numerical value is accurate to three significant figures.)

**Strategy**

We have a system with only conservative forces. Assuming the electron is accelerated in a vacuum, and neglecting the gravitational force (we will check on this assumption later), all of the electrical potential energy is converted into kinetic energy. We can identify the initial and final forms of energy to be \( KE_i = 0 \), \( KE_f = \frac{1}{2} m v^2 \), \( PE_i = qV \), and \( PE_f = 0 \).

**Solution**

Conservation of energy states that

\[ KE_i + PE_i = KE_f + PE_f. \tag{13.17} \]

Entering the forms identified above, we obtain

\[ qV = \frac{mv^2}{2}. \tag{13.18} \]

We solve this for \( v \):

\[ v = \sqrt{\frac{2qV}{m}}. \tag{13.19} \]

Entering values for \( q \), \( V \), and \( m \) gives
\[ v = \sqrt{\frac{2(-1.60 \times 10^{-19} \text{ C})(-100 \text{ J/C})}{9.11 \times 10^{-31} \text{ kg}}} \]
\[ = 5.93 \times 10^6 \text{ m/s}. \]  

**Discussion**

Note that both the charge and the initial voltage are negative, as in Figure 13.4. From the discussions in *Electric Charge and Electric Field*, we know that electrostatic forces on small particles are generally very large compared with the gravitational force. The large final speed confirms that the gravitational force is indeed negligible here. The large speed also indicates how easy it is to accelerate electrons with small voltages because of their very small mass. Voltages much higher than the 100 V in this problem are typically used in electron guns. Those higher voltages produce electron speeds so great that relativistic effects must be taken into account. That is why a low voltage is considered (accurately) in this example.

**13.2 Electric Potential in a Uniform Electric Field**

In the previous section, we explored the relationship between voltage and energy. In this section, we will explore the relationship between voltage and electric field. For example, a uniform electric field \( E \) is produced by placing a potential difference (or voltage) \( \Delta V \) across two parallel metal plates, labeled A and B. (See Figure 13.5.) Examining this will tell us what voltage is needed to produce a certain electric field strength; it will also reveal a more fundamental relationship between electric potential and electric field. From a physicist’s point of view, either \( \Delta V \) or \( E \) can be used to describe any charge distribution. \( \Delta V \) is most closely tied to energy, whereas \( E \) is most closely related to force. \( \Delta V \) is a scalar quantity and has no direction, while \( E \) is a vector quantity, having both magnitude and direction. (Note that the magnitude of the electric field strength, a scalar quantity, is represented by \( E \) below.) The relationship between \( \Delta V \) and \( E \) is revealed by calculating the work done by the force in moving a charge from point A to point B. But, as noted in *Electric Potential Energy: Potential Difference*, this is complex for arbitrary charge distributions, requiring calculus. We therefore look at a uniform electric field as an interesting special case.

**Figure 13.5** The relationship between \( V \) and \( E \) for parallel conducting plates is \( E = V / d \). (Note that \( \Delta V = V_{AB} \) in magnitude. For a charge that is moved from plate A at higher potential to plate B at lower potential, a minus sign needs to be included as follows:
\[ -\Delta V = V_A - V_B = V_{AB} \]. See the text for details.)

The work done by the electric field in Figure 13.5 to move a positive charge \( q \) from A, the positive plate, higher potential, to B, the negative plate, lower potential, is

\[ W = -\Delta PE = - q \Delta V. \]  

The potential difference between points A and B is

\[ -\Delta V = -(V_B - V_A) = V_A - V_B = V_{AB}. \]  

Entering this into the expression for work yields
\[ W = qV_{AB}. \]  

Work is \( W = Fd \cos \theta \); here \( \cos \theta = 1 \), since the path is parallel to the field, and so \( W = Fd \). Since \( F = qE \), we see that \( W = qEd \). Substituting this expression for work into the previous equation gives

\[ qEd = qV_{AB}. \]  

The charge cancels, and so the voltage between points A and B is seen to be

\[ V_{AB} = Ed \]  

\[ E = \frac{V_{AB}}{d} \]  

(13.25)

where \( d \) is the distance from A to B, or the distance between the plates in Figure 13.5. Note that the above equation implies the units for electric field are volts per meter. We already know the units for electric field are newtons per coulomb; thus the following relation among units is valid:

\[ 1 \text{ N/C} = 1 \text{ V/m}. \]  

(13.26)

**Voltage between Points A and B**

\[ V_{AB} = Ed \]  

\[ E = \frac{V_{AB}}{d} \]  

(13.27)

where \( d \) is the distance from A to B, or the distance between the plates.

---

**Example 13.4 What Is the Highest Voltage Possible between Two Plates?**

Dry air will support a maximum electric field strength of about \( 3.0 \times 10^6 \text{ V/m} \). Above that value, the field creates enough ionization in the air to make the air a conductor. This allows a discharge or spark that reduces the field. What, then, is the maximum voltage between two parallel conducting plates separated by 2.5 cm of dry air?

**Strategy**

We are given the maximum electric field \( E \) between the plates and the distance \( d \) between them. The equation \( V_{AB} = Ed \) can thus be used to calculate the maximum voltage.

**Solution**

The potential difference or voltage between the plates is

\[ V_{AB} = Ed. \]  

(13.28)

Entering the given values for \( E \) and \( d \) gives

\[ V_{AB} = (3.0 \times 10^6 \text{ V/m})(0.025 \text{ m}) = 7.5 \times 10^4 \text{ V} \]  

(13.29)

or

\[ V_{AB} = 75 \text{ kV}. \]  

(13.30)

(The answer is quoted to only two digits, since the maximum field strength is approximate.)

**Discussion**

One of the implications of this result is that it takes about 75 kV to make a spark jump across a 2.5 cm (1 in.) gap, or 150 kV for a 5 cm spark. This limits the voltages that can exist between conductors, perhaps on a power transmission line. A smaller voltage will cause a spark if there are points on the surface, since points create greater fields than smooth surfaces. Humid air breaks down at a lower field strength, meaning that a smaller voltage will make a spark jump through humid air. The largest voltages can be built up, say with static electricity, on dry days.
Figure 13.6 A spark chamber is used to trace the paths of high-energy particles. Ionization created by the particles as they pass through the gas between the plates allows a spark to jump. The sparks are perpendicular to the plates, following electric field lines between them. The potential difference between adjacent plates is not high enough to cause sparks without the ionization produced by particles from accelerator experiments (or cosmic rays). (credit: Daderot, Wikimedia Commons)

Example 13.5 Field and Force inside an Electron Gun

(a) An electron gun has parallel plates separated by 4.00 cm and gives electrons 25.0 keV of energy. What is the electric field strength between the plates? (b) What force would this field exert on a piece of plastic with a 0.500 μC charge that gets between the plates?

Strategy
Since the voltage and plate separation are given, the electric field strength can be calculated directly from the expression \( E = \frac{V_{AB}}{d} \). Once the electric field strength is known, the force on a charge is found using \( F = qE \). Since the electric field is in only one direction, we can write this equation in terms of the magnitudes, \( F = qE \).

Solution for (a)
The expression for the magnitude of the electric field between two uniform metal plates is

\[
E = \frac{V_{AB}}{d}
\]  

(13.31)

Since the electron is a single charge and is given 25.0 keV of energy, the potential difference must be 25.0 kV. Entering this value for \( V_{AB} \) and the plate separation of 0.0400 m, we obtain

\[
E = \frac{25.0 \text{ kV}}{0.0400 \text{ m}} = 6.25 \times 10^5 \text{ V/m}.
\]

(13.32)

Solution for (b)
The magnitude of the force on a charge in an electric field is obtained from the equation

\[
F = qE.
\]

(13.33)

Substituting known values gives

\[
F = (0.500 \times 10^{-6} \text{ C})(6.25 \times 10^5 \text{ V/m}) = 0.313 \text{ N}.
\]

(13.34)

Discussion
Note that the units are newtons, since 1 V/m = 1 N/C. The force on the charge is the same no matter where the charge is located between the plates. This is because the electric field is uniform between the plates.

In more general situations, regardless of whether the electric field is uniform, it points in the direction of decreasing potential, because the force on a positive charge is in the direction of \( \mathbf{E} \) and also in the direction of lower potential \( V \). Furthermore, the magnitude of \( \mathbf{E} \) equals the rate of decrease of \( V \) with distance. The faster \( V \) decreases over distance, the greater the electric field. In equation form, the general relationship between voltage and electric field is
where $\Delta s$ is the distance over which the change in potential, $\Delta V$, takes place. The minus sign tells us that E points in the direction of decreasing potential. The electric field is said to be the gradient (as in grade or slope) of the electric potential.

### Relationship between Voltage and Electric Field

In equation form, the general relationship between voltage and electric field is

$$E = -\frac{\Delta V}{\Delta s},$$

(13.36)

where $\Delta s$ is the distance over which the change in potential, $\Delta V$, takes place. The minus sign tells us that E points in the direction of decreasing potential. The electric field is said to be the gradient (as in grade or slope) of the electric potential.

For continually changing potentials, $\Delta V$ and $\Delta s$ become infinitesimals and differential calculus must be employed to determine the electric field.

### 13.3 Electrical Potential Due to a Point Charge

Point charges, such as electrons, are among the fundamental building blocks of matter. Furthermore, spherical charge distributions (like on a metal sphere) create external electric fields exactly like a point charge. The electric potential due to a point charge is, thus, a case we need to consider. Using calculus to find the work needed to move a test charge $q$ from a large distance away to a distance of $r$ from a point charge $Q$, and noting the connection between work and potential $W = -q\Delta V$, it can be shown that the electric potential $V$ of a point charge is

$$V = \frac{kQ}{r},$$

(13.37)

where $k$ is a constant equal to $9.0 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$.

### Electric Potential $V$ of a Point Charge

The electric potential $V$ of a point charge is given by

$$V = \frac{kQ}{r},$$

(13.38)

The potential at infinity is chosen to be zero. Thus $V$ for a point charge decreases with distance, whereas $E$ for a point charge decreases with distance squared:

$$E = \frac{F}{q} = \frac{kQ}{r^2}.$$  

(13.39)

Recall that the electric potential $V$ is a scalar and has no direction, whereas the electric field $E$ is a vector. To find the voltage due to a combination of point charges, you add the individual voltages as numbers. To find the total electric field, you must add the individual fields as vectors, taking magnitude and direction into account. This is consistent with the fact that $V$ is closely associated with energy, a scalar, whereas $E$ is closely associated with force, a vector.

### Example 13.6 What Voltage Is Produced by a Small Charge on a Metal Sphere?

Charges in static electricity are typically in the nanocoulomb ($\text{nC}$) to microcoulomb ($\mu\text{C}$) range. What is the voltage 5.00 cm away from the center of a 1-cm diameter metal sphere that has a $-3.00 \text{ nC}$ static charge?

**Strategy**

As we have discussed in Electric Charge and Electric Field, charge on a metal sphere spreads out uniformly and produces a field like that of a point charge located at its center. Thus we can find the voltage using the equation $V = kQ/r$.

**Solution**

Entering known values into the expression for the potential of a point charge, we obtain
\[ V = \frac{kQ}{r} \]

\[ = \left(8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2\right)\left(-3.00 \times 10^{-9} \text{ C}\right) \]

\[ = -539 \text{ V}. \]

**Discussion**

The negative value for voltage means a positive charge would be attracted from a larger distance, since the potential is lower (more negative) than at larger distances. Conversely, a negative charge would be repelled, as expected.

**Example 13.7 What Is the Excess Charge on a Van de Graaff Generator**

A demonstration Van de Graaff generator has a 25.0 cm diameter metal sphere that produces a voltage of 100 kV near its surface. (See Figure 13.7.) What excess charge resides on the sphere? (Assume that each numerical value here is shown with three significant figures.)

**Strategy**

The potential on the surface will be the same as that of a point charge at the center of the sphere, 12.5 cm away. (The radius of the sphere is 12.5 cm.) We can thus determine the excess charge using the equation

\[ V = \frac{kQ}{r}. \]

**Solution**

Solving for \( Q \) and entering known values gives

\[ Q = \frac{rV}{k} \]

\[ = \frac{(0.125 \text{ m})(100 \times 10^3 \text{ V})}{8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2} \]

\[ = 1.39 \times 10^{-6} \text{ C} = 1.39 \mu\text{C}. \]

**Discussion**

This is a relatively small charge, but it produces a rather large voltage. We have another indication here that it is difficult to store isolated charges.

The voltages in both of these examples could be measured with a meter that compares the measured potential with ground potential. Ground potential is often taken to be zero (instead of taking the potential at infinity to be zero). It is the potential difference between two points that is of importance, and very often there is a tacit assumption that some reference point, such as
Earth or a very distant point, is at zero potential. As noted in Electric Potential Energy: Potential Difference, this is analogous to taking sea level as \( h = 0 \) when considering gravitational potential energy, \( PE_g = mgh \).

### 13.4 Equipotential Lines

We can represent electric potentials (voltages) pictorially, just as we drew pictures to illustrate electric fields. Of course, the two are related. Consider Figure 13.8, which shows an isolated positive point charge and its electric field lines. Electric field lines radiate out from a positive charge and terminate on negative charges. While we use blue arrows to represent the magnitude and direction of the electric field, we use green lines to represent places where the electric potential is constant. These are called **equipotential lines** in two dimensions, or **equipotential surfaces** in three dimensions. The term **equipotential** is also used as a noun, referring to an equipotential line or surface. The potential for a point charge is the same anywhere on an imaginary sphere of radius \( r \) surrounding the charge. This is true since the potential for a point charge is given by \( V = kQ/r \) and, thus, has the same value at any point that is a given distance \( r \) from the charge. An equipotential sphere is a circle in the two-dimensional view of Figure 13.8. Since the electric field lines point radially away from the charge, they are perpendicular to the equipotential lines.

**Figure 13.8** An isolated point charge \( Q \) with its electric field lines in blue and equipotential lines in green. The potential is the same along each equipotential line, meaning that no work is required to move a charge anywhere along one of those lines. Work is needed to move a charge from one equipotential line to another. Equipotential lines are perpendicular to electric field lines in every case.

It is important to note that **equipotential lines are always perpendicular to electric field lines**. No work is required to move a charge along an equipotential, since \( \Delta V = 0 \). Thus the work is

\[
W = -\Delta PE = -q\Delta V = 0.
\]

![Equation (13.43)](https://legacy.cnx.org/content/col11951/1.1)

Work is zero if force is perpendicular to motion. Force is in the same direction as \( E \), so that motion along an equipotential must be perpendicular to \( E \). More precisely, work is related to the electric field by

\[
W = Fd \cos \theta = qEd \cos \theta = 0.
\]

![Equation (13.44)](https://legacy.cnx.org/content/col11951/1.1)

Note that in the above equation, \( E \) and \( F \) symbolize the magnitudes of the electric field strength and force, respectively. Neither \( q \) nor \( E \), nor \( d \) is zero, and so \( \cos \theta \) must be 0, meaning \( \theta \) must be 90°. In other words, motion along an equipotential is perpendicular to \( E \).

One of the rules for static electric fields and conductors is that the electric field must be perpendicular to the surface of any conductor. This implies that a conductor is an equipotential surface in static situations. There can be no voltage difference across the surface of a conductor, or charges will flow. One of the uses of this fact is that a conductor can be fixed at zero volts by connecting it to the earth with a good conductor—a process called **grounding**. Grounding can be a useful safety tool. For example, grounding the metal case of an electrical appliance ensures that it is at zero volts relative to the earth.

**Grounding**

A conductor can be fixed at zero volts by connecting it to the earth with a good conductor—a process called grounding.

Because a conductor is an equipotential, it can replace any equipotential surface. For example, in Figure 13.8 a charged spherical conductor can replace the point charge, and the electric field and potential surfaces outside of it will be unchanged, confirming the contention that a spherical charge distribution is equivalent to a point charge at its center.

**Figure 13.9** shows the electric field and equipotential lines for two equal and opposite charges. Given the electric field lines, the equipotential lines can be drawn simply by making them perpendicular to the electric field lines. Conversely, given the equipotential lines, as in **Figure 13.10**(a), the electric field lines can be drawn by making them perpendicular to the equipotentials, as in **Figure 13.10**(b).
Figure 13.9 The electric field lines and equipotential lines for two equal but opposite charges. The equipotential lines can be drawn by making them perpendicular to the electric field lines, if those are known. Note that the potential is greatest (most positive) near the positive charge and least (most negative) near the negative charge.

Figure 13.10 (a) These equipotential lines might be measured with a voltmeter in a laboratory experiment. (b) The corresponding electric field lines are found by drawing them perpendicular to the equipotentials. Note that these fields are consistent with two equal negative charges.

One of the most important cases is that of the familiar parallel conducting plates shown in Figure 13.11. Between the plates, the equipotentials are evenly spaced and parallel. The same field could be maintained by placing conducting plates at the equipotential lines at the potentials shown.

Figure 13.11 The electric field and equipotential lines between two metal plates.

An important application of electric fields and equipotential lines involves the heart. The heart relies on electrical signals to maintain its rhythm. The movement of electrical signals causes the chambers of the heart to contract and relax. When a person has a heart attack, the movement of these electrical signals may be disturbed. An artificial pacemaker and a defibrillator can be used to initiate the rhythm of electrical signals. The equipotential lines around the heart, the thoracic region, and the axis of the heart are useful ways of monitoring the structure and functions of the heart. An electrocardiogram (ECG) measures the small electric signals being generated during the activity of the heart. More about the relationship between electric fields and the heart is discussed in Energy Stored in Capacitors.
Charges and Fields

Move point charges around on the playing field and then view the electric field, voltages, equipotential lines, and more. It's colorful, it's dynamic, it's free.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42331/1.13/#fs-id1167066831384)

Figure 13.12

13.5 Capacitors and Dielectrics

A capacitor is a device used to store electric charge. Capacitors have applications ranging from filtering static out of radio reception to energy storage in heart defibrillators. Typically, commercial capacitors have two conducting parts close to one another, but not touching, such as those in Figure 13.13. (Most of the time an insulator is used between the two plates to provide separation—see the discussion on dielectrics below.) When battery terminals are connected to an initially uncharged capacitor, equal amounts of positive and negative charge, \( +Q \) and \( -Q \), are separated into its two plates. The capacitor remains neutral overall, but we refer to it as storing a charge \( Q \) in this circumstance.

**Capacitor**

A capacitor is a device used to store electric charge.

![Diagram of a capacitor](https://legacy.cnx.org/content/m42331/1.13/#fs-id1167066831384)

**Figure 13.13** Both capacitors shown here were initially uncharged before being connected to a battery. They now have separated charges of \( +Q \) and \( -Q \) on their two halves. (a) A parallel plate capacitor. (b) A rolled capacitor with an insulating material between its two conducting sheets.

The amount of charge \( Q \) a capacitor can store depends on two major factors—the voltage applied and the capacitor’s physical characteristics, such as its size.

**The Amount of Charge \( Q \) a Capacitor Can Store**

The amount of charge \( Q \) a capacitor can store depends on two major factors—the voltage applied and the capacitor’s physical characteristics, such as its size.

A system composed of two identical, parallel conducting plates separated by a distance, as in Figure 13.14, is called a parallel plate capacitor. It is easy to see the relationship between the voltage and the stored charge for a parallel plate capacitor, as
shown in Figure 13.14. Each electric field line starts on an individual positive charge and ends on a negative one, so that there will be more field lines if there is more charge. (Drawing a single field line per charge is a convenience, only. We can draw many field lines for each charge, but the total number is proportional to the number of charges.) The electric field strength is, thus, directly proportional to $Q$.

\[ E \propto Q, \quad (13.45) \]

where the symbol $\propto$ means "proportional to." From the discussion in Electric Potential in a Uniform Electric Field, we know that the voltage across parallel plates is $V = Ed$. Thus,

\[ V \propto E. \quad (13.46) \]

It follows, then, that $V \propto Q$, and conversely,

\[ Q \propto V. \quad (13.47) \]

This is true in general: The greater the voltage applied to any capacitor, the greater the charge stored in it.

Different capacitors will store different amounts of charge for the same applied voltage, depending on their physical characteristics. We define their capacitance $C$ to be such that the charge $Q$ stored in a capacitor is proportional to $C$. The charge stored in a capacitor is given by

\[ Q = CV. \quad (13.48) \]

This equation expresses the two major factors affecting the amount of charge stored. Those factors are the physical characteristics of the capacitor, $C$, and the voltage, $V$. Rearranging the equation, we see that capacitance $C$ is the amount of charge stored per volt, or

\[ C = \frac{Q}{V}. \quad (13.49) \]

**Capacitance**

Capacitance $C$ is the amount of charge stored per volt, or
The unit of capacitance is the farad (F), named for Michael Faraday (1791–1867), an English scientist who contributed to the fields of electromagnetism and electrochemistry. Since capacitance is charge per unit voltage, we see that a farad is a coulomb per volt, or

$$1 \text{ F} = \frac{1 \text{ C}}{1 \text{ V}}.$$  \hspace{1cm} (13.51)

A 1-farad capacitor would be able to store 1 coulomb (a very large amount of charge) with the application of only 1 volt. One farad is, thus, a very large capacitance. Typical capacitors range from fractions of a picofarad \(\left(1 \text{ pF} = 10^{-12} \text{ F}\right)\) to millifarads \(\left(1 \text{ mF} = 10^{-3} \text{ F}\right)\).

*Figure 13.15* shows some common capacitors. Capacitors are primarily made of ceramic, glass, or plastic, depending upon purpose and size. Insulating materials, called dielectrics, are commonly used in their construction, as discussed below.

**Parallel Plate Capacitor**

The parallel plate capacitor shown in *Figure 13.16* has two identical conducting plates, each having a surface area \(A\), separated by a distance \(d\) (with no material between the plates). When a voltage \(V\) is applied to the capacitor, it stores a charge \(Q\), as shown. We can see how its capacitance depends on \(A\) and \(d\) by considering the characteristics of the Coulomb force. We know that like charges repel, unlike charges attract, and the force between charges decreases with distance. So it seems quite reasonable that the bigger the plates are, the more charge they can store—because the charges can spread out more. Thus \(C\) should be greater for larger \(A\). Similarly, the closer the plates are together, the greater the attraction of the opposite charges on them. So \(C\) should be greater for smaller \(d\).

*Figure 13.16* Parallel plate capacitor with plates separated by a distance \(d\). Each plate has an area \(A\).

It can be shown that for a parallel plate capacitor there are only two factors \((A\) and \(d\)) that affect its capacitance \(C\). The

$$C = \frac{Q}{V}.$$  \hspace{1cm} (13.50)
Capacitance of a parallel plate capacitor in equation form is given by

\[ C = \varepsilon_0 \frac{A}{d} \]  

(13.52)

**Capacitance of a Parallel Plate Capacitor**

\[ C = \varepsilon_0 \frac{A}{d} \]  

(13.53)

\( A \) is the area of one plate in square meters, and \( d \) is the distance between the plates in meters. The constant \( \varepsilon_0 \) is the permittivity of free space; its numerical value in SI units is \( \varepsilon_0 = 8.85 \times 10^{-12} \) F/m. The units of F/m are equivalent to C\(^2\)/N\cdot m\(^2\). The small numerical value of \( \varepsilon_0 \) is related to the large size of the farad. A parallel plate capacitor must have a large area to have a capacitance approaching a farad. (Note that the above equation is valid when the parallel plates are separated by air or free space. When another material is placed between the plates, the equation is modified, as discussed below.)

### Example 13.8 Capacitance and Charge Stored in a Parallel Plate Capacitor

**(a)** What is the capacitance of a parallel plate capacitor with metal plates, each of area 1.00 m\(^2\), separated by 1.00 mm?

**(b)** What charge is stored in this capacitor if a voltage of 3.00\( \times \)10\(^3\) V is applied to it?

**Strategy**

Finding the capacitance \( C \) is a straightforward application of the equation \( C = \varepsilon_0 \frac{A}{d} \). Once \( C \) is found, the charge stored can be found using the equation \( Q = CV \).

**Solution for (a)**

Entering the given values into the equation for the capacitance of a parallel plate capacitor yields

\[ C = \varepsilon_0 \frac{A}{d} = \left(8.85 \times 10^{-12} \text{ F/m}\right) \frac{1.00 \text{ m}^2}{1.00 \times 10^{-3} \text{ m}} \]

\[ = 8.85 \times 10^{-9} \text{ F} = 8.85 \text{ nF}. \]

**Discussion for (a)**

This small value for the capacitance indicates how difficult it is to make a device with a large capacitance. Special techniques help, such as using very large area thin foils placed close together.

**Solution for (b)**

The charge stored in any capacitor is given by the equation \( Q = CV \). Entering the known values into this equation gives

\[ Q = CV = \left(8.85 \times 10^{-9} \text{ F}\right) \left(3.00 \times 10^3 \text{ V}\right) \]

\[ = 26.6 \mu\text{C}. \]

**Discussion for (b)**

This charge is only slightly greater than those found in typical static electricity. Since air breaks down at about 3.00\( \times \)10\(^9\) V/m, more charge cannot be stored on this capacitor by increasing the voltage.

Another interesting biological example dealing with electric potential is found in the cell’s plasma membrane. The membrane sets a cell off from its surroundings and also allows ions to selectively pass in and out of the cell. There is a potential difference across the membrane of about –70 mV. This is due to the mainly negatively charged ions in the cell and the predominance of positively charged sodium (Na\(^+\)) ions outside. Things change when a nerve cell is stimulated. Na\(^+\) ions are allowed to pass through the membrane into the cell, producing a positive membrane potential—the nerve signal. The cell membrane is about 7 to 10 nm thick. An approximate value of the electric field across it is given by

\[ E = \frac{V}{d} = \frac{-70 \times 10^{-3} \text{ V}}{8 \times 10^{-9} \text{ m}} = -9 \times 10^6 \text{ V/m}. \]

This electric field is enough to cause a breakdown in air.

**Dielectric**

The previous example highlights the difficulty of storing a large amount of charge in capacitors. If \( d \) is made smaller to produce
a larger capacitance, then the maximum voltage must be reduced proportionally to avoid breakdown (since \( E = \frac{V}{d} \)). An important solution to this difficulty is to put an insulating material, called a **dielectric**, between the plates of a capacitor and allow \( d \) to be as small as possible. Not only does the smaller \( d \) make the capacitance greater, but many insulators can withstand greater electric fields than air before breaking down.

There is another benefit to using a dielectric in a capacitor. Depending on the material used, the capacitance is greater than that given by the equation \( C = \epsilon_0 \frac{A}{d} \) by a factor \( \kappa \), called the **dielectric constant**. A parallel plate capacitor with a dielectric between its plates has a capacitance given by

\[
C = \kappa \epsilon_0 \frac{A}{d} \quad \text{(parallel plate capacitor with dielectric)}.
\]

Values of the dielectric constant \( \kappa \) for various materials are given in Table 13.1. Note that \( \kappa \) for vacuum is exactly 1, and so the above equation is valid in that case, too. If a dielectric is used, perhaps by placing Teflon between the plates of the capacitor in Example 13.8, then the capacitance is greater by the factor \( \kappa \), which for Teflon is 2.1.

**Take-Home Experiment: Building a Capacitor**

How large a capacitor can you make using a chewing gum wrapper? The plates will be the aluminum foil, and the separation (dielectric) in between will be the paper.

**Table 13.1 Dielectric Constants and Dielectric Strengths for Various Materials at 20°C**

<table>
<thead>
<tr>
<th>Material</th>
<th>Dielectric constant ( \kappa )</th>
<th>Dielectric strength (V/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum</td>
<td>1.00000</td>
<td>—</td>
</tr>
<tr>
<td>Air</td>
<td>1.00059</td>
<td>( 3 \times 10^6 )</td>
</tr>
<tr>
<td>Bakelite</td>
<td>4.9</td>
<td>( 24 \times 10^6 )</td>
</tr>
<tr>
<td>Fused quartz</td>
<td>3.78</td>
<td>( 8 \times 10^6 )</td>
</tr>
<tr>
<td>Neoprene rubber</td>
<td>6.7</td>
<td>( 12 \times 10^6 )</td>
</tr>
<tr>
<td>Nylon</td>
<td>3.4</td>
<td>( 14 \times 10^6 )</td>
</tr>
<tr>
<td>Paper</td>
<td>3.7</td>
<td>( 16 \times 10^6 )</td>
</tr>
<tr>
<td>Polystyrene</td>
<td>2.56</td>
<td>( 24 \times 10^6 )</td>
</tr>
<tr>
<td>Pyrex glass</td>
<td>5.6</td>
<td>( 14 \times 10^6 )</td>
</tr>
<tr>
<td>Silicon oil</td>
<td>2.5</td>
<td>( 15 \times 10^6 )</td>
</tr>
<tr>
<td>Strontium titanate</td>
<td>233</td>
<td>( 8 \times 10^6 )</td>
</tr>
<tr>
<td>Teflon</td>
<td>2.1</td>
<td>( 60 \times 10^6 )</td>
</tr>
<tr>
<td>Water</td>
<td>80</td>
<td>—</td>
</tr>
</tbody>
</table>

Note also that the dielectric constant for air is very close to 1, so that air-filled capacitors act much like those with vacuum between their plates except that the air can become conductive if the electric field strength becomes too great. (Recall that \( E = \frac{V}{d} \) for a parallel plate capacitor.) Also shown in Table 13.1 are maximum electric field strengths in V/m, called **dielectric strengths**, for several materials. These are the fields above which the material begins to break down and conduct. The dielectric strength imposes a limit on the voltage that can be applied for a given plate separation. For instance, in Example 13.8, the separation is 1.00 mm, and so the voltage limit for air is

\[
V = E \cdot d = (3 \times 10^6 \ \text{V/m})(1.00 \times 10^{-3} \text{ m}) = 3000 \text{ V}.
\]

However, the limit for a 1.00 mm separation filled with Teflon is 60,000 V, since the dielectric strength of Teflon is \( 60 \times 10^6 \) V/m. So the same capacitor filled with Teflon has a greater capacitance and can be subjected to a much greater voltage. Using the
capacitance we calculated in the above example for the air-filled parallel plate capacitor, we find that the Teflon-filled capacitor can store a maximum charge of

\[
Q = CV = \kappa C_{\text{air}} V = (2.1)(8.85 \text{ nF})(6.0 \times 10^4 \text{ V}) = 1.1 \text{ mC}.
\]

This is 42 times the charge of the same air-filled capacitor.

**Dielectric Strength**

The maximum electric field strength above which an insulating material begins to break down and conduct is called its dielectric strength.

Microscopically, how does a dielectric increase capacitance? Polarization of the insulator is responsible. The more easily it is polarized, the greater its dielectric constant \( \kappa \). Water, for example, is a polar molecule because one end of the molecule has a slight positive charge and the other end has a slight negative charge. The polarity of water causes it to have a relatively large dielectric constant of 80. The effect of polarization can be best explained in terms of the characteristics of the Coulomb force.

**Figure 13.17** shows the separation of charge schematically in the molecules of a dielectric material placed between the charged plates of a capacitor. The Coulomb force between the closest ends of the molecules and the charge on the plates is attractive and very strong, since they are very close together. This attracts more charge onto the plates than if the space were empty and the opposite charges were a distance \( d \) away.

Another way to understand how a dielectric increases capacitance is to consider its effect on the electric field inside the capacitor. **Figure 13.17**(b) shows the electric field lines with a dielectric in place. Since the field lines end on charges in the dielectric, there are fewer of them going from one side of the capacitor to the other. So the electric field strength is less than if there were a vacuum between the plates, even though the same charge is on the plates. The voltage between the plates is \( V = Ed \), so it too is reduced by the dielectric. Thus there is a smaller voltage \( V \) for the same charge \( Q \); since \( C = Q/V \), the capacitance \( C \) is greater.

The dielectric constant is generally defined to be \( \kappa = E_0/E \), or the ratio of the electric field in a vacuum to that in the dielectric.
material, and is intimately related to the polarizability of the material.

Things Great and Small

The Submicroscopic Origin of Polarization

Polarization is a separation of charge within an atom or molecule. As has been noted, the planetary model of the atom pictures it as having a positive nucleus orbited by negative electrons, analogous to the planets orbiting the Sun. Although this model is not completely accurate, it is very helpful in explaining a vast range of phenomena and will be refined elsewhere, such as in Atomic Physics (https://legacy.cnx.org/content/m42585/latest). The submicroscopic origin of polarization can be modeled as shown in Figure 13.18.

![Figure 13.18](https://legacy.cnx.org/content/m42585/latest/)

We will find in Atomic Physics (https://legacy.cnx.org/content/m42585/latest) that the orbits of electrons are more properly viewed as electron clouds with the density of the cloud related to the probability of finding an electron in that location (as opposed to the definite locations and paths of planets in their orbits around the Sun). This cloud is shifted by the Coulomb force so that the atom on average has a separation of charge. Although the atom remains neutral, it can now be the source of a Coulomb force, since a charge brought near the atom will be closer to one type of charge than the other.

Some molecules, such as those of water, have an inherent separation of charge and are thus called polar molecules. Figure 13.19 illustrates the separation of charge in a water molecule, which has two hydrogen atoms and one oxygen atom (H₂O).

![Figure 13.19](https://legacy.cnx.org/content/m42585/latest/)

The water molecule is not symmetric—the hydrogen atoms are repelled to one side, giving the molecule a boomerang shape. The electrons in a water molecule are more concentrated around the more highly charged oxygen nucleus than around the hydrogen nuclei. This makes the oxygen end of the molecule slightly negative and leaves the hydrogen ends slightly positive. The inherent separation of charge in polar molecules makes it easier to align them with external fields and charges. Polar molecules therefore exhibit greater polarization effects and have greater dielectric constants. Those who study chemistry will find that the polar nature of water has many effects. For example, water molecules gather ions much more effectively because they have an electric field and a separation of charge to attract charges of both signs. Also, as brought out in the previous chapter, polar water provides a shield or screening of the electric fields in the highly charged molecules of interest in biological systems.

Capacitor Lab

Explore how a capacitor works! Change the size of the plates and add a dielectric to see the effect on capacitance. Change the voltage and see charges built up on the plates. Observe the electric field in the capacitor. Measure the voltage and the electric field. (This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/)
### 13.6 Capacitors in Series and Parallel

Several capacitors may be connected together in a variety of applications. Multiple connections of capacitors act like a single equivalent capacitor. The total capacitance of this equivalent single capacitor depends both on the individual capacitors and how they are connected. There are two simple and common types of connections, called *series* and *parallel*, for which we can easily calculate the total capacitance. Certain more complicated connections can also be related to combinations of series and parallel.

#### Capacitance in Series

![Series Capacitors](image)

**Figure 13.20** (a) shows a series connection of three capacitors with a voltage applied. As for any capacitor, the capacitance of the combination is related to charge and voltage by \( C = \frac{Q}{V} \).

Note in **Figure 13.20** that opposite charges of magnitude \( Q \) flow to either side of the originally uncharged combination of capacitors when the voltage \( V \) is applied. Conservation of charge requires that equal-magnitude charges be created on the plates of the individual capacitors, since charge is only being separated in these originally neutral devices. The end result is that the combination resembles a single capacitor with an effective plate separation greater than that of the individual capacitors alone. (See **Figure 13.20** (b).) Larger plate separation means smaller capacitance. It is a general feature of series connections of capacitors that the total capacitance is less than any of the individual capacitances.

We can find an expression for the total capacitance by considering the voltage across the individual capacitors shown in **Figure 13.20**. Solving \( C = \frac{Q}{V} \) for \( V \) gives \( V = \frac{Q}{C} \). The voltages across the individual capacitors are thus \( V_1 = \frac{Q}{C_1} \), \( V_2 = \frac{Q}{C_2} \), and \( V_3 = \frac{Q}{C_3} \). The total voltage is the sum of the individual voltages:
\[ V = V_1 + V_2 + V_3. \]  
(13.60)

Now, calling the total capacitance \( C_S \) for series capacitance, consider that

\[ V = \frac{Q}{C_S} = V_1 + V_2 + V_3. \]  
(13.61)

Entering the expressions for \( V_1 \), \( V_2 \), and \( V_3 \), we get

\[ \frac{Q}{C_S} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}. \]  
(13.62)

Canceling the \( Q \) s, we obtain the equation for the total capacitance in series \( C_S \) to be

\[ \frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots, \]  
(13.63)

where “...” indicates that the expression is valid for any number of capacitors connected in series. An expression of this form always results in a total capacitance \( C_S \) that is less than any of the individual capacitances \( C_1, C_2, \ldots \), as the next example illustrates.

### Total Capacitance in Series, \( C_S \)

Total capacitance in series:

\[ \frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \]

### Example 13.9 What Is the Series Capacitance?

Find the total capacitance for three capacitors connected in series, given their individual capacitances are 1.000, 5.000, and 8.000 µF.

**Strategy**

With the given information, the total capacitance can be found using the equation for capacitance in series.

**Solution**

Entering the given capacitances into the expression for \( \frac{1}{C_S} \) gives

\[ \frac{1}{C_S} = \frac{1}{1.000 \, \mu F} + \frac{1}{5.000 \, \mu F} + \frac{1}{8.000 \, \mu F} = \frac{1.325}{\mu F} \]  
(13.64)

Inverting to find \( C_S \) yields

\[ C_S = \frac{1.325}{\mu F} = 0.755 \, \mu F. \]

**Discussion**

The total series capacitance \( C_S \) is less than the smallest individual capacitance, as promised. In series connections of capacitors, the sum is less than the parts. In fact, it is less than any individual. Note that it is sometimes possible, and more convenient, to solve an equation like the above by finding the least common denominator, which in this case (showing only whole-number calculations) is 40. Thus,

\[ \frac{1}{C_S} = \frac{40}{40 \, \mu F} + \frac{8}{40 \, \mu F} + \frac{5}{40 \, \mu F} = \frac{53}{40 \, \mu F}, \]  
(13.65)

so that

\[ C_S = \frac{40 \, \mu F}{53} = 0.755 \, \mu F. \]  
(13.66)

### Capacitors in Parallel

Figure 13.21(a) shows a parallel connection of three capacitors with a voltage applied. Here the total capacitance is easier to find than in the series case. To find the equivalent total capacitance \( C_P \), we first note that the voltage across each capacitor is \( V \), the same as that of the source, since they are connected directly to it through a conductor. (Conductors are equipotentials, and so the voltage across the capacitors is the same as that across the voltage source.) Thus the capacitors have the same
charges on them as they would have if connected individually to the voltage source. The total charge \( Q \) is the sum of the individual charges:

\[
Q = Q_1 + Q_2 + Q_3.
\]  

(13.67)

Figure 13.21 (a) Capacitors in parallel. Each is connected directly to the voltage source just as if it were all alone, and so the total capacitance in parallel is just the sum of the individual capacitances. (b) The equivalent capacitor has a larger plate area and can therefore hold more charge than the individual capacitors.

Using the relationship \( Q = CV \), we see that the total charge is \( Q = C_p V \), and the individual charges are \( Q_1 = C_1 V \), \( Q_2 = C_2 V \), and \( Q_3 = C_3 V \). Entering these into the previous equation gives

\[
C_p V = C_1 V + C_2 V + C_3 V.
\]

(13.68)

Canceling \( V \) from the equation, we obtain the equation for the total capacitance in parallel \( C_p \):

\[
C_p = C_1 + C_2 + C_3 + ....
\]

(13.69)

Total capacitance in parallel is simply the sum of the individual capacitances. (Again the “...” indicates the expression is valid for any number of capacitors connected in parallel.) So, for example, if the capacitors in the example above were connected in parallel, their capacitance would be

\[
C_p = 1.000 \, \mu F + 5.000 \, \mu F + 8.000 \, \mu F = 14.000 \, \mu F.
\]

(13.70)

The equivalent capacitor for a parallel connection has an effectively larger plate area and, thus, a larger capacitance, as illustrated in Figure 13.21(b).

Total Capacitance in Parallel, \( C_p \)

Total capacitance in parallel \( C_p = C_1 + C_2 + C_3 + ... \)

More complicated connections of capacitors can sometimes be combinations of series and parallel. (See Figure 13.22.) To find the total capacitance of such combinations, we identify series and parallel parts, compute their capacitances, and then find the total.
Figure 13.22 (a) This circuit contains both series and parallel connections of capacitors. See Example 13.10 for the calculation of the overall capacitance of the circuit. (b) \( C_1 \) and \( C_2 \) are in series; their equivalent capacitance \( C_S \) is less than either of them. (c) Note that \( C_S \) is in parallel with \( C_3 \). The total capacitance is, thus, the sum of \( C_S \) and \( C_3 \).

**Example 13.10 A Mixture of Series and Parallel Capacitance**

Find the total capacitance of the combination of capacitors shown in Figure 13.22. Assume the capacitances in Figure 13.22 are known to three decimal places (\( C_1 = 1.000 \mu F \), \( C_2 = 5.000 \mu F \), and \( C_3 = 8.000 \mu F \)), and round your answer to three decimal places.

**Strategy**

To find the total capacitance, we first identify which capacitors are in series and which are in parallel. Capacitors \( C_1 \) and \( C_2 \) are in series. Their combination, labeled \( C_S \) in the figure, is in parallel with \( C_3 \).

**Solution**

Since \( C_1 \) and \( C_2 \) are in series, their total capacitance is given by

\[
\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{1.000 \mu F} + \frac{1}{5.000 \mu F} = \frac{1.200}{\mu F}.
\]

Inverting gives

\[
C_S = 0.833 \mu F.
\]

This equivalent series capacitance is in parallel with the third capacitor; thus, the total is the sum

\[
C_{\text{tot}} = C_S + C_3 = 0.833 \mu F + 8.000 \mu F = 8.833 \mu F.
\]

**Discussion**

This technique of analyzing the combinations of capacitors piece by piece until a total is obtained can be applied to larger combinations of capacitors.

**13.7 Energy Stored in Capacitors**

Most of us have seen dramatizations in which medical personnel use a defibrillator to pass an electric current through a patient’s heart to get it to beat normally. (Review Figure 13.23.) Often realistic in detail, the person applying the shock directs another person to “make it 400 joules this time.” The energy delivered by the defibrillator is stored in a capacitor and can be adjusted to fit the situation. SI units of joules are often employed. Less dramatic is the use of capacitors in microelectronics, such as certain handheld calculators, to supply energy when batteries are charged. (See Figure 13.23.) Capacitors are also used to supply energy for flash lamps on cameras.
Energy stored in a capacitor is electrical potential energy, and it is thus related to the charge $Q$ and voltage $V$ on the capacitor. We must be careful when applying the equation for electrical potential energy $\Delta \text{PE} = q \Delta V$ to a capacitor. Remember that $\Delta \text{PE}$ is the potential energy of a charge $q$ going through a voltage $\Delta V$. But the capacitor starts with zero voltage and gradually comes up to its full voltage as it is charged. The first charge placed on a capacitor experiences a change in voltage $\Delta V = 0$, since the capacitor has zero voltage when uncharged. The final charge placed on a capacitor experiences $\Delta V = V$, since the capacitor now has its full voltage $V$ on it. The average voltage on the capacitor during the charging process is $V/2$, and so the average voltage experienced by the full charge $q$ is $V/2$. Thus the energy stored in a capacitor, $E_{\text{cap}}$, is

$$E_{\text{cap}} = \frac{QV}{2},$$

where $Q$ is the charge on a capacitor with a voltage $V$ applied. (Note that the energy is not $QV$, but $QV/2$.) Charge and voltage are related to the capacitance $C$ of a capacitor by $Q = CV$, and so the expression for $E_{\text{cap}}$ can be algebraically manipulated into three equivalent expressions:

$$E_{\text{cap}} = \frac{QV}{2} = \frac{CV^2}{2} = \frac{Q^2}{2C},$$

where $Q$ is the charge and $V$ the voltage on a capacitor $C$. The energy is in joules for a charge in coulombs, voltage in volts, and capacitance in farads. 

**Energy Stored in Capacitors**

The energy stored in a capacitor can be expressed in three ways:

$$E_{\text{cap}} = \frac{QV}{2} = \frac{CV^2}{2} = \frac{Q^2}{2C},$$

where $Q$ is the charge, $V$ is the voltage, and $C$ is the capacitance of the capacitor. The energy is in joules for a charge in coulombs, voltage in volts, and capacitance in farads.

In a defibrillator, the delivery of a large charge in a short burst to a set of paddles across a person’s chest can be a lifesaver. The person’s heart attack might have arisen from the onset of fast, irregular beating of the heart—cardiac or ventricular fibrillation. The application of a large shock of electrical energy can terminate the arrhythmia and allow the body’s pacemaker to resume normal patterns. Today it is common for ambulances to carry a defibrillator, which also uses an electrocardiogram to analyze the patient’s heartbeat pattern. Automated external defibrillators (AED) are found in many public places (Figure 13.24). These are designed to be used by lay persons. The device automatically diagnoses the patient’s heart condition and then applies the shock with appropriate energy and waveform. CPR is recommended in many cases before use of an AED.
Figure 13.24 Automated external defibrillators are found in many public places. These portable units provide verbal instructions for use in the important first few minutes for a person suffering a cardiac attack. (credit: Owain Davies, Wikimedia Commons)

Example 13.11 Capacitance in a Heart Defibrillator

A heart defibrillator delivers \(4.00 \times 10^2\) J of energy by discharging a capacitor initially at \(1.00 \times 10^4\) V. What is its capacitance?

**Strategy**

We are given \(E_{\text{cap}}\) and \(V\), and we are asked to find the capacitance \(C\). Of the three expressions in the equation for \(E_{\text{cap}}\), the most convenient relationship is

\[E_{\text{cap}} = CV^2/2.\]  

**Solution**

Solving this expression for \(C\) and entering the given values yields

\[C = \frac{2E_{\text{cap}}}{V^2} = \frac{2(4.00 \times 10^2 \text{ J})}{(1.00 \times 10^4 \text{ V})^2} = 8.00 \times 10^{-6} \text{ F}\]

\[= 8.00 \mu\text{F}.\]

**Discussion**

This is a fairly large, but manageable, capacitance at \(1.00 \times 10^4\) V.

**Glossary**

- **capacitance**: amount of charge stored per unit volt
- **capacitor**: a device that stores electric charge
- **defibrillator**: a machine used to provide an electrical shock to a heart attack victim's heart in order to restore the heart's normal rhythmic pattern
- **dielectric**: an insulating material
- **dielectric strength**: the maximum electric field above which an insulating material begins to break down and conduct
- **electric potential**: potential energy per unit charge
- **electron volt**: the energy given to a fundamental charge accelerated through a potential difference of one volt
**equipotential line**: a line along which the electric potential is constant

**grounding**: fixing a conductor at zero volts by connecting it to the earth or ground

**mechanical energy**: sum of the kinetic energy and potential energy of a system; this sum is a constant

**parallel plate capacitor**: two identical conducting plates separated by a distance

**polar molecule**: a molecule with inherent separation of charge

**potential difference (or voltage)**: change in potential energy of a charge moved from one point to another, divided by the charge; units of potential difference are joules per coulomb, known as volt

**scalar**: physical quantity with magnitude but no direction

**vector**: physical quantity with both magnitude and direction

**Section Summary**

**13.1 Electric Potential Energy: Potential Difference**

- Electric potential is potential energy per unit charge.
- The potential difference between points A and B, \( V_B - V_A \), defined to be the change in potential energy of a charge \( q \) moved from A to B, is equal to the change in potential energy divided by the charge, Potential difference is commonly called voltage, represented by the symbol \( \Delta V \).

\[
\Delta V = \frac{\Delta PE}{q} \quad \text{and} \quad \Delta PE = q \Delta V.
\]

- An electron volt is the energy given to a fundamental charge accelerated through a potential difference of 1 V. In equation form,

\[
1 \text{ eV} = (1.60 \times 10^{-19} \text{ C})(1 \text{ V}) = (1.60 \times 10^{-19} \text{ C})(1 \text{ J/C})
\]

\[= 1.60 \times 10^{-19} \text{ J}.\]

- Mechanical energy is the sum of the kinetic energy and potential energy of a system, that is, \( KE + PE \). This sum is a constant.

**13.2 Electric Potential in a Uniform Electric Field**

- The voltage between points A and B is

\[
V_{AB} = Ed
\]

\[E = \frac{V_{AB}}{d}\] (uniform \( E \) field only),

where \( d \) is the distance from A to B, or the distance between the plates.

- In equation form, the general relationship between voltage and electric field is

\[E = \frac{-\Delta V}{\Delta s},\]

where \( \Delta s \) is the distance over which the change in potential, \( \Delta V \), takes place. The minus sign tells us that \( E \) points in the direction of decreasing potential.) The electric field is said to be the gradient (as in grade or slope) of the electric potential.

**13.3 Electrical Potential Due to a Point Charge**

- Electric potential of a point charge is \( V = kQ/r \).

- Electric potential is a scalar, and electric field is a vector. Addition of voltages as numbers gives the voltage due to a combination of point charges, whereas addition of individual fields as vectors gives the total electric field.

**13.4 Equipotential Lines**

- An equipotential line is a line along which the electric potential is constant.

- An equipotential surface is a three-dimensional version of equipotential lines.

- Equipotential lines are always perpendicular to electric field lines.

- The process by which a conductor can be fixed at zero volts by connecting it to the earth with a good conductor is called grounding.

**13.5 Capacitors and Dielectrics**

- A capacitor is a device used to store charge.
The amount of charge \( Q \) a capacitor can store depends on two major factors—the voltage applied and the capacitor’s physical characteristics, such as its size.

The capacitance \( C \) is the amount of charge stored per volt, or

\[
C = \frac{Q}{V}.
\]

The capacitance of a parallel plate capacitor is \( C = \varepsilon_0 \frac{A}{d} \), when the plates are separated by air or free space. \( \varepsilon_0 \) is called the permittivity of free space.

A parallel plate capacitor with a dielectric between its plates has a capacitance given by

\[
C = \kappa \varepsilon_0 \frac{A}{d},
\]

where \( \kappa \) is the dielectric constant of the material.

The maximum electric field strength above which an insulating material begins to break down and conduct is called dielectric strength.

13.6 Capacitors in Series and Parallel

- Total capacitance in series
  \[
  \frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots
  \]

- Total capacitance in parallel
  \[
  C_p = C_1 + C_2 + C_3 + \ldots
  \]

- If a circuit contains a combination of capacitors in series and parallel, identify series and parallel parts, compute their capacitances, and then find the total.

13.7 Energy Stored in Capacitors

- Capacitors are used in a variety of devices, including defibrillators, microelectronics such as calculators, and flash lamps, to supply energy.

- The energy stored in a capacitor can be expressed in three ways:
  \[
  E_{\text{cap}} = \frac{QV}{2} = CV^2 = \frac{Q^2}{2C},
  \]
  where \( Q \) is the charge, \( V \) is the voltage, and \( C \) is the capacitance of the capacitor. The energy is in joules when the charge is in coulombs, voltage is in volts, and capacitance is in farads.

Conceptual Questions


1. Voltage is the common word for potential difference. Which term is more descriptive, voltage or potential difference?
2. If the voltage between two points is zero, can a test charge be moved between them with zero net work being done? Can this necessarily be done without exerting a force? Explain.
3. What is the relationship between voltage and energy? More precisely, what is the relationship between potential difference and electric potential energy?
4. Voltages are always measured between two points. Why?
5. How are units of volts and electron volts related? How do they differ?

13.2 Electric Potential in a Uniform Electric Field

6. Discuss how potential difference and electric field strength are related. Give an example.
7. What is the strength of the electric field in a region where the electric potential is constant?
8. Will a negative charge, initially at rest, move toward higher or lower potential? Explain why.

13.3 Electrical Potential Due to a Point Charge

9. In what region of space is the potential due to a uniformly charged sphere the same as that of a point charge? In what region does it differ from that of a point charge?
10. Can the potential of a non-uniformly charged sphere be the same as that of a point charge? Explain.

13.4 Equipotential Lines

11. What is an equipotential line? What is an equipotential surface?
12. Explain in your own words why equipotential lines and surfaces must be perpendicular to electric field lines.
13. Can different equipotential lines cross? Explain.

13.5 Capacitors and Dielectrics

14. Does the capacitance of a device depend on the applied voltage? What about the charge stored in it?

15. Use the characteristics of the Coulomb force to explain why capacitance should be proportional to the plate area of a capacitor. Similarly, explain why capacitance should be inversely proportional to the separation between plates.

16. Give the reason why a dielectric material increases capacitance compared with what it would be with air between the plates of a capacitor. What is the independent reason that a dielectric material also allows a greater voltage to be applied to a capacitor? (The dielectric thus increases $C$ and permits a greater $V$.)

17. How does the polar character of water molecules help to explain water’s relatively large dielectric constant? (Figure 13.19)

18. Sparks will occur between the plates of an air-filled capacitor at lower voltage when the air is humid than when dry. Explain why, considering the polar character of water molecules.

19. Water has a large dielectric constant, but it is rarely used in capacitors. Explain why.

20. Membranes in living cells, including those in humans, are characterized by a separation of charge across the membrane. Effectively, the membranes are thus charged capacitors with important functions related to the potential difference across the membrane. Is energy required to separate these charges in living membranes and, if so, is its source the metabolization of food energy or some other source?

21. If you wish to store a large amount of energy in a capacitor bank, would you connect capacitors in series or parallel? Explain.

13.6 Capacitors in Series and Parallel

22. How does the energy contained in a charged capacitor change when a dielectric is inserted, assuming the capacitor is isolated and its charge is constant? Does this imply that work was done?

23. What happens to the energy stored in a capacitor connected to a battery when a dielectric is inserted? Was work done in the process?
9. Integrated Concepts
A battery-operated car utilizes a 12.0 V system. Find the charge the batteries must be able to move in order to accelerate the 750 kg car from rest to 25.0 m/s, make it climb a 2.00 \times 10^{-2} \text{ m} high hill, and then cause it to travel at a constant 25.0 m/s by exerting a 5.00 \times 10^{2} \text{ N} force for an hour.

10. Integrated Concepts
Fusion probability is greatly enhanced when appropriate nuclei are brought close together, but mutual Coulomb repulsion must be overcome. This can be done using the kinetic energy of high-temperature gas ions or by accelerating the nuclei toward one another. (a) Calculate the potential energy of two singly charged nuclei separated by 1.00 \times 10^{-12} \text{ m} by finding the voltage of one at that distance and multiplying by the charge of the other. (b) At what temperature will atoms of a gas have an average kinetic energy equal to this needed electrical potential energy?

11. Unreasonable Results
(a) Find the voltage near a 10.0 cm diameter metal sphere that has 8.00 C of excess positive charge on it. (b) What is unreasonable about this result? (c) Which assumptions are responsible?

12. Construct Your Own Problem
Consider a battery used to supply energy to a cellular phone. Construct a problem in which you determine the energy that charge the batteries must be able to move in order to supply this energy. Among the things to be considered are the amount of charge it must be able to move in order to supply this energy. Among the things to be considered are the energy needs and battery voltage. You may need to look ahead to interpret manufacturer’s battery ratings in ampere-hours as energy in joules.

13.2 Electric Potential in a Uniform Electric Field
13. Show that units of V/m and N/C for electric field strength are indeed equivalent.

14. What is the strength of the electric field between two parallel conducting plates separated by 1.00 cm and having a potential difference (voltage) between them of 1.50 \times 10^{4} \text{ V} ?

15. The electric field strength between two parallel conducting plates separated by 4.00 cm is 7.50 \times 10^{4} \text{ V/m} . (a) What is the potential difference between the plates? (b) The plate with the lowest potential is taken to be at zero volts. What is the potential 1.00 cm from that plate (and 3.00 cm from the other)?

16. How far apart are two conducting plates that have an electric field strength of 4.50 \times 10^{3} \text{ V/m} between them, if their potential difference is 15.0 kV?

17. (a) Will the electric field strength between two parallel conducting plates exceed the breakdown strength for air (3.0 \times 10^{6} \text{ V/m}) if the plates are separated by 2.00 mm and a potential difference of 5.0 \times 10^{3} \text{ V} is applied? (b) How close together can the plates be with this applied voltage?
18. The voltage across a membrane forming a cell wall is 80.0 mV and the membrane is 9.00 nm thick. What is the electric field strength? (The value is surprisingly large, but correct. Membranes are discussed in Capacitors and Dielectrics and Nerve Conduction—Electrocardiograms (https://legacy.cnx.org/content/m42352/latest/). You may assume a uniform electric field.

19. Membrane walls of living cells have surprisingly large electric fields across them due to separation of ions. (Membranes are discussed in some detail in Nerve Conduction—Electrocardiograms (https://legacy.cnx.org/content/m42352/latest/).) What is the voltage across an 8.00 nm–thick membrane if the electric field strength across it is 5.50 MV/m? You may assume a uniform electric field.

20. Two parallel conducting plates are separated by 10.0 cm, and one of them is taken to be at zero volts. (a) What is the electric field strength between them, if the potential 8.00 cm from the zero volt plate (and 2.00 cm from the other) is 450 V? (b) What is the voltage between the plates?

21. Find the maximum potential difference between two parallel conducting plates separated by 0.500 cm of air, given the average sustainable electric field strength in air to be 3.0×10^6 V/m.

22. A doubly charged ion is accelerated to an energy of 32.0 keV by the electric field between two parallel conducting plates separated by 2.00 cm. What is the electric field strength between the plates?

23. An electron is to be accelerated in a uniform electric field having a strength of 2.00×10^5 V/m. (a) What energy in keV is given to the electron if it is accelerated through 0.400 m? (b) Over what distance would it have to be accelerated to increase its energy by 50.0 GeV?

13.3 Electrical Potential Due to a Point Charge

24. A 0.500 cm diameter plastic sphere, used in a static electricity demonstration, has a uniformly distributed 40.0 pC charge on its surface. What is the potential near its surface?

25. What is the potential 0.530×10^{-10} m from a proton (the average distance between the proton and electron in a hydrogen atom)?

26. (a) A sphere has a surface uniformly charged with 1.00 C. At what distance from its center is the potential 5.00 MV? (b) What does your answer imply about the practical aspect of isolating such a large charge?

27. How far from a 1.00 µC point charge will the potential be 100 V? At what distance will it be 2.00×10^2 V?

28. What are the sign and magnitude of a point charge that produces a potential of −2.00 V at a distance of 1.00 mm?

29. If the potential due to a point charge is 5.00×10^2 V at a distance of 15.0 m, what are the sign and magnitude of the charge?

30. In nuclear fission, a nucleus splits roughly in half. (a) What is the potential 2.00×10^{-14} m from a fragment that has 46 protons in it? (b) What is the potential energy in MeV of a similarly charged fragment at this distance?

31. A research Van de Graaff generator has a 2.00-m-diameter metal sphere with a charge of 5.00 mC on it. (a) What is the potential near its surface? (b) At what distance from its center is the potential 1.00 MV? (c) An oxygen atom with three missing electrons is released near the Van de Graaff generator. What is its energy in MeV at this distance?

32. An electrostatic paint sprayer has a 0.200-m-diameter metal sphere at a potential of 25.0 kV that repels paint droplets onto a grounded object. (a) What charge is on the sphere? (b) What charge must a 0.100-mg drop of paint have to arrive at the object with a speed of 10.0 m/s?

33. In one of the classic nuclear physics experiments at the beginning of the 20th century, an alpha particle was accelerated toward a gold nucleus, and its path was substantially deflected by the Coulomb interaction. If the energy of the doubly charged alpha nucleus was 5.00 MeV, how close to the gold nucleus (79 protons) could it come before being deflected?

34. (a) What is the potential between two points situated 10 cm and 20 cm from a 3.0 µC point charge? (b) To what location should the point at 20 cm be moved to increase this potential difference by a factor of two?

35. Unreasonable Results

(a) What is the final speed of an electron accelerated from rest through a voltage of 25.0 MV by a negatively charged Van de Graaff terminal?

(b) What is unreasonable about this result?

(c) Which assumptions are responsible?

13.4 Equipotential Lines

36. (a) Sketch the equipotential lines near a point charge + q . Indicate the direction of increasing potential. (b) Do the same for a point charge − 3q .

37. Sketch the equipotential lines for the two equal positive charges shown in Figure 13.26. Indicate the direction of increasing potential.

38. Figure 13.27 shows the electric field lines near two charges q_1 and q_2, the first having a magnitude four times that of the second. Sketch the equipotential lines for these two charges, and indicate the direction of increasing potential.
39. Sketch the equipotential lines a long distance from the charges shown in Figure 13.27. Indicate the direction of increasing potential.

Figure 13.27 The electric field near two charges.

40. Sketch the equipotential lines in the vicinity of two opposite charges, where the negative charge is three times as great in magnitude as the positive. See Figure 13.27 for a similar situation. Indicate the direction of increasing potential.

41. Sketch the equipotential lines in the vicinity of the negatively charged conductor in Figure 13.28. How will these equipotentials look a long distance from the object?

Figure 13.28 A negatively charged conductor.

42. Sketch the equipotential lines surrounding the two conducting plates shown in Figure 13.29, given the top plate is positive and the bottom plate has an equal amount of negative charge. Be certain to indicate the distribution of charge on the plates. Is the field strongest where the plates are closest? Why should it be?

Figure 13.29

43. (a) Sketch the electric field lines in the vicinity of the charged insulator in Figure 13.30. Note its non-uniform charge distribution. (b) Sketch equipotential lines surrounding the insulator. Indicate the direction of increasing potential.

Figure 13.30 A charged insulating rod such as might be used in a classroom demonstration.

44. The naturally occurring electrical field on the ground to an open sky point 3.00 m above is $1.13 \times 10^2 \text{ N/C}$. This open point in the sky is at a greater electric potential than the ground. (a) Calculate the electric potential at this height. (b) Sketch electric field and equipotential lines for this scenario. Calculate the electric potential at this height. (c) Sketch electric field and equipotential lines for this scenario.

45. The lesser electric ray (Narcine bancroftii) maintains an incredible charge on its head and a charge equal in magnitude but opposite in sign on its tail (Figure 13.31). (a) Sketch the equipotential lines surrounding the ray. (b) Sketch the equipotentials when the ray is near a ship with a conducting surface. (c) How could this charge distribution be of use to the ray?

Figure 13.31 Lesser electric ray (Narcine bancroftii) (credit: National Oceanic and Atmospheric Administration, NOAA’s Fisheries Collection).

13.5 Capacitors and Dielectrics

46. What charge is stored in a $180 \ \mu\text{F}$ capacitor when 120 V is applied to it?

47. Find the charge stored when 5.50 V is applied to an 8.00 pF capacitor.

48. What charge is stored in the capacitor in Example 13.8?

49. Calculate the voltage applied to a $2.00 \ \mu\text{F}$ capacitor when it holds $3.10 \ \mu\text{C}$ of charge.
50. What voltage must be applied to an 8.00 nF capacitor to store 0.160 mC of charge?

51. What capacitance is needed to store 3.00 µC of charge at a voltage of 120 V?

52. What is the capacitance of a large Van de Graaff generator's terminal, given that it stores 8.00 mC of charge at a voltage of 12.0 MV?

53. Find the capacitance of a parallel plate capacitor having plates of area 5.00 m² that are separated by 0.100 mm of Teflon.

54. (a) What is the capacitance of a parallel plate capacitor having plates of area 1.50 m² that are separated by 0.0200 mm of neoprene rubber? (b) What charge does it hold when 9.00 V is applied to it?

55. Integrated Concepts
A prankster applies 450 V to an 80.0 µF capacitor and then tosses it to an unsuspecting victim. The victim's finger is burned by the discharge of the capacitor through 0.200 g of flesh. What is the temperature increase of the flesh? Is it reasonable to assume no phase change?

56. Unreasonable Results
(a) A certain parallel plate capacitor has plates of area 4.00 m², separated by 0.0100 mm of nylon, and stores 0.170 C of charge. What is the applied voltage? (b) What is unreasonable about this result? (c) Which assumptions are responsible or inconsistent?

### 13.6 Capacitors in Series and Parallel

57. Find the total capacitance of the combination of capacitors in Figure 13.32.

58. Suppose you want a capacitor bank with a total capacitance of 0.750 F and you possess numerous 1.50 mF capacitors. What is the smallest number you could hook together to achieve your goal, and how would you connect them?

59. What total capacitances can you make by connecting a 5.00 µF and an 8.00 µF capacitor together?

60. Find the total capacitance of the combination of capacitors shown in Figure 13.33.

61. Find the total capacitance of the combination of capacitors shown in Figure 13.34.

### 13.7 Energy Stored in Capacitors

62. Unreasonable Results
(a) An 8.00 µF capacitor is connected in parallel to another capacitor, producing a total capacitance of 5.00 µF. What is the capacitance of the second capacitor? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

63. (a) What is the energy stored in the 10.0 µF capacitor of a heart defibrillator charged to 9.00 × 10³ V? (b) Find the amount of stored charge.

64. In open heart surgery, a much smaller amount of energy will defibrillate the heart. (a) What voltage is applied to the 8.00 µF capacitor of a heart defibrillator that stores 40.0 J of energy? (b) Find the amount of stored charge.

65. A 165 µF capacitor is used in conjunction with a motor. How much energy is stored in it when 119 V is applied?

66. Suppose you have a 9.00 V battery, a 2.00 µF capacitor, and a 7.40 µF capacitor. (a) Find the charge and energy stored if the capacitors are connected to the battery in series. (b) Do the same for a parallel connection.
67. A nervous physicist worries that the two metal shelves of his wood frame bookcase might obtain a high voltage if charged by static electricity, perhaps produced by friction. (a) What is the capacitance of the empty shelves if they have area \(1.00 \times 10^2 \text{ m}^2\) and are 0.200 m apart? (b) What is the voltage between them if opposite charges of magnitude 2.00 nC are placed on them? (c) To show that this voltage poses a small hazard, calculate the energy stored.

68. Show that for a given dielectric material the maximum energy a parallel plate capacitor can store is directly proportional to the volume of dielectric (Volume = \(A \cdot d\)). Note that the applied voltage is limited by the dielectric strength.

69. Construct Your Own Problem

Consider a heart defibrillator similar to that discussed in Example 13.11. Construct a problem in which you examine the charge stored in the capacitor of a defibrillator as a function of stored energy. Among the things to be considered are the applied voltage and whether it should vary with energy to be delivered, the range of energies involved, and the capacitance of the defibrillator. You may also wish to consider the much smaller energy needed for defibrillation during open-heart surgery as a variation on this problem.

70. Unreasonable Results

(a) On a particular day, it takes \(9.60 \times 10^3 \text{ J}\) of electric energy to start a truck’s engine. Calculate the capacitance of a capacitor that could store that amount of energy at 12.0 V. (b) What is unreasonable about this result? (c) Which assumptions are responsible?
Introduction to Electromagnetic Waves

The beauty of a coral reef, the warm radiance of sunshine, the sting of sunburn, the X-ray revealing a broken bone, even microwave popcorn—all are brought to us by electromagnetic waves. The list of the various types of electromagnetic waves, ranging from radio transmission waves to nuclear gamma-ray (γ-ray) emissions, is interesting in itself.

Even more intriguing is that all of these widely varied phenomena are different manifestations of the same thing—electromagnetic waves. (See Figure 14.2.) What are electromagnetic waves? How are they created, and how do they...
travel? How can we understand and organize their widely varying properties? What is their relationship to electric and magnetic effects? These and other questions will be explored.

**Misconception Alert: Sound Waves vs. Radio Waves**

Many people confuse sound waves with radio waves, one type of electromagnetic (EM) wave. However, sound and radio waves are completely different phenomena. Sound creates pressure variations (waves) in matter, such as air or water, or your eardrum. Conversely, radio waves are electromagnetic waves, like visible light, infrared, ultraviolet, X-rays, and gamma rays. EM waves don't need a medium in which to propagate; they can travel through a vacuum, such as outer space.

A radio works because sound waves played by the D.J. at the radio station are converted into electromagnetic waves, then encoded and transmitted in the radio-frequency range. The radio in your car receives the radio waves, decodes the information, and uses a speaker to change it back into a sound wave, bringing sweet music to your ears.

**Discovering a New Phenomenon**

It is worth noting at the outset that the general phenomenon of electromagnetic waves was predicted by theory before it was realized that light is a form of electromagnetic wave. The prediction was made by James Clerk Maxwell in the mid-19th century when he formulated a single theory combining all the electric and magnetic effects known by scientists at that time. “Electromagnetic waves” was the name he gave to the phenomena his theory predicted.

Such a theoretical prediction followed by experimental verification is an indication of the power of science in general, and physics in particular. The underlying connections and unity of physics allow certain great minds to solve puzzles without having all the pieces. The prediction of electromagnetic waves is one of the most spectacular examples of this power. Certain others, such as the prediction of antimatter, will be discussed in later modules.

![Figure 14.2](https://example.com/figure14.2.jpg)

*Figure 14.2* The electromagnetic waves sent and received by this 50-foot radar dish antenna at Kennedy Space Center in Florida are not visible, but help track expendable launch vehicles with high-definition imagery. The first use of this C-band radar dish was for the launch of the Atlas V rocket sending the New Horizons probe toward Pluto. (credit: NASA)

**14.1 Maxwell’s Equations: Electromagnetic Waves Predicted and Observed**

The Scotsman James Clerk Maxwell (1831–1879) is regarded as the greatest theoretical physicist of the 19th century. (See **Figure 14.3**.) Although he died young, Maxwell not only formulated a complete electromagnetic theory, represented by Maxwell’s equations, he also developed the kinetic theory of gases and made significant contributions to the understanding of color vision and the nature of Saturn’s rings.
Maxwell brought together all the work that had been done by brilliant physicists such as Oersted, Coulomb, Gauss, and Faraday, and added his own insights to develop the overarching theory of electromagnetism. Maxwell’s equations are paraphrased here in words because their mathematical statement is beyond the level of this text. However, the equations illustrate how apparently simple mathematical statements can elegantly unite and express a multitude of concepts—why mathematics is the language of science.

Maxwell’s Equations

1. Electric field lines originate on positive charges and terminate on negative charges. The electric field is defined as the force per unit charge on a test charge, and the strength of the force is related to the electric constant $\varepsilon_0$, also known as the permittivity of free space. From Maxwell’s first equation we obtain a special form of Coulomb’s law known as Gauss’s law for electricity.

2. Magnetic field lines are continuous, having no beginning or end. No magnetic monopoles are known to exist. The strength of the magnetic force is related to the magnetic constant $\mu_0$, also known as the permeability of free space. This second of Maxwell’s equations is known as Gauss’s law for magnetism.

3. A changing magnetic field induces an electromotive force (emf) and, hence, an electric field. The direction of the emf opposes the change. This third of Maxwell’s equations is Faraday’s law of induction, and includes Lenz’s law.

4. Magnetic fields are generated by moving charges or by changing electric fields. This fourth of Maxwell’s equations encompasses Ampere’s law and adds another source of magnetism—changing electric fields.

Maxwell’s equations encompass the major laws of electricity and magnetism. What is not so apparent is the symmetry that Maxwell introduced in his mathematical framework. Especially important is his addition of the hypothesis that changing electric fields create magnetic fields. This is exactly analogous (and symmetric) to Faraday’s law of induction and had been suspected for some time, but fits beautifully into Maxwell’s equations.

Symmetry is apparent in nature in a wide range of situations. In contemporary research, symmetry plays a major part in the search for sub-atomic particles using massive multinational particle accelerators such as the new Large Hadron Collider at CERN.

Making Connections: Unification of Forces

Maxwell’s complete and symmetric theory showed that electric and magnetic forces are not separate, but different manifestations of the same thing—the electromagnetic force. This classical unification of forces is one motivation for current attempts to unify the four basic forces in nature—the gravitational, electrical, strong, and weak nuclear forces.

Since changing electric fields create relatively weak magnetic fields, they could not be easily detected at the time of Maxwell’s hypothesis. Maxwell realized, however, that oscillating charges, like those in AC circuits, produce changing electric fields. He predicted that these changing fields would propagate from the source like waves generated on a lake by a jumping fish.

The waves predicted by Maxwell would consist of oscillating electric and magnetic fields—defined to be an electromagnetic wave (EM wave). Electromagnetic waves would be capable of exerting forces on charges great distances from their source, and they might thus be detectable. Maxwell calculated that electromagnetic waves would propagate at a speed given by the equation

$$c = \sqrt{\frac{1}{\mu_0 \varepsilon_0}}.$$  \hspace{1cm} (14.1)

When the values for $\mu_0$ and $\varepsilon_0$ are entered into the equation for $c$, we find that
\[
c = \frac{1}{\sqrt{(8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N} \cdot \text{m}^2})(4\pi \times 10^{-7} \frac{T}{\text{m}})}} = 3.00 \times 10^8 \text{ m/s},
\]

which is the speed of light. In fact, Maxwell concluded that light is an electromagnetic wave having such wavelengths that it can be detected by the eye.

Other wavelengths should exist—it remained to be seen if they did. If so, Maxwell’s theory and remarkable predictions would be verified, the greatest triumph of physics since Newton. Experimental verification came within a few years, but not before Maxwell’s death.

**Hertz’s Observations**

The German physicist Heinrich Hertz (1857–1894) was the first to generate and detect certain types of electromagnetic waves in the laboratory. Starting in 1887, he performed a series of experiments that not only confirmed the existence of electromagnetic waves, but also verified that they travel at the speed of light.

Hertz used an AC RLC (resistor-inductor-capacitor) circuit that resonates at a known frequency \( f_0 = \frac{1}{2\pi \sqrt{LC}} \) and connected it to a loop of wire as shown in Figure 14.4. High voltages induced across the gap in the loop produced sparks that were visible evidence of the current in the circuit and that helped generate electromagnetic waves. Across the laboratory, Hertz had another loop attached to another RLC circuit, which could be tuned (as the dial on a radio) to the same resonant frequency as the first and could, thus, be made to receive electromagnetic waves. This loop also had a gap across which sparks were generated, giving solid evidence that electromagnetic waves had been received.

![Figure 14.4](https://example.com/figure14_4.png)

**Figure 14.4** The apparatus used by Hertz in 1887 to generate and detect electromagnetic waves. An RLC circuit connected to the first loop caused sparks across a gap in the wire loop and generated electromagnetic waves. Sparks across a gap in the second loop located across the laboratory gave evidence that the waves had been received.

Hertz also studied the reflection, refraction, and interference patterns of the electromagnetic waves he generated, verifying their wave character. He was able to determine wavelength from the interference patterns, and knowing their frequency, he could calculate the propagation speed using the equation \( v = f\lambda \) (velocity—or speed—equals frequency times wavelength). Hertz was thus able to prove that electromagnetic waves travel at the speed of light. The SI unit for frequency, the hertz (1 Hz = 1 cycle/s), is named in his honor.

### 14.2 Production of Electromagnetic Waves

We can get a good understanding of electromagnetic waves (EM) by considering how they are produced. Whenever a current varies, associated electric and magnetic fields vary, moving out from the source like waves. Perhaps the easiest situation to visualize is a varying current in a long straight wire, produced by an AC generator at its center, as illustrated in Figure 14.5.
This long straight gray wire with an AC generator at its center becomes a broadcast antenna for electromagnetic waves. Shown here are the charge distributions at four different times. The electric field ($E$) propagates away from the antenna at the speed of light, forming part of an electromagnetic wave.

The electric field ($E$) shown surrounding the wire is produced by the charge distribution on the wire. Both the $E$ and the charge distribution vary as the current changes. The changing field propagates outward at the speed of light.

There is an associated magnetic field ($B$) which propagates outward as well (see Figure 14.6). The electric and magnetic fields are closely related and propagate as an electromagnetic wave. This is what happens in broadcast antennae such as those in radio and TV stations.

Closer examination of the one complete cycle shown in Figure 14.5 reveals the periodic nature of the generator-driven charges oscillating up and down in the antenna and the electric field produced. At time $t = 0$, there is the maximum separation of charge, with negative charges at the top and positive charges at the bottom, producing the maximum magnitude of the electric field (or $E$-field) in the upward direction. One-fourth of a cycle later, there is no charge separation and the field next to the antenna is zero, while the maximum $E$-field has moved away at speed $c$.

As the process continues, the charge separation reverses and the field reaches its maximum downward value, returns to zero, and rises to its maximum upward value at the end of one complete cycle. The outgoing wave has an amplitude proportional to the maximum separation of charge. Its wavelength ($\lambda$) is proportional to the period of the oscillation and, hence, is smaller for short periods or high frequencies. (As usual, wavelength and frequency ($f$) are inversely proportional.)

**Electric and Magnetic Waves: Moving Together**

Following Ampere's law, current in the antenna produces a magnetic field, as shown in Figure 14.6. The relationship between $E$ and $B$ is shown at one instant in Figure 14.6 (a). As the current varies, the magnetic field varies in magnitude and direction.

The magnetic field lines also propagate away from the antenna at the speed of light, forming the other part of the electromagnetic wave, as seen in Figure 14.6 (b). The magnetic part of the wave has the same period and wavelength as the...
electric part, since they are both produced by the same movement and separation of charges in the antenna.

The electric and magnetic waves are shown together at one instant in time in Figure 14.7. The electric and magnetic fields produced by a long straight wire antenna are exactly in phase. Note that they are perpendicular to one another and to the direction of propagation, making this a transverse wave.

![Figure 14.7 A part of the electromagnetic wave sent out from the antenna at one instant in time. The electric and magnetic fields (E and B) are in phase, and they are perpendicular to one another and the direction of propagation. For clarity, the waves are shown only along one direction, but they propagate out in other directions too.](image)

Electromagnetic waves generally propagate out from a source in all directions, sometimes forming a complex radiation pattern. A linear antenna like this one will not radiate parallel to its length, for example. The wave is shown in one direction from the antenna in Figure 14.7 to illustrate its basic characteristics.

Instead of the AC generator, the antenna can also be driven by an AC circuit. In fact, charges radiate whenever they are accelerated. But while a current in a circuit needs a complete path, an antenna has a varying charge distribution forming a standing wave, driven by the AC. The dimensions of the antenna are critical for determining the frequency of the radiated electromagnetic waves. This is a resonant phenomenon and when we tune radios or TV, we vary electrical properties to achieve appropriate resonant conditions in the antenna.

**Receiving Electromagnetic Waves**

Electromagnetic waves carry energy away from their source, similar to a sound wave carrying energy away from a standing wave on a guitar string. An antenna for receiving EM signals works in reverse. And like antennas that produce EM waves, receiver antennas are specially designed to resonate at particular frequencies.

An incoming electromagnetic wave accelerates electrons in the antenna, setting up a standing wave. If the radio or TV is switched on, electrical components pick up and amplify the signal formed by the accelerating electrons. The signal is then converted to audio and/or video format. Sometimes big receiver dishes are used to focus the signal onto an antenna.

In fact, charges radiate whenever they are accelerated. When designing circuits, we often assume that energy does not quickly escape AC circuits, and mostly this is true. A broadcast antenna is specially designed to enhance the rate of electromagnetic radiation, and shielding is necessary to keep the radiation close to zero. Some familiar phenomena are based on the production of electromagnetic waves by varying currents. Your microwave oven, for example, sends electromagnetic waves, called microwaves, from a concealed antenna that has an oscillating current imposed on it.

**Relating E-Field and B-Field Strengths**

There is a relationship between the E- and B-field strengths in an electromagnetic wave. This can be understood by again considering the antenna just described. The stronger the E-field created by a separation of charge, the greater the current and, hence, the greater the B-field created.

Since current is directly proportional to voltage (Ohm's law) and voltage is directly proportional to E-field strength, the two should be directly proportional. It can be shown that the magnitudes of the fields do have a constant ratio, equal to the speed of light. That is,

\[
\frac{E}{B} = c
\]

is the ratio of E-field strength to B-field strength in any electromagnetic wave. This is true at all times and at all locations in space. A simple and elegant result.

**Example 14.1 Calculating B-Field Strength in an Electromagnetic Wave**

What is the maximum strength of the B-field in an electromagnetic wave that has a maximum E-field strength of
1000 V/m?

Strategy
To find the $B$-field strength, we rearrange the above equation to solve for $B$, yielding

$$B = \frac{E}{c}. \quad (14.4)$$

Solution
We are given $E$, and $c$ is the speed of light. Entering these into the expression for $B$ yields

$$B = \frac{1000 \text{ V/m}}{3.00 \times 10^8 \text{ m/s}} = 3.33 \times 10^{-6} \text{T}. \quad (14.5)$$

Where T stands for Tesla, a measure of magnetic field strength.

Discussion
The $B$-field strength is less than a tenth of the Earth's admittedly weak magnetic field. This means that a relatively strong electric field of 1000 V/m is accompanied by a relatively weak magnetic field. Note that as this wave spreads out, say with distance from an antenna, its field strengths become progressively weaker.

The result of this example is consistent with the statement made in the module Maxwell’s Equations: Electromagnetic Waves Predicted and Observed that changing electric fields create relatively weak magnetic fields. They can be detected in electromagnetic waves, however, by taking advantage of the phenomenon of resonance, as Hertz did. A system with the same natural frequency as the electromagnetic wave can be made to oscillate. All radio and TV receivers use this principle to pick up and then amplify weak electromagnetic waves, while rejecting all others not at their resonant frequency.

Take-Home Experiment: Antennas
For your TV or radio at home, identify the antenna, and sketch its shape. If you don't have cable, you might have an outdoor or indoor TV antenna. Estimate its size. If the TV signal is between 60 and 216 MHz for basic channels, then what is the wavelength of those EM waves?

Try tuning the radio and note the small range of frequencies at which a reasonable signal for that station is received. (This is easier with digital readout.) If you have a car with a radio and extendable antenna, note the quality of reception as the length of the antenna is changed.

Radio Waves and Electromagnetic Fields
Broadcast radio waves from KPhET. Wiggle the transmitter electron manually or have it oscillate automatically. Display the field as a curve or vectors. The strip chart shows the electron positions at the transmitter and at the receiver.

(Here media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42440/1.17/#fs-id1167065661124)

Figure 14.8

14.3 The Electromagnetic Spectrum
In this module we examine how electromagnetic waves are classified into categories such as radio, infrared, ultraviolet, and so on, so that we can understand some of their similarities as well as some of their differences. We will also find that there are many connections with previously discussed topics, such as wavelength and resonance. A brief overview of the production and utilization of electromagnetic waves is found in Table 14.1.
Table 14.1 Electromagnetic Waves

<table>
<thead>
<tr>
<th>Type of EM wave</th>
<th>Production</th>
<th>Applications</th>
<th>Life sciences aspect</th>
<th>Issues</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radio &amp; TV</td>
<td>Accelerating charges</td>
<td>Communications Remote controls</td>
<td>MRI</td>
<td>Requires controls for band use</td>
</tr>
<tr>
<td>Microwaves</td>
<td>Accelerating charges &amp; thermal agitation</td>
<td>Communications Ovens Radar</td>
<td>Deep heating</td>
<td>Cell phone use</td>
</tr>
<tr>
<td>Infrared</td>
<td>Thermal agitations &amp; electronic transitions</td>
<td>Thermal imaging Heating</td>
<td>Absorbed by atmosphere</td>
<td>Greenhouse effect</td>
</tr>
<tr>
<td>Visible light</td>
<td>Thermal agitations &amp; electronic transitions</td>
<td>All pervasive</td>
<td>Photosynthesis Human vision</td>
<td></td>
</tr>
<tr>
<td>Ultraviolet</td>
<td>Thermal agitations &amp; electronic transitions</td>
<td>Sterilization Cancer control</td>
<td>Vitamin D production</td>
<td>Ozone depletion Cancer causing</td>
</tr>
<tr>
<td>X-rays</td>
<td>Inner electronic transitions and fast collisions</td>
<td>Medical Security</td>
<td>Medical diagnosis Cancer therapy</td>
<td>Cancer causing</td>
</tr>
<tr>
<td>Gamma rays</td>
<td>Nuclear decay</td>
<td>Nuclear medicineSecurity</td>
<td>Medical diagnosis Cancer therapy</td>
<td>Cancer causing Radiation damage</td>
</tr>
</tbody>
</table>

Connections: Waves

There are many types of waves, such as water waves and even earthquakes. Among the many shared attributes of waves are propagation speed, frequency, and wavelength. These are always related by the expression \( v_W = f \lambda \). This module concentrates on EM waves, but other modules contain examples of all of these characteristics for sound waves and submicroscopic particles.

As noted before, an electromagnetic wave has a frequency and a wavelength associated with it and travels at the speed of light, or \( c \). The relationship among these wave characteristics can be described by \( v_W = f \lambda \), where \( v_W \) is the propagation speed of the wave, \( f \) is the frequency, and \( \lambda \) is the wavelength. Here \( v_W = c \), so that for all electromagnetic waves,

\[
c = f \lambda.
\]

(14.6)

Thus, for all electromagnetic waves, the greater the frequency, the smaller the wavelength.

Figure 14.9 shows how the various types of electromagnetic waves are categorized according to their wavelengths and frequencies—that is, it shows the electromagnetic spectrum. Many of the characteristics of the various types of electromagnetic waves are related to their frequencies and wavelengths, as we shall see.

Electromagnetic Spectrum: Rules of Thumb

Three rules that apply to electromagnetic waves in general are as follows:

- High-frequency electromagnetic waves are more energetic and are more able to penetrate than low-frequency waves.
- High-frequency electromagnetic waves can carry more information per unit time than low-frequency waves.
The shorter the wavelength of any electromagnetic wave probing a material, the smaller the detail it is possible to resolve. Note that there are exceptions to these rules of thumb.

Transmission, Reflection, and Absorption

What happens when an electromagnetic wave impinges on a material? If the material is transparent to the particular frequency, then the wave can largely be transmitted. If the material is opaque to the frequency, then the wave can be totally reflected. The wave can also be absorbed by the material, indicating that there is some interaction between the wave and the material, such as the thermal agitation of molecules.

Of course it is possible to have partial transmission, reflection, and absorption. We normally associate these properties with visible light, but they do apply to all electromagnetic waves. What is not obvious is that something that is transparent to light may be opaque at other frequencies. For example, ordinary glass is transparent to visible light but largely opaque to ultraviolet radiation. Human skin is opaque to visible light—we cannot see through people—but transparent to X-rays.

Radio and TV Waves

The broad category of radio waves is defined to contain any electromagnetic wave produced by currents in wires and circuits. Its name derives from their most common use as a carrier of audio information (i.e., radio). The name is applied to electromagnetic waves of similar frequencies regardless of source. Radio waves from outer space, for example, do not come from alien radio stations. They are created by many astronomical phenomena, and their study has revealed much about nature on the largest scales.

There are many uses for radio waves, and so the category is divided into many subcategories, including microwaves and those electromagnetic waves used for AM and FM radio, cellular telephones, and TV.

The lowest commonly encountered radio frequencies are produced by high-voltage AC power transmission lines at frequencies of 50 or 60 Hz. (See Figure 14.10.) These extremely long wavelength electromagnetic waves (about 6000 km!) are one means of energy loss in long-distance power transmission.

Figure 14.10 This high-voltage traction power line running to Eutingen Railway Substation in Germany radiates electromagnetic waves with very long wavelengths. (credit: Zonk43, Wikimedia Commons)

There is an ongoing controversy regarding potential health hazards associated with exposure to these electromagnetic fields (E-fields). Some people suspect that living near such transmission lines may cause a variety of illnesses, including cancer. But demographic data are either inconclusive or simply do not support the hazard theory. Recent reports that have looked at many European and American epidemiological studies have found no increase in risk for cancer due to exposure to E-fields.

Extremely low frequency (ELF) radio waves of about 1 kHz are used to communicate with submerged submarines. The ability of radio waves to penetrate salt water is related to their wavelength (much like ultrasound penetrating tissue)—the longer the wavelength, the farther they penetrate. Since salt water is a good conductor, radio waves are strongly absorbed by it, and very long wavelengths are needed to reach a submarine under the surface. (See Figure 14.11.)
Very long wavelength radio waves are needed to reach this submarine, requiring extremely low frequency signals (ELF). Shorter wavelengths do not penetrate to any significant depth.

AM radio waves are used to carry commercial radio signals in the frequency range from 540 to 1600 kHz. The abbreviation AM stands for **amplitude modulation**, which is the method for placing information on these waves. (See Figure 14.12.) A **carrier wave** having the basic frequency of the radio station, say 1530 kHz, is varied or modulated in amplitude by an audio signal. The resulting wave has a constant frequency, but a varying amplitude.

A radio receiver tuned to have the same resonant frequency as the carrier wave can pick up the signal, while rejecting the many other frequencies impinging on its antenna. The receiver’s circuitry is designed to respond to variations in amplitude of the carrier wave to replicate the original audio signal. That audio signal is amplified to drive a speaker or perhaps to be recorded.

**FM Radio Waves**

FM radio waves are also used for commercial radio transmission, but in the frequency range of 88 to 108 MHz. FM stands for **frequency modulation**, another method of carrying information. (See Figure 14.13.) Here a carrier wave having the basic frequency of the radio station, perhaps 105.1 MHz, is modulated in frequency by the audio signal, producing a wave of constant amplitude but varying frequency.

Since audible frequencies range up to 20 kHz (or 0.020 MHz) at most, the frequency of the FM radio wave can vary from the carrier by as much as 0.020 MHz. Thus the carrier frequencies of two different radio stations cannot be closer than 0.020 MHz. An FM receiver is tuned to resonate at the carrier frequency and has circuitry that responds to variations in frequency, reproducing the audio information.
FM radio is inherently less subject to noise from stray radio sources than AM radio. The reason is that amplitudes of waves add. So an AM receiver would interpret noise added onto the amplitude of its carrier wave as part of the information. An FM receiver can be made to reject amplitudes other than that of the basic carrier wave and only look for variations in frequency. It is thus easier to reject noise from FM, since noise produces a variation in amplitude.

Television is also broadcast on electromagnetic waves. Since the waves must carry a great deal of visual as well as audio information, each channel requires a larger range of frequencies than simple radio transmission. TV channels utilize frequencies in the range of 54 to 88 MHz and 174 to 222 MHz. (The entire FM radio band lies between channels 88 MHz and 174 MHz.) These TV channels are called VHF (for very high frequency). Other channels called UHF (for ultra high frequency) utilize an even higher frequency range of 470 to 1000 MHz.

The TV video signal is AM, while the TV audio is FM. Note that these frequencies are those of free transmission with the user utilizing an old-fashioned roof antenna. Satellite dishes and cable transmission of TV occurs at significantly higher frequencies and is rapidly evolving with the use of the high-definition or HD format.

### Example 14.2 Calculating Wavelengths of Radio Waves

Calculate the wavelengths of a 1530-kHz AM radio signal, a 105.1-MHz FM radio signal, and a 1.90-GHz cell phone signal.

**Strategy**

The relationship between wavelength and frequency is $\lambda = \frac{c}{f}$, where $c = 3.00 \times 10^8 \text{ m/s}$ is the speed of light (the speed of light is only very slightly smaller in air than it is in a vacuum). We can rearrange this equation to find the wavelength for all three frequencies.

**Solution**

Rearranging gives

$$\lambda = \frac{c}{f} \quad (14.7)$$

(a) For the $f = 1530$ kHz AM radio signal, then,

$$\lambda = \frac{3.00 \times 10^8 \text{ m/s}}{1530 \times 10^3 \text{ cycles/s}} = 196 \text{ m}. \quad (14.8)$$

(b) For the $f = 105.1$ MHz FM radio signal,

$$\lambda = \frac{3.00 \times 10^8 \text{ m/s}}{105.1 \times 10^6 \text{ cycles/s}} = 2.85 \text{ m}. \quad (14.9)$$

(c) And for the $f = 1.90$ GHz cell phone,

$$\lambda = \frac{3.00 \times 10^8 \text{ m/s}}{1.90 \times 10^9 \text{ cycles/s}} = 0.158 \text{ m}. \quad (14.10)$$

**Discussion**

These wavelengths are consistent with the spectrum in Figure 14.9. The wavelengths are also related to other properties of these electromagnetic waves, as we shall see.

The wavelengths found in the preceding example are representative of AM, FM, and cell phones, and account for some of the differences in how they are broadcast and how well they travel. The most efficient length for a linear antenna, such as discussed in Production of Electromagnetic Waves, is $\lambda / 2$, half the wavelength of the electromagnetic wave. Thus a very large antenna is needed to efficiently broadcast typical AM radio with its carrier wavelengths on the order of hundreds of meters.

One benefit to these long AM wavelengths is that they can go over and around rather large obstacles (like buildings and hills), just as ocean waves can go around large rocks. FM and TV are best received when there is a line of sight between the broadcast antenna and receiver, and they are often sent from very tall structures. FM, TV, and mobile phone antennas themselves are much smaller than those used for AM, but they are elevated to achieve an unobstructed line of sight. (See Figure 14.14.)
Radio Wave Interference

Astronomers and astrophysicists collect signals from outer space using electromagnetic waves. A common problem for astrophysicists is the “pollution” from electromagnetic radiation pervading our surroundings from communication systems in general. Even everyday gadgets like our car keys having the facility to lock car doors remotely and being able to turn TVs on and off using remotes involve radio-wave frequencies. In order to prevent interference between all these electromagnetic signals, strict regulations are drawn up for different organizations to utilize different radio frequency bands.

One reason why we are sometimes asked to switch off our mobile phones (operating in the range of 1.9 GHz) on airplanes and in hospitals is that important communications or medical equipment often uses similar radio frequencies and their operation can be affected by frequencies used in the communication devices.

For example, radio waves used in magnetic resonance imaging (MRI) have frequencies on the order of 100 MHz, although this varies significantly depending on the strength of the magnetic field used and the nuclear type being scanned. MRI is an important medical imaging and research tool, producing highly detailed two- and three-dimensional images. Radio waves are broadcast, absorbed, and reemitted in a resonance process that is sensitive to the density of nuclei (usually protons or hydrogen nuclei).

The wavelength of 100-MHz radio waves is 3 m, yet using the sensitivity of the resonant frequency to the magnetic field strength, details smaller than a millimeter can be imaged. This is a good example of an exception to a rule of thumb (in this case, the rubric that details much smaller than the probe’s wavelength cannot be detected). The intensity of the radio waves used in MRI presents little or no hazard to human health.

Microwaves

Microwaves are the highest-frequency electromagnetic waves that can be produced by currents in macroscopic circuits and devices. Microwave frequencies range from about $10^9$ Hz to the highest practical $LC$ resonance at nearly $10^{12}$ Hz. Since they have high frequencies, their wavelengths are short compared with those of other radio waves—hence the name “microwave.”

Microwaves can also be produced by atoms and molecules. They are, for example, a component of electromagnetic radiation generated by thermal agitation. The thermal motion of atoms and molecules in any object at a temperature above absolute zero causes them to emit and absorb radiation.

Since it is possible to carry more information per unit time on high frequencies, microwaves are quite suitable for communications. Most satellite-transmitted information is carried on microwaves, as are land-based long-distance transmissions. A clear line of sight between transmitter and receiver is needed because of the short wavelengths involved.

Radar is a common application of microwaves that was first developed in World War II. By detecting and timing microwave echoes, radar systems can determine the distance to objects as diverse as clouds and aircraft. A Doppler shift in the radar echo can be used to determine the speed of a car or the intensity of a rainstorm. Sophisticated radar systems are used to map the Earth and other planets, with a resolution limited by wavelength. (See Figure 14.15.) The shorter the wavelength of any probe, the smaller the detail it is possible to observe.
Heating with Microwaves

How does the ubiquitous microwave oven produce microwaves electronically, and why does food absorb them preferentially? Microwaves at a frequency of 2.45 GHz are produced by accelerating electrons. The microwaves are then used to induce an alternating electric field in the oven.

Water and some other constituents of food have a slightly negative charge at one end and a slightly positive charge at one end (called polar molecules). The range of microwave frequencies is specially selected so that the polar molecules, in trying to keep orienting themselves with the electric field, absorb these energies and increase their temperatures—called dielectric heating.

The energy thereby absorbed results in thermal agitation heating food and not the plate, which does not contain water. Hot spots in the food are related to constructive and destructive interference patterns. Rotating antennas and food turntables help spread out the hot spots.

Another use of microwaves for heating is within the human body. Microwaves will penetrate more than shorter wavelengths into tissue and so can accomplish “deep heating” (called microwave diathermy). This is used for treating muscular pains, spasms, tendinitis, and rheumatoid arthritis.

Making Connections: Take-Home Experiment—Microwave Ovens

1. Look at the door of a microwave oven. Describe the structure of the door. Why is there a metal grid on the door? How does the size of the holes in the grid compare with the wavelengths of microwaves used in microwave ovens? What is this wavelength?

2. Place a glass of water (about 250 ml) in the microwave and heat it for 30 seconds. Measure the temperature gain (the $\Delta T$). Assuming that the power output of the oven is 1000 W, calculate the efficiency of the heat-transfer process.

3. Remove the rotating turntable or moving plate and place a cup of water in several places along a line parallel with the opening. Heat for 30 seconds and measure the $\Delta T$ for each position. Do you see cases of destructive interference?

Microwaves generated by atoms and molecules far away in time and space can be received and detected by electronic circuits. Deep space acts like a blackbody with a 2.7 K temperature, radiating most of its energy in the microwave frequency range. In 1964, Penzias and Wilson detected this radiation and eventually recognized that it was the radiation of the Big Bang’s cooled remnants.

Infrared Radiation

The microwave and infrared regions of the electromagnetic spectrum overlap (see Figure 14.9). Infrared radiation is generally produced by thermal motion and the vibration and rotation of atoms and molecules. Electronic transitions in atoms and molecules can also produce infrared radiation.

The range of infrared frequencies extends up to the lower limit of visible light, just below red. In fact, infrared means “below red.” Frequencies at its upper limit are too high to be produced by accelerating electrons in circuits, but small systems, such as atoms and molecules, can vibrate fast enough to produce these waves.

Water molecules rotate and vibrate particularly well at infrared frequencies, emitting and absorbing them so efficiently that the emissivity for skin is $\varepsilon = 0.97$ in the infrared. Night-vision scopes can detect the infrared emitted by various warm objects, including humans, and convert it to visible light.

We can examine radiant heat transfer from a house by using a camera capable of detecting infrared radiation. Reconnaissance satellites can detect buildings, vehicles, and even individual humans by their infrared emissions, whose power radiation is proportional to the fourth power of the absolute temperature. More mundanely, we use infrared lamps, some of which are called quartz heaters, to preferentially warm us because we absorb infrared better than our surroundings.

The Sun radiates like a nearly perfect blackbody (that is, it has $\varepsilon = 1$), with a 6000 K surface temperature. About half of the
solar energy arriving at the Earth is in the infrared region, with most of the rest in the visible part of the spectrum, and a relatively small amount in the ultraviolet. On average, 50 percent of the incident solar energy is absorbed by the Earth.

The relatively constant temperature of the Earth is a result of the energy balance between the incoming solar radiation and the energy radiated from the Earth. Most of the infrared radiation emitted from the Earth is absorbed by \( \text{CO}_2 \) and \( \text{H}_2\text{O} \) in the atmosphere and then radiated back to Earth or into outer space. This radiation back to Earth is known as the greenhouse effect, and it maintains the surface temperature of the Earth about 40ºC higher than it would be if there is no absorption. Some scientists think that the increased concentration of \( \text{CO}_2 \) and other greenhouse gases in the atmosphere, resulting from increases in fossil fuel burning, has increased global average temperatures.

Visible Light

Visible light is the narrow segment of the electromagnetic spectrum to which the normal human eye responds. Visible light is produced by vibrations and rotations of atoms and molecules, as well as by electronic transitions within atoms and molecules. The receivers or detectors of light largely utilize electronic transitions. We say the atoms and molecules are excited when they absorb and relax when they emit through electronic transitions.

*Figure 14.16* shows this part of the spectrum, together with the colors associated with particular pure wavelengths. We usually refer to visible light as having wavelengths of between 400 nm and 750 nm. (The retina of the eye actually responds to the lowest ultraviolet frequencies, but these do not normally reach the retina because they are absorbed by the cornea and lens of the eye.)

Red light has the lowest frequencies and longest wavelengths, while violet has the highest frequencies and shortest wavelengths. Blackbody radiation from the Sun peaks in the visible part of the spectrum but is more intense in the red than in the violet, making the Sun yellowish in appearance.

![Visible light spectrum](image)

*Figure 14.16* A small part of the electromagnetic spectrum that includes its visible components. The divisions between infrared, visible, and ultraviolet are not perfectly distinct, nor are those between the seven rainbow colors.

Living things—plants and animals—have evolved to utilize and respond to parts of the electromagnetic spectrum they are embedded in. Visible light is the most predominant and we enjoy the beauty of nature through visible light. Plants are more selective. Photosynthesis makes use of parts of the visible spectrum to make sugars.

**Example 14.3 Integrated Concept Problem: Correcting Vision with Lasers**

During laser vision correction, a brief burst of 193-nm ultraviolet light is projected onto the cornea of a patient. It makes a spot 0.80 mm in diameter and evaporates a layer of cornea 0.30 \( \mu\text{m} \) thick. Calculate the energy absorbed, assuming the corneal tissue has the same properties as water; it is initially at 34ºC. Assume the evaporated tissue leaves at a temperature of 100ºC.

**Strategy**

The energy from the laser light goes toward raising the temperature of the tissue and also toward evaporating it. Thus we have two amounts of heat to add together. Also, we need to find the mass of corneal tissue involved.

**Solution**

To figure out the heat required to raise the temperature of the tissue to 100ºC, we can apply concepts of thermal energy. We know that

\[
Q = mc\Delta T, \tag{14.11}
\]

where \( Q \) is the heat required to raise the temperature, \( \Delta T \) is the desired change in temperature, \( m \) is the mass of tissue to be heated, and \( c \) is the specific heat of water equal to 4186 J/kg/K.

Without knowing the mass \( m \) at this point, we have

\[
Q = m(4186 \text{ J/kg/K})(100^\circ\text{C} - 34^\circ\text{C}) = m(276.276 \text{ J/kg}) = m(276 \text{ kJ/kg}). \tag{14.12}
\]

The latent heat of vaporization of water is 2256 kJ/kg, so that the energy needed to evaporate mass \( m \) is

\[
Q_v = mL_v = m(2256 \text{ kJ/kg}). \tag{14.13}
\]

To find the mass \( m \), we use the equation \( \rho = m/V \), where \( \rho \) is the density of the tissue and \( V \) is its volume. For this
\[ m = \rho V \]
\[ = (1000 \text{ kg/m}^3)(\text{area} \times \text{thickness}(m^3)) \]
\[ = (1000 \text{ kg/m}^3)\pi(0.80 \times 10^{-3} \text{ m})^2/(0.30 \times 10^{-6} \text{ m}) \]
\[ = 0.151 \times 10^{-9} \text{ kg}. \]

Therefore, the total energy absorbed by the tissue in the eye is the sum of \( Q \) and \( Q_v \) :
\[ Q_{\text{tot}} = m(c\Delta T + L_v) = (0.151 \times 10^{-9} \text{ kg})(276 \text{ kJ/kg} + 2256 \text{ kJ/kg}) = 382 \times 10^{-9} \text{ kJ}. \]

Discussion

The lasers used for this eye surgery are excimer lasers, whose light is well absorbed by biological tissue. They evaporate rather than burn the tissue, and can be used for precision work. Most lasers used for this type of eye surgery have an average power rating of about one watt. For our example, if we assume that each laser burst from this pulsed laser lasts for 10 ns, and there are 400 bursts per second, then the average power is \( Q_{\text{tot}} \times 400 = 150 \text{ mW} \).

Optics is the study of the behavior of visible light and other forms of electromagnetic waves. Optics falls into two distinct categories. When electromagnetic radiation, such as visible light, interacts with objects that are large compared with its wavelength, its motion can be represented by straight lines like rays. Ray optics is the study of such situations and includes lenses and mirrors.

When electromagnetic radiation interacts with objects about the same size as the wavelength or smaller, its wave nature becomes apparent. For example, observable detail is limited by the wavelength, and so visible light can never detect individual atoms, because they are so much smaller than its wavelength. Physical or wave optics is the study of such situations and includes all wave characteristics.

**Take-Home Experiment: Colors That Match**

When you light a match you see largely orange light; when you light a gas stove you see blue light. Why are the colors different? What other colors are present in these?

**Ultraviolet Radiation**

Ultraviolet means “above violet.” The electromagnetic frequencies of ultraviolet radiation (UV) extend upward from violet, the highest-frequency visible light. Ultraviolet is also produced by atomic and molecular motions and electronic transitions. The wavelengths of ultraviolet extend from 400 nm down to about 10 nm at its highest frequencies, which overlap with the lowest X-ray frequencies. It was recognized as early as 1801 by Johann Ritter that the solar spectrum had an invisible component beyond the violet range.

Solar UV radiation is broadly subdivided into three regions: UV-A (320–400 nm), UV-B (290–320 nm), and UV-C (220–290 nm), ranked from long to shorter wavelengths (from smaller to larger energies). Most UV-B and all UV-C is absorbed by ozone (O\(_3\)) molecules in the upper atmosphere. Consequently, 99% of the solar UV radiation reaching the Earth’s surface is UV-A.

**Human Exposure to UV Radiation**

It is largely exposure to UV-B that causes skin cancer. It is estimated that as many as 20% of adults will develop skin cancer over the course of their lifetime. Again, treatment is often successful if caught early. Despite very little UV-B reaching the Earth’s surface, there are substantial increases in skin-cancer rates in countries such as Australia, indicating how important it is that UV-B and UV-C continue to be absorbed by the upper atmosphere.

All UV radiation can damage collagen fibers, resulting in an acceleration of the aging process of skin and the formation of wrinkles. Because there is so little UV-B and UV-C reaching the Earth’s surface, sunburn is caused by large exposures, and skin cancer from repeated exposure. Some studies indicate a link between overexposure to the Sun when young and melanoma later in life.

The tanning response is a defense mechanism in which the body produces pigments to absorb future exposures in inert skin layers above living cells. Basically UV-B radiation excites DNA molecules, distorting the DNA helix, leading to mutations and the possible formation of cancerous cells.

Repeated exposure to UV-B may also lead to the formation of cataracts in the eyes—a cause of blindness among people living in the equatorial belt where medical treatment is limited. Cataracts, clouding in the eye’s lens and a loss of vision, are age related; 60% of those between the ages of 65 and 74 will develop cataracts. However, treatment is easy and successful, as one replaces the lens of the eye with a plastic lens. Prevention is important. Eye protection from UV is more effective with plastic sunglasses than those made of glass.

A major acute effect of extreme UV exposure is the suppression of the immune system, both locally and throughout the body.
Low-intensity ultraviolet is used to sterilize haircutting implements, implying that the energy associated with ultraviolet is deposited in a manner different from lower-frequency electromagnetic waves. (Actually this is true for all electromagnetic waves with frequencies greater than visible light.)

Flash photography is generally not allowed of precious artworks and colored prints because the UV radiation from the flash can cause photo-degradation in the artworks. Often artworks will have an extra-thick layer of glass in front of them, which is especially designed to absorb UV radiation.

**UV Light and the Ozone Layer**

If all of the Sun’s ultraviolet radiation reached the Earth’s surface, there would be extremely grave effects on the biosphere from the severe cell damage it causes. However, the layer of ozone ($O_3$) in our upper atmosphere (10 to 50 km above the Earth) protects life by absorbing most of the dangerous UV radiation.

Unfortunately, today we are observing a depletion in ozone concentrations in the upper atmosphere. This depletion has led to the formation of an “ozone hole” in the upper atmosphere. The hole is more centered over the southern hemisphere, and changes with the seasons, being largest in the spring. This depletion is attributed to the breakdown of ozone molecules by refrigerant gases called chlorofluorocarbons (CFCs).

The UV radiation helps dissociate the CFC’s, releasing highly reactive chlorine (Cl) atoms, which catalyze the destruction of the ozone layer. For example, the reaction of $\text{CFCl}_3$ with a photon of light ($h\nu$) can be written as:

$$\text{CFCl}_3 + h\nu \rightarrow \text{CFCl}_2 + \text{Cl}.$$  \hspace{1cm} (14.16)

The Cl atom then catalyzes the breakdown of ozone as follows:

$$\text{Cl} + \text{O}_3 \rightarrow \text{ClO} + \text{O}_2 \text{ and } \text{ClO} + \text{O}_3 \rightarrow \text{Cl} + 2\text{O}_2.$$  \hspace{1cm} (14.17)

A single chlorine atom could destroy ozone molecules for up to two years before being transported down to the surface. The CFCs are relatively stable and will contribute to ozone depletion for years to come. CFCs are found in refrigerants, air conditioning systems, foams, and aerosols.

International concern over this problem led to the establishment of the “Montreal Protocol” agreement (1987) to phase out CFC production in most countries. However, developing-country participation is needed if worldwide production and elimination of CFCs is to be achieved. Probably the largest contributor to CFC emissions today is India. But the protocol seems to be working, as there are signs of an ozone recovery. (See Figure 14.17.)

![Figure 14.17](https://legacy.cnx.org/content/col11951/1.1)

**Benefits of UV Light**

Besides the adverse effects of ultraviolet radiation, there are also benefits of exposure in nature and uses in technology. Vitamin D production in the skin (epidermis) results from exposure to UVB radiation, generally from sunlight. A number of studies indicate lack of vitamin D can result in the development of a range of cancers (prostate, breast, colon), so a certain amount of UV exposure is helpful. Lack of vitamin D is also linked to osteoporosis. Exposures (with no sunscreen) of 10 minutes a day to arms, face, and legs might be sufficient to provide the accepted dietary level. However, in the winter time north of about 37º latitude, most UVB gets blocked by the atmosphere.

UV radiation is used in the treatment of infantile jaundice and in some skin conditions. It is also used in sterilizing workspaces and tools, and killing germs in a wide range of applications. It is also used as an analytical tool to identify substances.

When exposed to ultraviolet, some substances, such as minerals, glow in characteristic visible wavelengths, a process called
fluorescence. So-called black lights emit ultraviolet to cause posters and clothing to fluoresce in the visible. Ultraviolet is also used in special microscopes to detect details smaller than those observable with longer-wavelength visible-light microscopes.

### Things Great and Small: A Submicroscopic View of X-Ray Production

X-rays can be created in a high-voltage discharge. They are emitted in the material struck by electrons in the discharge current. There are two mechanisms by which the electrons create X-rays.

The first method is illustrated in Figure 14.18. An electron is accelerated in an evacuated tube by a high positive voltage. The electron strikes a metal plate (e.g., copper) and produces X-rays. Since this is a high-voltage discharge, the electron gains sufficient energy to ionize the atom.

![Figure 14.18](image)

**Figure 14.18** Artist’s conception of an electron ionizing an atom followed by the recapture of an electron and emission of an X-ray. An energetic electron strikes an atom and knocks an electron out of one of the orbits closest to the nucleus. Later, the atom captures another electron, and the energy released by its fall into a low orbit generates a high-energy EM wave called an X-ray.

In the case shown, an inner-shell electron (one in an orbit relatively close to and tightly bound to the nucleus) is ejected. A short time later, another electron is captured and falls into the orbit in a single great plunge. The energy released by this fall is given to an EM wave known as an X-ray. Since the orbits of the atom are unique to the type of atom, the energy of the X-ray is characteristic of the atom, hence the name characteristic X-ray.

The second method by which an energetic electron creates an X-ray when it strikes a material is illustrated in Figure 14.19. The electron interacts with charges in the material as it penetrates. These collisions transfer kinetic energy from the electron to the electrons and atoms in the material.

![Figure 14.19](image)

**Figure 14.19** Artist’s conception of an electron being slowed by collisions in a material and emitting X-ray radiation. This energetic electron makes numerous collisions with electrons and atoms in a material it penetrates. An accelerated charge radiates EM waves, a second method by which X-rays are created.

A loss of kinetic energy implies an acceleration, in this case decreasing the electron’s velocity. Whenever a charge is accelerated, it radiates EM waves. Given the high energy of the electron, these EM waves can have high energy. We call them X-rays. Since the process is random, a broad spectrum of X-ray energy is emitted that is more characteristic of the electron energy than the type of material the electron encounters. Such EM radiation is called “bremsstrahlung” (German for “braking radiation”).
X-Rays

In the 1850s, scientists (such as Faraday) began experimenting with high-voltage electrical discharges in tubes filled with rarefied gases. It was later found that these discharges created an invisible, penetrating form of very high frequency electromagnetic radiation. This radiation was called an X-ray, because its identity and nature were unknown.

As described in Things Great and Small, there are two methods by which X-rays are created—both are submicroscopic processes and can be caused by high-voltage discharges. While the low-frequency end of the X-ray range overlaps with the ultraviolet, X-rays extend to much higher frequencies (and energies).

X-rays have adverse effects on living cells similar to those of ultraviolet radiation, and they have the additional liability of being more penetrating, affecting more than the surface layers of cells. Cancer and genetic defects can be induced by exposure to X-rays. Because of their effect on rapidly dividing cells, X-rays can also be used to treat and even cure cancer.

The widest use of X-rays is for imaging objects that are opaque to visible light, such as the human body or aircraft parts. In humans, the risk of cell damage is weighed carefully against the benefit of the diagnostic information obtained. However, questions have risen in recent years as to accidental overexposure of some people during CT scans—a mistake at least in part due to poor monitoring of radiation dose.

The ability of X-rays to penetrate matter depends on density, and so an X-ray image can reveal very detailed density information. Figure 14.20 shows an example of the simplest type of X-ray image, an X-ray shadow on film. The amount of information in a simple X-ray image is impressive, but more sophisticated techniques, such as CT scans, can reveal three-dimensional information with details smaller than a millimeter.

![Figure 14.20 This shadow X-ray image shows many interesting features, such as artificial heart valves, a pacemaker, and the wires used to close the sternum. (credit: P. P. Urone)](image)

The use of X-ray technology in medicine is called radiology—an established and relatively cheap tool in comparison to more sophisticated technologies. Consequently, X-rays are widely available and used extensively in medical diagnostics. During World War I, mobile X-ray units, advocated by Madame Marie Curie, were used to diagnose soldiers.

Because they can have wavelengths less than 0.01 nm, X-rays can be scattered (a process called X-ray diffraction) to detect the shape of molecules and the structure of crystals. X-ray diffraction was crucial to Crick, Watson, and Wilkins in the determination of the shape of the double-helix DNA molecule.

X-rays are also used as a precise tool for trace-metal analysis in X-ray induced fluorescence, in which the energy of the X-ray emissions are related to the specific types of elements and amounts of materials present.

Gamma Rays

Soon after nuclear radioactivity was first detected in 1896, it was found that at least three distinct types of radiation were being emitted. The most penetrating nuclear radiation was called a gamma ray (γ ray) (again a name given because its identity and character were unknown), and it was later found to be an extremely high frequency electromagnetic wave.

In fact, γ rays are any electromagnetic radiation emitted by a nucleus. This can be from natural nuclear decay or induced nuclear processes in nuclear reactors and weapons. The lower end of the γ-ray frequency range overlaps the upper end of the X-ray range, but γ rays can have the highest frequency of any electromagnetic radiation.

Gamma rays have characteristics identical to X-rays of the same frequency—they differ only in source. At higher frequencies, γ rays are more penetrating and more damaging to living tissue. They have many of the same uses as X-rays, including cancer therapy. Gamma radiation from radioactive materials is used in nuclear medicine.

Figure 14.21 shows a medical image based on γ rays. Food spoilage can be greatly inhibited by exposing it to large doses of γ radiation, thereby obliterating responsible microorganisms. Damage to food cells through irradiation occurs as well, and the long-term hazards of consuming radiation-preserved food are unknown and controversial for some groups. Both X-ray and γ-ray technologies are also used in scanning luggage at airports.
Detecting Electromagnetic Waves from Space

A final note on star gazing. The entire electromagnetic spectrum is used by researchers for investigating stars, space, and time. As noted earlier, Penzias and Wilson detected microwaves to identify the background radiation originating from the Big Bang. Radio telescopes such as the Arecibo Radio Telescope in Puerto Rico and Parkes Observatory in Australia were designed to detect radio waves.

Infrared telescopes need to have their detectors cooled by liquid nitrogen to be able to gather useful signals. Since infrared radiation is predominantly from thermal agitation, if the detectors were not cooled, the vibrations of the molecules in the antenna would be stronger than the signal being collected.

The most famous of these infrared sensitive telescopes is the James Clerk Maxwell Telescope in Hawaii. The earliest telescopes, developed in the seventeenth century, were optical telescopes, collecting visible light. Telescopes in the ultraviolet, X-ray, and $\gamma$-ray regions are placed outside the atmosphere on satellites orbiting the Earth.

The Hubble Space Telescope (launched in 1990) gathers ultraviolet radiation as well as visible light. In the X-ray region, there is the Chandra X-ray Observatory (launched in 1999), and in the $\gamma$-ray region, there is the new Fermi Gamma-ray Space Telescope (launched in 2008—taking the place of the Compton Gamma Ray Observatory, 1991–2000.).

Color Vision

Make a whole rainbow by mixing red, green, and blue light. Change the wavelength of a monochromatic beam or filter white light. View the light as a solid beam, or see the individual photons.

(This media type is not supported in this reader. Click to open media in browser.) (http://legacy.cnx.org/content/m42444/1.15/#fs-id1167064871798)
14.4 Energy in Electromagnetic Waves

Anyone who has used a microwave oven knows there is energy in electromagnetic waves. Sometimes this energy is obvious, such as in the warmth of the summer sun. Other times it is subtle, such as the unfelt energy of gamma rays, which can destroy living cells.

Electromagnetic waves can bring energy into a system by virtue of their electric and magnetic fields. These fields can exert forces and move charges in the system and, thus, do work on them. If the frequency of the electromagnetic wave is the same as the natural frequencies of the system (such as microwaves at the resonant frequency of water molecules), the transfer of energy is much more efficient.

Connections: Waves and Particles

The behavior of electromagnetic radiation clearly exhibits wave characteristics. But we shall find in later modules that at high frequencies, electromagnetic radiation also exhibits particle characteristics. These particle characteristics will be used to explain more of the properties of the electromagnetic spectrum and to introduce the formal study of modern physics.

Another startling discovery of modern physics is that particles, such as electrons and protons, exhibit wave characteristics. This simultaneous sharing of wave and particle properties for all submicroscopic entities is one of the great symmetries in nature.

Figure 14.23 Energy carried by a wave is proportional to its amplitude squared. With electromagnetic waves, larger $E$-fields and $B$-fields exert larger forces and can do more work.

But there is energy in an electromagnetic wave, whether it is absorbed or not. Once created, the fields carry energy away from a source. If absorbed, the field strengths are diminished and anything left travels on. Clearly, the larger the strength of the electric and magnetic fields, the more work they can do and the greater the energy the electromagnetic wave carries.

A wave’s energy is proportional to its amplitude squared ($E^2$ or $B^2$). This is true for waves on guitar strings, for water waves, and for sound waves, where amplitude is proportional to pressure. In electromagnetic waves, the amplitude is the maximum field strength of the electric and magnetic fields. (See Figure 14.23.)

Thus the energy carried and the intensity $I$ of an electromagnetic wave is proportional to $E^2$ and $B^2$. In fact, for a continuous sinusoidal electromagnetic wave, the average intensity $I_{\text{ave}}$ is given by

$$I_{\text{ave}} = \frac{c\varepsilon_0 E_0^2}{2},$$

where $c$ is the speed of light, $\varepsilon_0$ is the permittivity of free space, and $E_0$ is the maximum electric field strength; intensity, as always, is power per unit area (here in $\text{W/m}^2$).

The average intensity of an electromagnetic wave $I_{\text{ave}}$ can also be expressed in terms of the magnetic field strength by using the relationship $B = E/c$, and the fact that $\varepsilon_0 = 1/\mu_0 c^2$, where $\mu_0$ is the permeability of free space. Algebraic manipulation produces the relationship

$$I_{\text{ave}} = \frac{cB_0^2}{2\mu_0},$$

where $B_0$ is the maximum magnetic field strength.

One more expression for $I_{\text{ave}}$ in terms of both electric and magnetic field strengths is useful. Substituting the fact that $c \cdot B_0 = E_0$, the previous expression becomes
\[ I_{\text{ave}} = \frac{E_0 B_0}{2 \mu_0}. \]  

(14.20)

Whichever of the three preceding equations is most convenient can be used, since they are really just different versions of the same principle: Energy in a wave is related to amplitude squared. Furthermore, since these equations are based on the assumption that the electromagnetic waves are sinusoidal, peak intensity is twice the average; that is, \( I_0 = 2I_{\text{ave}} \).

Example 14.4 Calculate Microwave Intensities and Fields

On its highest power setting, a certain microwave oven projects 1.00 kW of microwaves onto a 30.0 by 40.0 cm area. (a) What is the intensity in W/m\(^2\)? (b) Calculate the peak electric field strength \( E_0 \) in these waves. (c) What is the peak magnetic field strength \( B_0 \)?

Strategy

In part (a), we can find intensity from its definition as power per unit area. Once the intensity is known, we can use the equations below to find the field strengths asked for in parts (b) and (c).

Solution for (a)

Entering the given power into the definition of intensity, and noting the area is 0.300 by 0.400 m, yields

\[ I = \frac{P}{A} = \frac{1.00 \text{ kW}}{0.300 \text{ m} \times 0.400 \text{ m}}. \]  

(14.21)

Here \( I = I_{\text{ave}} \), so that

\[ I_{\text{ave}} = \frac{1000 \text{ W}}{0.120 \text{ m}^2} = 8.33 \times 10^3 \text{ W/m}^2. \]  

(14.22)

Note that the peak intensity is twice the average:

\[ I_0 = 2I_{\text{ave}} = 1.67 \times 10^4 \text{ W/m}^2. \]  

(14.23)

Solution for (b)

To find \( E_0 \), we can rearrange the first equation given above for \( I_{\text{ave}} \) to give

\[ E_0 = \left( \frac{2I_{\text{ave}}}{c \epsilon_0} \right)^{1/2}. \]  

(14.24)

Entering known values gives

\[ E_0 = \left( \frac{2(8.33 \times 10^3 \text{ W/m}^2)}{(3.00 \times 10^8 \text{ m/s})(8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)} \right)^{1/2} = 2.51 \times 10^3 \text{ V/m}. \]  

(14.25)

Solution for (c)

Perhaps the easiest way to find magnetic field strength, now that the electric field strength is known, is to use the relationship given by

\[ B_0 = \frac{E_0}{c}. \]  

(14.26)

Entering known values gives

\[ B_0 = \frac{2.51 \times 10^3 \text{ V/m}}{3.0 \times 10^8 \text{ m/s}} = 8.35 \times 10^{-6} \text{ T}. \]  

(14.27)

Discussion

As before, a relatively strong electric field is accompanied by a relatively weak magnetic field in an electromagnetic wave, since \( B = E/c \), and \( c \) is a large number.

Glossary

amplitude: the height, or magnitude, of an electromagnetic wave
amplitude modulation (AM): a method for placing information on electromagnetic waves by modulating the amplitude of a carrier wave with an audio signal, resulting in a wave with constant frequency but varying amplitude

carrier wave: an electromagnetic wave that carries a signal by modulation of its amplitude or frequency

electric field: a vector quantity (E); the lines of electric force per unit charge, moving radially outward from a positive charge and toward a negative charge

electric field lines: a pattern of imaginary lines that extend between an electric source and charged objects in the surrounding area, with arrows pointed away from positively charged objects and toward negatively charged objects. The more lines in the pattern, the stronger the electric field in that region

electric field strength: the magnitude of the electric field, denoted E-field

electromagnetic spectrum: the full range of wavelengths or frequencies of electromagnetic radiation

electromagnetic waves: radiation in the form of waves of electric and magnetic energy

electromotive force (emf): energy produced per unit charge, drawn from a source that produces an electrical current

extremely low frequency (ELF): electromagnetic radiation with wavelengths usually in the range of 0 to 300 Hz, but also about 1kHz

frequency: the number of complete wave cycles (up-down-up) passing a given point within one second (cycles/second)

frequency modulation (FM): a method of placing information on electromagnetic waves by modulating the frequency of a carrier wave with an audio signal, producing a wave of constant amplitude but varying frequency

gamma ray: (γ ray); extremely high frequency electromagnetic radiation emitted by the nucleus of an atom, either from natural nuclear decay or induced nuclear processes in nuclear reactors and weapons. The lower end of the γ-ray frequency range overlaps the upper end of the X-ray range, but γ rays can have the highest frequency of any electromagnetic radiation

hertz: an SI unit denoting the frequency of an electromagnetic wave, in cycles per second

infrared radiation (IR): a region of the electromagnetic spectrum with a frequency range that extends from just below the red region of the visible light spectrum up to the microwave region, or from 0.74 μm to 300 μm

intensity: the power of an electric or magnetic field per unit area, for example, Watts per square meter

magnetic field: a vector quantity (B); can be used to determine the magnetic force on a moving charged particle

magnetic field lines: a pattern of continuous, imaginary lines that emerge from and enter into opposite magnetic poles. The density of the lines indicates the magnitude of the magnetic field

magnetic field strength: the magnitude of the magnetic field, denoted B-field

maximum field strength: the maximum amplitude an electromagnetic wave can reach, representing the maximum amount of electric force and/or magnetic flux that the wave can exert

Maxwell’s equations: a set of four equations that comprise a complete, overarching theory of electromagnetism

microwaves: electromagnetic waves with wavelengths in the range from 1 mm to 1 m; they can be produced by currents in macroscopic circuits and devices

oscillate: to fluctuate back and forth in a steady beat

radar: a common application of microwaves. Radar can determine the distance to objects as diverse as clouds and aircraft, as well as determine the speed of a car or the intensity of a rainstorm

radio waves: electromagnetic waves with wavelengths in the range from 1 mm to 100 km; they are produced by currents in wires and circuits and by astronomical phenomena

resonant: a system that displays enhanced oscillation when subjected to a periodic disturbance of the same frequency as its natural frequency

RLC circuit: an electric circuit that includes a resistor, capacitor and inductor

speed of light: in a vacuum, such as space, the speed of light is a constant 3 x 10^8 m/s

standing wave: a wave that oscillates in place, with nodes where no motion happens
thermal agitation: the thermal motion of atoms and molecules in any object at a temperature above absolute zero, which causes them to emit and absorb radiation

transverse wave: a wave, such as an electromagnetic wave, which oscillates perpendicular to the axis along the line of travel

TV: video and audio signals broadcast on electromagnetic waves

ultra-high frequency (UHF): TV channels in an even higher frequency range than VHF, of 470 to 1000 MHz

ultraviolet radiation (UV): electromagnetic radiation in the range extending upward in frequency from violet light and overlapping with the lowest X-ray frequencies, with wavelengths from 400 nm down to about 10 nm

very high frequency (VHF): TV channels utilizing frequencies in the two ranges of 54 to 88 MHz and 174 to 222 MHz

visible light: the narrow segment of the electromagnetic spectrum to which the normal human eye responds

wavelength: the distance from one peak to the next in a wave

X-ray: invisible, penetrating form of very high frequency electromagnetic radiation, overlapping both the ultraviolet range and the γ-ray range

Section Summary

14.1 Maxwell’s Equations: Electromagnetic Waves Predicted and Observed
- Electromagnetic waves consist of oscillating electric and magnetic fields and propagate at the speed of light \( c \). They were predicted by Maxwell, who also showed that
  \[
  c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}
  \]
  where \( \mu_0 \) is the permeability of free space and \( \varepsilon_0 \) is the permittivity of free space.
- Maxwell’s prediction of electromagnetic waves resulted from his formulation of a complete and symmetric theory of electricity and magnetism, known as Maxwell’s equations.
- These four equations are paraphrased in this text, rather than presented numerically, and encompass the major laws of electricity and magnetism. First is Gauss’s law for electricity, second is Gauss’s law for magnetism, third is Faraday’s law of induction, including Lenz’s law, and fourth is Ampere’s law in a symmetric formulation that adds another source of magnetism—changing electric fields.

14.2 Production of Electromagnetic Waves
- Electromagnetic waves are created by oscillating charges (which radiate whenever accelerated) and have the same frequency as the oscillation.
- Since the electric and magnetic fields in most electromagnetic waves are perpendicular to the direction in which the wave moves, it is ordinarily a transverse wave.
- The strengths of the electric and magnetic parts of the wave are related by
  \[
  \frac{E}{B} = c,
  \]
  which implies that the magnetic field \( B \) is very weak relative to the electric field \( E \).

14.3 The Electromagnetic Spectrum
- The relationship among the speed of propagation, wavelength, and frequency for any wave is given by \( v_W = f \lambda \), so that for electromagnetic waves,
  \[
  c = f \lambda,
  \]
  where \( f \) is the frequency, \( \lambda \) is the wavelength, and \( c \) is the speed of light.
- The electromagnetic spectrum is separated into many categories and subcategories, based on the frequency and wavelength, source, and uses of the electromagnetic waves.
- Any electromagnetic wave produced by currents in wires is classified as a radio wave, the lowest frequency electromagnetic waves. Radio waves are divided into many types, depending on their applications, ranging up to microwaves at their highest frequencies.
- Infrared radiation lies below visible light in frequency and is produced by thermal motion and the vibration and rotation of atoms and molecules. Infrared’s lower frequencies overlap with the highest-frequency microwaves.
- Visible light is largely produced by electronic transitions in atoms and molecules, and is defined as being detectable by the human eye. Its colors vary with frequency, from red at the lowest to violet at the highest.
- Ultraviolet radiation starts with frequencies just above violet in the visible range and is produced primarily by electronic transitions in atoms and molecules.
- X-rays are created in high-voltage discharges and by electron bombardment of metal targets. Their lowest frequencies
overlap the ultraviolet range but extend to much higher values, overlapping at the high end with gamma rays.

- Gamma rays are nuclear in origin and are defined to include the highest-frequency electromagnetic radiation of any type.

### 14.4 Energy in Electromagnetic Waves

- The energy carried by any wave is proportional to its amplitude squared. For electromagnetic waves, this means intensity can be expressed as

\[ I_{\text{ave}} = \frac{c\varepsilon_0 E_0^2}{2}, \]

where \( I_{\text{ave}} \) is the average intensity in \( \text{W/m}^2 \), and \( E_0 \) is the maximum electric field strength of a continuous sinusoidal wave.

- This can also be expressed in terms of the maximum magnetic field strength \( B_0 \) as

\[ I_{\text{ave}} = \frac{cB_0^2}{2\mu_0} \]

and in terms of both electric and magnetic fields as

\[ I_{\text{ave}} = \frac{E_0 B_0}{2\mu_0}. \]

- The three expressions for \( I_{\text{ave}} \) are all equivalent.

### Conceptual Questions

#### 14.2 Production of Electromagnetic Waves

1. The direction of the electric field shown in each part of Figure 14.5 is that produced by the charge distribution in the wire. Justify the direction shown in each part, using the Coulomb force law and the definition of \( E = F/q \), where \( q \) is a positive test charge.

2. Is the direction of the magnetic field shown in Figure 14.6 (a) consistent with the right-hand rule for current (RHR-2) in the direction shown in the figure?

3. Why is the direction of the current shown in each part of Figure 14.6 opposite to the electric field produced by the wire’s charge separation?

4. In which situation shown in Figure 14.24 will the electromagnetic wave be more successful in inducing a current in the wire? Explain.

![Figure 14.24](image1.png) Electromagnetic waves approaching long straight wires.

5. In which situation shown in Figure 14.25 will the electromagnetic wave be more successful in inducing a current in the loop? Explain.

![Figure 14.25](image2.png) Electromagnetic waves approaching a wire loop.
6. Should the straight wire antenna of a radio be vertical or horizontal to best receive radio waves broadcast by a vertical transmitter antenna? How should a loop antenna be aligned to best receive the signals? (Note that the direction of the loop that produces the best reception can be used to determine the location of the source. It is used for that purpose in tracking tagged animals in nature studies, for example.)

7. Under what conditions might wires in a DC circuit emit electromagnetic waves?

8. Give an example of interference of electromagnetic waves.

9. Figure 14.26 shows the interference pattern of two radio antennas broadcasting the same signal. Explain how this is analogous to the interference pattern for sound produced by two speakers. Could this be used to make a directional antenna system that broadcasts preferentially in certain directions? Explain.

10. Can an antenna be any length? Explain your answer.

14.3 The Electromagnetic Spectrum

11. If you live in a region that has a particular TV station, you can sometimes pick up some of its audio portion on your FM radio receiver. Explain how this is possible. Does it imply that TV audio is broadcast as FM?

12. Explain why people who have the lens of their eye removed because of cataracts are able to see low-frequency ultraviolet.

13. How do fluorescent soap residues make clothing look “brighter and whiter” in outdoor light? Would this be effective in candlelight?

14. Give an example of resonance in the reception of electromagnetic waves.

15. Illustrate that the size of details of an object that can be detected with electromagnetic waves is related to their wavelength, by comparing details observable with two different types (for example, radar and visible light or infrared and X-rays).

16. Why don’t buildings block radio waves as completely as they do visible light?

17. Make a list of some everyday objects and decide whether they are transparent or opaque to each of the types of electromagnetic waves.

18. Your friend says that more patterns and colors can be seen on the wings of birds if viewed in ultraviolet light. Would you agree with your friend? Explain your answer.

19. The rate at which information can be transmitted on an electromagnetic wave is proportional to the frequency of the wave. Is this consistent with the fact that laser telephone transmission at visible frequencies carries far more conversations per optical fiber than conventional electronic transmission in a wire? What is the implication for ELF radio communication with submarines?

20. Give an example of energy carried by an electromagnetic wave.

21. In an MRI scan, a higher magnetic field requires higher frequency radio waves to resonate with the nuclear type whose density and location is being imaged. What effect does going to a larger magnetic field have on the most efficient antenna to broadcast those radio waves? Does it favor a smaller or larger antenna?

22. Laser vision correction often uses an excimer laser that produces 193-nm electromagnetic radiation. This wavelength is extremely strongly absorbed by the cornea and ablates it in a manner that reshapes the cornea to correct vision defects. Explain how the strong absorption helps concentrate the energy in a thin layer and thus give greater accuracy in shaping the cornea. Also explain how this strong absorption limits damage to the lens and retina of the eye.
14.1 Maxwell’s Equations: Electromagnetic Waves Predicted and Observed

1. Verify that the correct value for the speed of light \( c \) is obtained when numerical values for the permeability and permittivity of free space (\( \mu_0 \) and \( \epsilon_0 \)) are entered into the equation \( c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \).

2. Show that, when SI units for \( \mu_0 \) and \( \epsilon_0 \) are entered, the units given by the right-hand side of the equation in the problem above are m/s.

14.2 Production of Electromagnetic Waves

3. What is the maximum electric field strength in an electromagnetic wave that has a maximum magnetic field strength of \( 5.00 \times 10^{-4} \) T (about 10 times the Earth’s)?

4. The maximum magnetic field strength of an electromagnetic field is \( 5 \times 10^{-6} \) T. Calculate the maximum electric field strength if the wave is traveling in a medium in which the speed of the wave is 0.75c.

5. Verify the units obtained for magnetic field strength \( B \) in Example 14.1 (using the equation \( B = \frac{E}{c} \)) are in fact teslas (T).

14.3 The Electromagnetic Spectrum

6. (a) Two microwave frequencies are authorized for use in microwave ovens: 900 and 2560 MHz. Calculate the wavelength of each. (b) Which frequency would produce smaller hot spots in foods due to interference effects?

7. (a) Calculate the range of wavelengths for AM radio given its frequency range is 540 to 1600 kHz. (b) Do the same for the FM frequency range of 88.0 to 108 MHz.

8. A radio station utilizes frequencies between commercial AM and FM. What is the frequency of a 11.12-m-wavelength channel?

9. Find the frequency range of visible light, given that it encompasses wavelengths from 380 to 760 nm.

10. Combing your hair leads to excess electrons on the comb. How fast would you have to move the comb up and down to produce red light?

11. Electromagnetic radiation having a 15.0-\( \mu \)m wavelength is classified as infrared radiation. What is its frequency?

12. Approximately what is the smallest detail observable with a microscope that uses ultraviolet light of frequency \( 1.20 \times 10^{15} \) Hz?

13. A radar used to detect the presence of aircraft receives a pulse that has reflected off an object \( 6 \times 10^{-5} \) s after it was transmitted. What is the distance from the radar station to the reflecting object?

14. Some radar systems detect the size and shape of objects such as aircraft and geological terrain. Approximately what is the smallest observable detail utilizing 500-MHz radar?

15. Determine the amount of time it takes for X-rays of frequency \( 3 \times 10^{18} \) Hz to travel (a) 1 mm and (b) 1 cm.

16. If you wish to detect details of the size of atoms (about \( 1 \times 10^{-10} \) m) with electromagnetic radiation, it must have a wavelength of about this size. (a) What is its frequency? (b) What type of electromagnetic radiation might this be?

17. If the Sun suddenly turned off, we would not know it until its light stopped coming. How long would that be, given that the Sun is \( 1.50 \times 10^{11} \) m away?

18. Distances in space are often quoted in units of light years, the distance light travels in one year. (a) How many meters is a light year? (b) How many meters is it to Andromeda, the nearest large galaxy, given that it is \( 2.00 \times 10^6 \) light years away? (c) The most distant galaxy yet discovered is \( 12.0 \times 10^9 \) light years away. How far is this in meters?

19. A certain 50.0-Hz AC power line radiates an electromagnetic wave having a maximum electric field strength of 13.0 kV/m. (a) What is the wavelength of this very low frequency electromagnetic wave? (b) What is its maximum magnetic field strength?

20. During normal beating, the heart creates a maximum 4.00-mV potential across 0.300 m of a person’s chest, creating a 1.00-Hz electromagnetic wave. (a) What is the maximum electric field strength created? (b) What is the corresponding maximum magnetic field strength in the electromagnetic wave? (c) What is the wavelength of the electromagnetic wave?

21. (a) The ideal size (most efficient) for a broadcast antenna with one end on the ground is one-fourth the wavelength (\( \lambda/4 \)) of the electromagnetic radiation being sent out. If a new radio station has such an antenna that is 50.0 m high, what frequency does it broadcast most efficiently? Is this in the AM or FM band? (b) Discuss the analogy of the fundamental resonant mode of an air column closed at one end to the resonance of currents on an antenna that is one-fourth their wavelength.

22. (a) What is the wavelength of 100-MHz radio waves used in an MRI unit? (b) If the frequencies are swept over a \( \pm 1.00 \) range centered on 100 MHz, what is the range of wavelengths broadcast?

23. (a) What is the frequency of the 193-nm ultraviolet radiation used in laser eye surgery? (b) Assuming the accuracy with which this EM radiation can ablate the cornea is directly proportional to wavelength, how much more accurate can this UV be than the shortest visible wavelength of light?
24. TV-reception antennas for VHF are constructed with cross wires supported at their centers, as shown in Figure 14.27. The ideal length for the cross wires is one-half the wavelength to be received, with the more expensive antennas having one for each channel. Suppose you measure the lengths of the wires for particular channels and find them to be 1.94 and 0.753 m long, respectively. What are the frequencies for these channels?

![Figure 14.27](image)

25. Conversations with astronauts on lunar walks had an echo that was used to estimate the distance to the Moon. The sound spoken by the person on Earth was transformed into a radio signal sent to the Moon, and transformed back into sound on a speaker inside the astronaut’s space suit. This sound was picked up by the microphone in the space suit (intended for the astronaut’s voice) and sent back to Earth as a radio echo of sorts. If the round-trip time was 2.60 s, what was the approximate distance to the Moon, neglecting any delays in the electronics?

26. Lunar astronauts placed a reflector on the Moon’s surface, off which a laser beam is periodically reflected. The distance to the Moon is calculated from the round-trip time. (a) To what accuracy in meters can the distance to the Moon be determined, if this time can be measured to 0.100 ns? (b) What percent accuracy is this, given the average distance to the Moon is $3.84 \times 10^8$ m?

27. Radar is used to determine distances to various objects by measuring the round-trip time for an echo from the object. (a) How far away is the planet Venus if the echo time is 1000 s? (b) What is the echo time for a car 75.0 m from a Highway Police radar unit? (c) How accurately (in nanoseconds) must you be able to measure the echo time to an airplane 12.0 km away to determine its distance within 10.0 m?

28. Integrated Concepts
(a) Calculate the ratio of the highest to lowest frequencies of electromagnetic waves the eye can see, given the wavelength range of visible light is from 380 to 760 nm. (b) Compare this with the ratio of highest to lowest frequencies the ear can hear.

29. Integrated Concepts
(a) Calculate the rate in watts at which heat transfer through radiation occurs (almost entirely in the infrared) from 1.0 m$^2$ of the Earth’s surface at night. Assume the emissivity is 0.90, the temperature of the Earth is $15^\circ$C, and that of outer space is 2.7 K. (b) Compare the intensity of this radiation with that coming to the Earth from the Sun during the day, which averages about 800 W/m$^2$, only half of which is absorbed. (c) What is the maximum magnetic field strength in the outgoing radiation, assuming it is a continuous wave?

14.4 Energy in Electromagnetic Waves

30. What is the intensity of an electromagnetic wave with a peak electric field strength of 125 V/m?

31. Find the intensity of an electromagnetic wave having a peak magnetic field strength of $4.00 \times 10^{-9}$ T.

32. Assume the helium-neon lasers commonly used in student physics laboratories have power outputs of 0.250 mW. (a) If such a laser beam is projected onto a circular spot 1.00 mm in diameter, what is its intensity? (b) Find the peak magnetic field strength. (c) Find the peak electric field strength.

33. An AM radio transmitter broadcasts 50.0 kW of power uniformly in all directions. (a) Assuming all of the radio waves that strike the ground are completely absorbed, and that there is no absorption by the atmosphere or other objects, what is the intensity 30.0 km away? (Hint: Half the power will be spread over the area of a hemisphere.) (b) What is the maximum electric field strength at this distance?

34. Suppose the maximum safe intensity of microwaves for human exposure is taken to be 1.00 W/m$^2$. (a) If a radar unit leaks 10.0 W of microwaves (other than those sent by its antenna) uniformly in all directions, how far away must you be to be exposed to an intensity considered to be safe? Assume that the power spreads uniformly over the area of a sphere with no complications from absorption or reflection. (b) What is the maximum electric field strength at this intensity? (Note that early radar units leaked more than modern ones do. This caused identifiable health problems, such as cataracts, for people who worked near them.)
35. A 2.50-m-diameter university communications satellite dish receives TV signals that have a maximum electric field strength (for one channel) of 7.50 \( \mu \text{V/m} \). (See Figure 14.28.) (a) What is the intensity of this wave? (b) What is the power received by the antenna? (c) If the orbiting satellite broadcasts uniformly over an area of 1.50 \( \times 10^{13} \text{ m}^2 \) (a large fraction of North America), how much power does it radiate?

36. Lasers can be constructed that produce an extremely high intensity electromagnetic wave for a brief time—called pulsed lasers. They are used to ignite nuclear fusion, for example. Such a laser may produce an electromagnetic wave with a maximum electric field strength of 1.00 \( \times 10^{11} \text{ V/m} \) for a time of 1.00 ns. (a) What is the maximum magnetic field strength in the wave? (b) What is the intensity of the beam? (c) What energy does it deliver on a 1.00-mm \( ^2 \) area?

37. Show that for a continuous sinusoidal electromagnetic wave, the peak intensity is twice the average intensity (\( I_0 = 2I_{\text{ave}} \)), using either the fact that \( E_0 = \sqrt{2}E_{\text{rms}} \), or \( B_0 = \sqrt{2}B_{\text{rms}} \), where rms means average (actually root mean square, a type of average).

38. Suppose a source of electromagnetic waves radiates uniformly in all directions in empty space where there are no absorption or interference effects. (a) Show that the intensity is inversely proportional to \( r^2 \), the distance from the source squared. (b) Show that the magnitudes of the electric and magnetic fields are inversely proportional to \( r \).

39. Integrated Concepts
An LC circuit with a 5.00-pF capacitor oscillates in such a manner as to radiate at a wavelength of 3.30 m. (a) What is the resonant frequency? (b) What inductance is in series with the capacitor?

40. Integrated Concepts
What capacitance is needed in series with an 800 - \( \mu \text{H} \) inductor to form a circuit that radiates a wavelength of 196 m?

41. Integrated Concepts
Police radar determines the speed of motor vehicles using the same Doppler-shift technique employed for ultrasound in medical diagnostics. Beats are produced by mixing the double Doppler-shifted echo with the original frequency. If 1.50 \( \times 10^9 \text{ Hz} \) microwaves are used and a beat frequency of 150 Hz is produced, what is the speed of the vehicle? (Assume the same Doppler-shift formulas are valid with the speed of sound replaced by the speed of light.)

42. Integrated Concepts
Assume the mostly infrared radiation from a heat lamp acts like a continuous wave with wavelength 1.50 \( \mu \text{m} \). (a) If the lamp’s 200-W output is focused on a person’s shoulder, over a circular area 25.0 cm in diameter, what is the intensity in W/m \(^2 \)? (b) What is the peak electric field strength? (c) Find the peak magnetic field strength. (d) How long will it take to increase the temperature of the 4.00-kg shoulder by 2.00º C, assuming no other heat transfer and given that its specific heat is 3.47 \( \times 10^3 \text{ J/kg } ^º\text{C} \)?

43. Integrated Concepts
On its highest power setting, a microwave oven increases the temperature of 0.400 kg of spaghetti by 45.0º C in 120 s. (a) What was the rate of power absorption by the spaghetti, given that its specific heat is 3.76 \( \times 10^3 \text{ J/kg } ^º\text{C} \)? (b) Find the average intensity of the microwaves, given that they are absorbed over a circular area 20.0 cm in diameter. (c) What is the peak electric field strength of the microwave? (d) What is its peak magnetic field strength?

44. Integrated Concepts
Electromagnetic radiation from a 5.00-mW laser is concentrated on a 1.00-mm \( ^2 \) area. (a) What is the intensity in W/m \(^2 \)? (b) Suppose a 2.00-nC static charge is in the beam. What is the maximum electric force it experiences? (c) If the static charge moves at 400 m/s, what maximum magnetic force can it feel?

45. Integrated Concepts
A 200-turn flat coil of wire 30.0 cm in diameter acts as an antenna for FM radio at a frequency of 100 MHz. The magnetic field of the incoming electromagnetic wave is perpendicular to the coil and has a maximum strength of 1.00 \( \times 10^{-12} \text{ T} \). (a) What power is incident on the coil? (b) What average emf is induced in the coil over one-fourth of a cycle? (c) If the radio receiver has an inductance of 2.50 \( \mu \text{H} \), what capacitance must it have to resonate at 100 MHz?

46. Integrated Concepts
If electric and magnetic field strengths vary sinusoidally in time, being zero at \( t = 0 \), then \( E = E_0 \sin 2\pi ft \) and \( B = B_0 \sin 2\pi ft \). Let \( f = 1.00 \text{ GH} \) here. (a) When are the field strengths first zero? (b) When do they reach their most negative value? (c) How much time is needed for them to complete one cycle?
47. Unreasonable Results
A researcher measures the wavelength of a 1.20-GHz electromagnetic wave to be 0.500 m. (a) Calculate the speed at which this wave propagates. (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

48. Unreasonable Results
The peak magnetic field strength in a residential microwave oven is $9.20 \times 10^{-5}$ T. (a) What is the intensity of the microwave? (b) What is unreasonable about this result? (c) What is wrong about the premise?

49. Unreasonable Results
An $LC$ circuit containing a 2.00-H inductor oscillates at such a frequency that it radiates at a 1.00-m wavelength. (a) What is the capacitance of the circuit? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

50. Unreasonable Results
An $LC$ circuit containing a 1.00-pF capacitor oscillates at such a frequency that it radiates at a 300-nm wavelength. (a) What is the inductance of the circuit? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

51. Create Your Own Problem
Consider electromagnetic fields produced by high voltage power lines. Construct a problem in which you calculate the intensity of this electromagnetic radiation in $W/m^2$ based on the measured magnetic field strength of the radiation in a home near the power lines. Assume these magnetic field strengths are known to average less than a $\mu T$. The intensity is small enough that it is difficult to imagine mechanisms for biological damage due to it. Discuss how much energy may be radiating from a section of power line several hundred meters long and compare this to the power likely to be carried by the lines. An idea of how much power this is can be obtained by calculating the approximate current responsible for $\mu T$ fields at distances of tens of meters.

52. Create Your Own Problem
Consider the most recent generation of residential satellite dishes that are a little less than half a meter in diameter. Construct a problem in which you calculate the power received by the dish and the maximum electric field strength of the microwave signals for a single channel received by the dish. Among the things to be considered are the power broadcast by the satellite and the area over which the power is spread, as well as the area of the receiving dish.
Figure 15.1 The colors reflected by this compact disc vary with angle and are not caused by pigments. Colors such as these are direct evidence of the wave character of light. (credit: Reggie Mathalone)

Chapter Outline

15.1. The Wave Aspect of Light: Interference
- Discuss the wave character of light.
- Identify the changes when light enters a medium.

15.2. Huygens's Principle: Diffraction
- Discuss the propagation of transverse waves.
- Discuss Huygens's principle.
- Explain the bending of light.

15.3. Young's Double Slit Experiment
- Explain the phenomena of interference.
- Define constructive interference for a double slit and destructive interference for a double slit.

15.4. Multiple Slit Diffraction
- Discuss the pattern obtained from diffraction grating.
- Explain diffraction grating effects.

15.5. Single Slit Diffraction
- Discuss the single slit diffraction pattern.

15.6. Limits of Resolution: The Rayleigh Criterion
- Discuss the Rayleigh criterion.

15.7. Thin Film Interference
- Discuss the rainbow formation by thin films.

15.8. Polarization
- Discuss the meaning of polarization.
- Discuss the property of optical activity of certain materials.

15.9. *Extended Topic* Microscopy Enhanced by the Wave Characteristics of Light
- Discuss the different types of microscopes.

Introduction to Wave Optics
Examine a compact disc under white light, noting the colors observed and locations of the colors. Determine if the spectra are formed by diffraction from circular lines centered at the middle of the disc and, if so, what is their spacing. If not, determine the type of spacing. Also with the CD, explore the spectra of a few light sources, such as a candle flame, incandescent bulb, halogen light, and fluorescent light. Knowing the spacing of the rows of pits in the compact disc, estimate the maximum spacing that will allow the given number of megabytes of information to be stored.

If you have ever looked at the reds, blues, and greens in a sunlit soap bubble and wondered how straw-colored soapy water could produce them, you have hit upon one of the many phenomena that can only be explained by the wave character of light (see Figure 15.2). The same is true for the colors seen in an oil slick or in the light reflected from a compact disc. These and other interesting phenomena, such as the dispersion of white light into a rainbow of colors when passed through a narrow slit, cannot be explained fully by geometric optics. In these cases, light interacts with small objects and exhibits its wave characteristics. The branch of optics that considers the behavior of light when it exhibits wave characteristics (particularly when it interacts with small objects) is called wave optics (sometimes called physical optics). It is the topic of this chapter.
These soap bubbles exhibit brilliant colors when exposed to sunlight. How are the colors produced if they are not pigments in the soap? (credit: Scott Robinson, Flickr)

**15.1 The Wave Aspect of Light: Interference**

We know that visible light is the type of electromagnetic wave to which our eyes respond. Like all other electromagnetic waves, it obeys the equation

\[ c = f \lambda, \]

where \( c = 3 \times 10^8 \text{ m/s} \) is the speed of light in vacuum, \( f \) is the frequency of the electromagnetic waves, and \( \lambda \) is its wavelength. The range of visible wavelengths is approximately 380 to 760 nm. As is true for all waves, light travels in straight lines and acts like a ray when it interacts with objects several times as large as its wavelength. However, when it interacts with smaller objects, it displays its wave characteristics prominently. Interference is the hallmark of a wave, and in Figure 15.3 both the ray and wave characteristics of light can be seen. The laser beam emitted by the observatory epitomizes a ray, traveling in a straight line. However, passing a pure-wavelength beam through vertical slits with a size close to the wavelength of the beam reveals the wave character of light, as the beam spreads out horizontally into a pattern of bright and dark regions caused by systematic constructive and destructive interference. Rather than spreading out, a ray would continue traveling straight ahead after passing through slits.

**Making Connections: Waves**

The most certain indication of a wave is interference. This wave characteristic is most prominent when the wave interacts with an object that is not large compared with the wavelength. Interference is observed for water waves, sound waves, light waves, and (as we will see in Special Relativity) for matter waves, such as electrons scattered from a crystal.
Figure 15.3 (a) The laser beam emitted by an observatory acts like a ray, traveling in a straight line. This laser beam is from the Paranal Observatory of the European Southern Observatory. (credit: Yuri Beletsky, European Southern Observatory) (b) A laser beam passing through a grid of vertical slits produces an interference pattern—characteristic of a wave. (credit: Shim'on and Slava Rybka, Wikimedia Commons)

Light has wave characteristics in various media as well as in a vacuum. When light goes from a vacuum to some medium, like water, its speed and wavelength change, but its frequency remains the same. (We can think of light as a forced oscillation that must have the frequency of the original source.) The speed of light in a medium is \( v = c/n \), where \( n \) is its index of refraction. If we divide both sides of equation \( c = f\lambda \) by \( n \), we get \( c/n = v = f\lambda/n \). This implies that \( v = f\lambda_n \), where \( \lambda_n \) is the wavelength in a medium and that

\[
\lambda_n = \frac{\lambda}{n}.
\] (15.2)

where \( \lambda \) is the wavelength in vacuum and \( n \) is the medium’s index of refraction. Therefore, the wavelength of light is smaller in any medium than it is in vacuum. In water, for example, which has \( n = 1.333 \), the range of visible wavelengths is (380 nm)/1.333 to (760 nm)/1.333, or \( \lambda_n = 285 \text{ to } 570 \text{ nm} \). Although wavelengths change while traveling from one medium to another, colors do not, since colors are associated with frequency.

15.2 Huygens’s Principle: Diffraction

Figure 15.4 shows how a transverse wave looks as viewed from above and from the side. A light wave can be imagined to propagate like this, although we do not actually see it wiggling through space. From above, we view the wavefronts (or wave crests) as we would by looking down on the ocean waves. The side view would be a graph of the electric or magnetic field. The view from above is perhaps the most useful in developing concepts about wave optics.
A transverse wave, such as an electromagnetic wave like light, as viewed from above and from the side. The direction of propagation is perpendicular to the wavefronts (or wave crests) and is represented by an arrow like a ray.

The Dutch scientist Christiaan Huygens (1629–1695) developed a useful technique for determining in detail how and where waves propagate. Starting from some known position, Huygens's principle states that:

Every point on a wavefront is a source of wavelets that spread out in the forward direction at the same speed as the wave itself. The new wavefront is a line tangent to all of the wavelets.

Figure 15.5 shows how Huygens's principle is applied. A wavefront is the long edge that moves, for example, the crest or the trough. Each point on the wavefront emits a semicircular wave that moves at the propagation speed \( v \). These are drawn at a time \( t \) later, so that they have moved a distance \( s = vt \). The new wavefront is a line tangent to the wavelets and is where we would expect the wave to be a time \( t \) later. Huygens's principle works for all types of waves, including water waves, sound waves, and light waves. We will find it useful not only in describing how light waves propagate, but also in explaining the laws of reflection and refraction. In addition, we will see that Huygens's principle tells us how and where light rays interfere.

Figure 15.5 Huygens's principle applied to a straight wavefront. Each point on the wavefront emits a semicircular wavelet that moves a distance \( s = vt \). The new wavefront is a line tangent to the wavelets.

Figure 15.6 shows how a mirror reflects an incoming wave at an angle equal to the incident angle, verifying the law of reflection. As the wavefront strikes the mirror, wavelets are first emitted from the left part of the mirror and then the right. The wavelets closer to the left have had time to travel farther, producing a wavefront traveling in the direction shown.
The law of refraction can be explained by applying Huygens’s principle to a wavefront passing from one medium to another (see Figure 15.7). Each wavelet in the figure was emitted when the wavefront crossed the interface between the media. Since the speed of light is smaller in the second medium, the waves do not travel as far in a given time, and the new wavefront changes direction as shown. This explains why a ray changes direction to become closer to the perpendicular when light slows down. Snell’s law can be derived from the geometry in Figure 15.7, but this is left as an exercise for ambitious readers.

What happens when a wave passes through an opening, such as light shining through an open door into a dark room? For light, we expect to see a sharp shadow of the doorway on the floor of the room, and we expect no light to bend around corners into other parts of the room. When sound passes through a door, we expect to hear it everywhere in the room and, thus, expect that sound spreads out when passing through such an opening (see Figure 15.8). What is the difference between the behavior of sound waves and light waves in this case? The answer is that light has very short wavelengths and acts like a ray. Sound has wavelengths on the order of the size of the door and bends around corners (for frequency of 1000 Hz,
\[\lambda = \frac{c}{f} = \frac{330 \text{ m/s}}{1000 \text{ s}^{-1}} = 0.33 \text{ m},\] about three times smaller than the width of the doorway).

If we pass light through smaller openings, often called slits, we can use Huygens’s principle to see that light bends as sound does (see Figure 15.9). The bending of a wave around the edges of an opening or an obstacle is called diffraction. Diffraction is a wave characteristic and occurs for all types of waves. If diffraction is observed for some phenomenon, it is evidence that the
phenomenon is a wave. Thus the horizontal diffraction of the laser beam after it passes through slits in Figure 15.3 is evidence that light is a wave.

Figure 15.3 Huygens’s principle applied to a straight wavefront striking an opening. The edges of the wavefront bend after passing through the opening, a process called diffraction. The amount of bending is more extreme for a small opening, consistent with the fact that wave characteristics are most noticeable for interactions with objects about the same size as the wavelength.

15.3 Young’s Double Slit Experiment

Although Christiaan Huygens thought that light was a wave, Isaac Newton did not. Newton felt that there were other explanations for color, and for the interference and diffraction effects that were observable at the time. Owing to Newton’s tremendous stature, his view generally prevailed. The fact that Huygens’s principle worked was not considered evidence that was direct enough to prove that light is a wave. The acceptance of the wave character of light came many years later when, in 1801, the English physicist and physician Thomas Young (1773–1829) did his now-classic double slit experiment (see Figure 15.10).

Figure 15.10 Young’s double slit experiment. Here pure-wavelength light sent through a pair of vertical slits is diffracted into a pattern on the screen of numerous vertical lines spread out horizontally. Without diffraction and interference, the light would simply make two lines on the screen.

Why do we not ordinarily observe wave behavior for light, such as observed in Young’s double slit experiment? First, light must interact with something small, such as the closely spaced slits used by Young, to show pronounced wave effects. Furthermore, Young first passed light from a single source (the Sun) through a single slit to make the light somewhat coherent. By coherent, we mean waves are in phase or have a definite phase relationship. Incoherent means the waves have random phase relationships. Why did Young then pass the light through a double slit? The answer to this question is that two slits provide two coherent light sources that then interfere constructively or destructively. Young used sunlight, where each wavelength forms its own pattern, making the effect more difficult to see. We illustrate the double slit experiment with monochromatic (single \( \lambda \)) light to clarify the effect. Figure 15.11 shows the pure constructive and destructive interference of two waves having the same wavelength and amplitude.
Figure 15.11 The amplitudes of waves add. (a) Pure constructive interference is obtained when identical waves are in phase. (b) Pure destructive interference occurs when identical waves are exactly out of phase, or shifted by half a wavelength.

When light passes through narrow slits, it is diffracted into semicircular waves, as shown in Figure 15.12(a). Pure constructive interference occurs where the waves are crest to crest or trough to trough. Pure destructive interference occurs where they are crest to trough. The light must fall on a screen and be scattered into our eyes for us to see the pattern. An analogous pattern for water waves is shown in Figure 15.12(b). Note that regions of constructive and destructive interference move out from the slits at well-defined angles to the original beam. These angles depend on wavelength and the distance between the slits, as we shall see below.

Figure 15.12 Double slits produce two coherent sources of waves that interfere. (a) Light spreads out (diffracts) from each slit, because the slits are narrow. These waves overlap and interfere constructively (bright lines) and destructively (dark regions). We can only see this if the light falls onto a screen and is scattered into our eyes. (b) Double slit interference pattern for water waves are nearly identical to that for light. Wave action is greatest in regions of constructive interference and least in regions of destructive interference. (c) When light that has passed through double slits falls on a screen, we see a pattern such as this. (credit: PASCO)

To understand the double slit interference pattern, we consider how two waves travel from the slits to the screen, as illustrated in Figure 15.13. Each slit is a different distance from a given point on the screen. Thus different numbers of wavelengths fit into each path. Waves start out from the slits in phase (crest to crest), but they may end up out of phase (crest to trough) at the screen if the paths differ in length by half a wavelength, interfering destructively as shown in Figure 15.13(a). If the paths differ by a whole wavelength, then the waves arrive in phase (crest to crest) at the screen, interfering constructively as shown in
Figure 15.13(b). More generally, if the paths taken by the two waves differ by any half-integral number of wavelengths \((1/2)\lambda, (3/2)\lambda, (5/2)\lambda,\) etc., then destructive interference occurs. Similarly, if the paths taken by the two waves differ by any integral number of wavelengths \(\lambda, 2\lambda, 3\lambda,\) etc., then constructive interference occurs.

Take-Home Experiment: Using Fingers as Slits

Look at a light, such as a street lamp or incandescent bulb, through the narrow gap between two fingers held close together. What type of pattern do you see? How does it change when you allow the fingers to move a little farther apart? Is it more distinct for a monochromatic source, such as the yellow light from a sodium vapor lamp, than for an incandescent bulb?

Figure 15.14 shows how to determine the path length difference for waves traveling from two slits to a common point on a screen. If the screen is a large distance away compared with the distance between the slits, then the angle \(\theta\) between the path and a line from the slits to the screen (see the figure) is nearly the same for each path. The difference between the paths is shown in the figure; simple trigonometry shows it to be \(d \sin \theta\), where \(d\) is the distance between the slits. To obtain constructive interference for a double slit, the path length difference must be an integral multiple of the wavelength, or

\[
d \sin \theta = m\lambda, \quad \text{for } m = 0, 1, -1, 2, -2, \ldots \quad (\text{constructive}).
\]

Similarly, to obtain destructive interference for a double slit, the path length difference must be a half-integral multiple of the wavelength, or

\[
d \sin \theta = \left( m + \frac{1}{2} \right)\lambda, \quad \text{for } m = 0, 1, -1, 2, -2, \ldots \quad (\text{destructive}),
\]

where \(\lambda\) is the wavelength of the light, \(d\) is the distance between slits, and \(\theta\) is the angle from the original direction of the beam as discussed above. We call \(m\) the order of the interference. For example, \(m = 4\) is fourth-order interference.

Figure 15.14 The paths from each slit to a common point on the screen differ by an amount \(d \sin \theta\), assuming the distance to the screen is much greater than the distance between slits (not to scale here).

The equations for double slit interference imply that a series of bright and dark lines are formed. For vertical slits, the light spreads out horizontally on either side of the incident beam into a pattern called interference fringes, illustrated in Figure 15.15. The intensity of the bright fringes falls off on either side, being brightest at the center. The closer the slits are, the more is the spreading of the bright fringes. We can see this by examining the equation.
\[ d \sin \theta = m\lambda, \text{ for } m = 0, 1, -1, 2, -2, \ldots . \] (15.5)

For fixed \( \lambda \) and \( m \), the smaller \( d \) is, the larger \( \theta \) must be, since \( \sin \theta = m\lambda / d \). This is consistent with our contention that wave effects are most noticeable when the object the wave encounters (here, slits a distance \( d \) apart) is small. Small \( d \) gives large \( \theta \), hence a large effect.

**Figure 15.15** The interference pattern for a double slit has an intensity that falls off with angle. The photograph shows multiple bright and dark lines, or fringes, formed by light passing through a double slit.

### Example 15.1 Finding a Wavelength from an Interference Pattern

Suppose you pass light from a He-Ne laser through two slits separated by 0.0100 mm and find that the third bright line on a screen is formed at an angle of 10.95° relative to the incident beam. What is the wavelength of the light?

**Strategy**
The third bright line is due to third-order constructive interference, which means that \( m = 3 \). We are given \( d = 0.0100 \text{ mm} \) and \( \theta = 10.95^\circ \). The wavelength can thus be found using the equation \( d \sin \theta = m\lambda \) for constructive interference.

**Solution**
The equation is \( d \sin \theta = m\lambda \). Solving for the wavelength \( \lambda \) gives

\[ \lambda = \frac{d \sin \theta}{m}. \] (15.6)

Substituting known values yields

\[ \lambda = \frac{(0.0100 \text{ mm})(\sin 10.95^\circ)}{3} = 6.33 \times 10^{-4} \text{ mm} = 633 \text{ nm}. \] (15.7)

**Discussion**
To three digits, this is the wavelength of light emitted by the common He-Ne laser. Not by coincidence, this red color is similar to that emitted by neon lights. More important, however, is the fact that interference patterns can be used to measure wavelength. Young did this for visible wavelengths. This analytical technique is still widely used to measure electromagnetic spectra. For a given order, the angle for constructive interference increases with \( \lambda \), so that spectra (measurements of intensity versus wavelength) can be obtained.
Example 15.2 Calculating Highest Order Possible

Interference patterns do not have an infinite number of lines, since there is a limit to how big $m$ can be. What is the highest-order constructive interference possible with the system described in the preceding example?

Strategy and Concept

The equation $d \sin \theta = m\lambda$ (for $m = 0, 1, -1, 2, -2, \ldots$) describes constructive interference. For fixed values of $d$ and $\lambda$, the larger $m$ is, the larger $\sin \theta$ is. However, the maximum value that $\sin \theta$ can have is 1, for an angle of 90º. (Larger angles imply that light goes backward and does not reach the screen at all.) Let us find which $m$ corresponds to this maximum diffraction angle.

Solution

Solving the equation $d \sin \theta = m\lambda$ for $m$ gives

$m = \frac{d \sin \theta}{\lambda}. \tag{15.8}$

Taking $\sin \theta = 1$ and substituting the values of $d$ and $\lambda$ from the preceding example gives

$m = \frac{(0.0100 \text{ mm})(1)}{633 \text{ nm}} \approx 15.8. \tag{15.9}$

Therefore, the largest integer $m$ can be is 15, or

$m = 15. \tag{15.10}$

Discussion

The number of fringes depends on the wavelength and slit separation. The number of fringes will be very large for large slit separations. However, if the slit separation becomes much greater than the wavelength, the intensity of the interference pattern changes so that the screen has two bright lines cast by the slits, as expected when light behaves like a ray. We also note that the fringes get fainter further away from the center. Consequently, not all 15 fringes may be observable.

15.4 Multiple Slit Diffraction

An interesting thing happens if you pass light through a large number of evenly spaced parallel slits, called a diffraction grating. An interference pattern is created that is very similar to the one formed by a double slit (see Figure 15.16). A diffraction grating can be manufactured by scratching glass with a sharp tool in a number of precisely positioned parallel lines, with the untouched regions acting like slits. These can be photographically mass produced rather cheaply. Diffraction gratings work both for transmission of light, as in Figure 15.16, and for reflection of light, as on butterfly wings and the Australian opal in Figure 15.17 or the CD pictured in the opening photograph of this chapter, Figure 15.1. In addition to their use as novelty items, diffraction gratings are commonly used for spectroscopic dispersion and analysis of light. What makes them particularly useful is the fact that they form a sharper pattern than double slits do. That is, their bright regions are narrower and brighter, while their dark regions are darker. Figure 15.18 shows idealized graphs demonstrating the sharper pattern. Natural diffraction gratings occur in the feathers of certain birds. Tiny, finger-like structures in regular patterns act as reflection gratings, producing constructive interference that gives the feathers colors not solely due to their pigmentation. This is called iridescence.

Figure 15.16 A diffraction grating is a large number of evenly spaced parallel slits. (a) Light passing through is diffracted in a pattern similar to a double slit, with bright regions at various angles. (b) The pattern obtained for white light incident on a grating. The central maximum is white, and the higher-order maxima disperse white light into a rainbow of colors.
Figure 15.17 (a) This Australian opal and (b) the butterfly wings have rows of reflectors that act like reflection gratings, reflecting different colors at different angles. (credits: (a) Opals-On-Black.com, via Flickr (b) whologwhy, Flickr)

Figure 15.18 Idealized graphs of the intensity of light passing through a double slit (a) and a diffraction grating (b) for monochromatic light. Maxima can be produced at the same angles, but those for the diffraction grating are narrower and hence sharper. The maxima become narrower and the regions between darker as the number of slits is increased.

The analysis of a diffraction grating is very similar to that for a double slit (see Figure 15.19). As we know from our discussion of double slits in Young’s Double Slit Experiment, light is diffracted by each slit and spreads out after passing through. Rays traveling in the same direction (at an angle $\theta$ relative to the incident direction) are shown in the figure. Each of these rays travels a different distance to a common point on a screen far away. The rays start in phase, and they can be in or out of phase when they reach a screen, depending on the difference in the path lengths traveled. As seen in the figure, each ray travels a distance $d \sin \theta$ different from that of its neighbor, where $d$ is the distance between slits. If this distance equals an integral number of wavelengths, the rays all arrive in phase, and constructive interference (a maximum) is obtained. Thus, the condition necessary to obtain constructive interference for a diffraction grating is

$$d \sin \theta = m\lambda, \quad \text{for } m = 0, 1, -1, 2, -2, \ldots \quad \text{(constructive),}$$

(15.11)

where $d$ is the distance between slits in the grating, $\lambda$ is the wavelength of light, and $m$ is the order of the maximum. Note that this is exactly the same equation as for double slits separated by $d$. However, the slits are usually closer in diffraction gratings than in double slits, producing fewer maxima at larger angles.
Figure 15.19 Diffraction grating showing light rays from each slit traveling in the same direction. Each ray travels a different distance to reach a common point on a screen (not shown). Each ray travels a distance $d \sin \theta$ different from that of its neighbor.

Where are diffraction gratings used? Diffraction gratings are key components of monochromators used, for example, in optical imaging of particular wavelengths from biological or medical samples. A diffraction grating can be chosen to specifically analyze a wavelength emitted by molecules in diseased cells in a biopsy sample or to help excite strategic molecules in the sample with a selected frequency of light. Another vital use is in optical fiber technologies where fibers are designed to provide optimum performance at specific wavelengths. A range of diffraction gratings are available for selecting specific wavelengths for such use.

Take-Home Experiment: Rainbows on a CD

The spacing $d$ of the grooves in a CD or DVD can be well determined by using a laser and the equation $d \sin \theta = m\lambda$, for $m = 0, 1, -1, 2, -2, \ldots$. However, we can still make a good estimate of this spacing by using white light and the rainbow of colors that comes from the interference. Reflect sunlight from a CD onto a wall and use your best judgment of the location of a strongly diffracted color to find the separation $d$.

Example 15.3 Calculating Typical Diffraction Grating Effects

Diffraction gratings with 10,000 lines per centimeter are readily available. Suppose you have one, and you send a beam of white light through it to a screen 2.00 m away. (a) Find the angles for the first-order diffraction of the shortest and longest wavelengths of visible light (380 and 760 nm). (b) What is the distance between the ends of the rainbow of visible light produced on the screen for first-order interference? (See Figure 15.20.)
Figure 15.20 The diffraction grating considered in this example produces a rainbow of colors on a screen a distance $x = 2.00$ m from the grating. The distances along the screen are measured perpendicular to the $x$-direction. In other words, the rainbow pattern extends out of the page.

**Strategy**

The angles can be found using the equation

$$d \sin \theta = m\lambda \quad \text{(for } m = 0, 1, -1, 2, -2, \ldots)$$

once a value for the slit spacing $d$ has been determined. Since there are 10,000 lines per centimeter, each line is separated by $1/10,000$ of a centimeter. Once the angles are found, the distances along the screen can be found using simple trigonometry.

**Solution for (a)**

The distance between slits is $d = 1 \text{ cm} / 10,000 = 1.00 \times 10^{-4} \text{ cm}$ or $1.00 \times 10^{-6} \text{ m}$. Let us call the two angles $\theta_V$ for violet (380 nm) and $\theta_R$ for red (760 nm). Solving the equation $d \sin \theta = m\lambda$ for $\sin \theta$,

$$\sin \theta = \frac{m\lambda}{d},$$

where $m = 1$ for first order and $\lambda_V = 380 \text{ nm} = 3.80 \times 10^{-7} \text{ m}$. Substituting these values gives

$$\sin \theta = \frac{3.80 \times 10^{-7} \text{ m}}{1.00 \times 10^{-6} \text{ m}} = 0.380.$$  

Thus the angle $\theta_V$ is

$$\theta_V = \sin^{-1} 0.380 = 22.33^\circ. \quad \text{(15.15)}$$

Similarly,

$$\sin \theta = \frac{7.60 \times 10^{-7} \text{ m}}{1.00 \times 10^{-6} \text{ m}}$$

Thus the angle $\theta_R$ is

$$\theta_R = \sin^{-1} 0.760 = 49.46^\circ. \quad \text{(15.17)}$$
Notice that in both equations, we reported the results of these intermediate calculations to four significant figures to use with the calculation in part (b).

**Solution for (b)**

The distances on the screen are labeled $y_V$ and $y_R$ in Figure 15.20. Noting that $\tan \theta = y / x$, we can solve for $y_V$ and $y_R$. That is,

$$y_V = x \tan \theta_V = (2.00 \text{ m})(\tan 22.33^\circ) = 0.815 \text{ m} \quad (15.18)$$

and

$$y_R = x \tan \theta_R = (2.00 \text{ m})(\tan 49.46^\circ) = 2.338 \text{ m} \quad (15.19)$$

The distance between them is therefore

$$y_R - y_V = 1.52 \text{ m}. \quad (15.20)$$

**Discussion**

The large distance between the red and violet ends of the rainbow produced from the white light indicates the potential this diffraction grating has as a spectroscopic tool. The more it can spread out the wavelengths (greater dispersion), the more detail can be seen in a spectrum. This depends on the quality of the diffraction grating—it must be very precisely made in addition to having closely spaced lines.

---

### 15.5 Single Slit Diffraction

Light passing through a single slit forms a diffraction pattern somewhat different from those formed by double slits or diffraction gratings. Figure 15.21 shows a single slit diffraction pattern. Note that the central maximum is larger than those on either side, and that the intensity decreases rapidly on either side. In contrast, a diffraction grating produces evenly spaced lines that dim slowly on either side of center.

![Figure 15.21](https://example.com/image.jpg)

(a) Single slit diffraction pattern. Monochromatic light passing through a single slit has a central maximum and many smaller and dimmer maxima on either side. The central maximum is six times higher than shown. (b) The drawing shows the bright central maximum and dimmer and thinner maxima on either side.

The analysis of single slit diffraction is illustrated in Figure 15.22. Here we consider light coming from different parts of the same slit. According to Huygens’s principle, every part of the wavefront in the slit emits wavelets. These are like rays that start out in phase and head in all directions. (Each ray is perpendicular to the wavefront of a wavelet.) Assuming the screen is very far away compared with the size of the slit, rays heading toward a common destination are nearly parallel. When they travel straight ahead, as in Figure 15.22(a), they remain in phase, and a central maximum is obtained. However, when rays travel at an angle $\theta$ relative to the original direction of the beam, each travels a different distance to a common location, and they can arrive in or out of phase. In Figure 15.22(b), the ray from the bottom travels a distance of one wavelength $\lambda$ farther than the ray from the top. Thus a ray from the center travels a distance $\lambda / 2$ farther than the one on the left, arrives out of phase, and interferes destructively. A ray from slightly above the center and one from slightly above the bottom will also cancel one another. In fact, each ray from the slit will have another to interfere destructively, and a minimum in intensity will occur at this angle. There will be another minimum at the same angle to the right of the incident direction of the light.
Figure 15.22 Light passing through a single slit is diffracted in all directions and may interfere constructively or destructively, depending on the angle. The difference in path length for rays from either side of the slit is seen to be $D \sin \theta$.

At the larger angle shown in Figure 15.22(c), the path lengths differ by $3\lambda/2$ for rays from the top and bottom of the slit. One ray travels a distance $\lambda$ different from the ray from the bottom and arrives in phase, interfering constructively. Two rays, each from slightly above those two, will also add constructively. Most rays from the slit will have another to interfere with constructively, and a maximum in intensity will occur at this angle. However, all rays do not interfere constructively for this situation, and so the maximum is not as intense as the central maximum. Finally, in Figure 15.22(d), the angle shown is large enough to produce a second minimum. As seen in the figure, the difference in path length for rays from either side of the slit is $D \sin \theta$, and we see that a destructive minimum is obtained when this distance is an integral multiple of the wavelength.
Thus, to obtain **destructive interference for a single slit**,  
\[ D \sin \theta = m\lambda, \text{ for } m = 1, -1, 2, -2, 3, \ldots \ (\text{destructive}), \]  

where \( D \) is the slit width, \( \lambda \) is the light's wavelength, \( \theta \) is the angle relative to the original direction of the light, and \( m \) is the order of the minimum. Figure 15.23 shows a graph of intensity for single slit interference, and it is apparent that the maxima on either side of the central maximum are much less intense and not as wide. This is consistent with the illustration in Figure 15.21(b).

### Example 15.4 Calculating Single Slit Diffraction

Visible light of wavelength 550 nm falls on a single slit and produces its second diffraction minimum at an angle of 45.0º relative to the incident direction of the light. (a) What is the width of the slit? (b) At what angle is the first minimum produced?

#### Strategy

From the given information, and assuming the screen is far away from the slit, we can use the equation \( D \sin \theta = m\lambda \) first to find \( D \), and again to find the angle for the first minimum \( \theta_1 \).

#### Solution for (a)

We are given that \( \lambda = 550 \text{ nm} \), \( m = 2 \), and \( \theta_2 = 45.0^\circ \). Solving the equation \( D \sin \theta = m\lambda \) for \( D \) and substituting known values gives

\[
D = \frac{m\lambda}{\sin \theta_2} = \frac{2(550 \text{ nm})}{\sin 45.0^\circ}
\]

\[
= \frac{1100 \times 10^{-9}}{0.707}
\]

\[
= 1.56 \times 10^{-6}.
\]
Solving the equation $D \sin \theta = m\lambda$ for $\sin \theta_1$ and substituting the known values gives

$$\sin \theta_1 = \frac{m\lambda}{D} = \frac{1(550 \times 10^{-9} \text{ m})}{1.56 \times 10^{-6} \text{ m}}$$

Thus the angle $\theta_1$ is

$$\theta_1 = \sin^{-1} 0.354 = 20.7^\circ.$$  \hspace{1cm} (15.24)

**Discussion**

We see that the slit is narrow (it is only a few times greater than the wavelength of light). This is consistent with the fact that light must interact with an object comparable in size to its wavelength in order to exhibit significant wave effects such as this single slit diffraction pattern. We also see that the central maximum extends 20.7° on either side of the original beam, for a width of about 41°. The angle between the first and second minima is only about 24° (45.0° − 20.7°). Thus the second maximum is only about half as wide as the central maximum.

**15.6 Limits of Resolution: The Rayleigh Criterion**

Light diffracts as it moves through space, bending around obstacles, interfering constructively and destructively. While this can be used as a spectroscopic tool—a diffraction grating disperses light according to wavelength, for example, and is used to produce spectra—diffraction also limits the detail we can obtain in images. Figure 15.25(a) shows the effect of passing light through a small circular aperture. Instead of a bright spot with sharp edges, a spot with a fuzzy edge surrounded by circles of light is obtained. This pattern is caused by diffraction similar to that produced by a single slit. Light from different parts of the circular aperture interferes constructively and destructively. The effect is most noticeable when the aperture is small, but the effect is there for large apertures, too.

![Figure 15.25](https://example.com/figure15_25.png)

(a) | (b) | (c)
--- | --- | ---
Monochromatic light passed through a small circular aperture produces this diffraction pattern. (b) Two point light sources that are close to one another produce overlapping images because of diffraction. (c) If they are closer together, they cannot be resolved or distinguished.

How does diffraction affect the detail that can be observed when light passes through an aperture? Figure 15.25(b) shows the diffraction pattern produced by two point light sources that are close to one another. The pattern is similar to that for a single point source, and it is just barely possible to tell that there are two light sources rather than one. If they were closer together, as in Figure 15.25(c), we could not distinguish them, thus limiting the detail or resolution we can obtain. This limit is an inescapable consequence of the wave nature of light.

There are many situations in which diffraction limits the resolution. The acuity of our vision is limited because light passes through the pupil, the circular aperture of our eye. Be aware that the diffraction-like spreading of light is due to the limited diameter of a light beam, not the interaction with an aperture. Thus light passing through a lens with a diameter $D$ shows this effect and spreads, blurring the image, just as light passing through an aperture of diameter $D$ does. So diffraction limits the resolution of any system having a lens or mirror. Telescopes are also limited by diffraction, because of the finite diameter $D$ of their primary mirror.

**Take-Home Experiment: Resolution of the Eye**

Draw two lines on a white sheet of paper (several mm apart). How far away can you be and still distinguish the two lines? What does this tell you about the size of the eye’s pupil? Can you be quantitative? (The size of an adult’s pupil is discussed in *Physics of the Eye* (https://legacy.cnx.org/content/m42482/latest).)

Just what is the limit? To answer that question, consider the diffraction pattern for a circular aperture, which has a central maximum that is wider and brighter than the maxima surrounding it (similar to a slit) (see Figure 15.26(a)). It can be shown that, for a circular aperture of diameter $D$, the first minimum in the diffraction pattern occurs at $\theta = 1.22 \frac{\lambda}{D}$ (providing the aperture is large compared with the wavelength of light, which is the case for most optical instruments). The accepted criterion for determining the diffraction limit to resolution based on this angle was developed by Lord Rayleigh in the 19th century. The Rayleigh criterion for the diffraction limit to resolution states that two images are just resolvable when the center of the
diffraction pattern of one is directly over the first minimum of the diffraction pattern of the other. See Figure 15.26(b). The first minimum is at an angle of $\theta = 1.22 \frac{\lambda}{D}$, so that two point objects are just resolvable if they are separated by the angle

$$\theta = 1.22 \frac{\lambda}{D}$$  \hspace{1cm} (15.25)$$

where $\lambda$ is the wavelength of light (or other electromagnetic radiation) and $D$ is the diameter of the aperture, lens, mirror, etc., with which the two objects are observed. In this expression, $\theta$ has units of radians.

Connections: Limits to Knowledge

All attempts to observe the size and shape of objects are limited by the wavelength of the probe. Even the small wavelength of light prohibits exact precision. When extremely small wavelength probes as with an electron microscope are used, the system is disturbed, still limiting our knowledge, much as making an electrical measurement alters a circuit. Heisenberg’s uncertainty principle asserts that this limit is fundamental and inescapable, as we shall see in quantum mechanics.

Example 15.5 Calculating Diffraction Limits of the Hubble Space Telescope

The primary mirror of the orbiting Hubble Space Telescope has a diameter of 2.40 m. Being in orbit, this telescope avoids the degrading effects of atmospheric distortion on its resolution. (a) What is the angle between two just-resolvable point light sources (perhaps two stars)? Assume an average light wavelength of 550 nm. (b) If these two stars are at the 2 million light year distance of the Andromeda galaxy, how close together can they be and still be resolved? (A light year, or ly, is the distance light travels in 1 year.)

Strategy

The Rayleigh criterion stated in the equation $\theta = 1.22 \frac{\lambda}{D}$ gives the smallest possible angle $\theta$ between point sources, or the best obtainable resolution. Once this angle is found, the distance between stars can be calculated, since we are given how far away they are.

Solution for (a)

The Rayleigh criterion for the minimum resolvable angle is

$$\theta = 1.22 \frac{\lambda}{D}$$  \hspace{1cm} (15.26)$$

Entering known values gives

$$\theta = 1.22 \frac{550 \times 10^{-9} \text{ m}}{2.40 \text{ m}}$$

$$= 2.80 \times 10^{-7} \text{ rad.}$$

Solution for (b)
The distance \( s \) between two objects a distance \( r \) away and separated by an angle \( \theta \) is \( s = r\theta \).

Substituting known values gives

\[
s = (2.0 \times 10^6 \text{ ly})(2.80 \times 10^{-7} \text{ rad}) = 0.56 \text{ ly}.
\] (15.28)

**Discussion**

The angle found in part (a) is extraordinarily small (less than 1/50,000 of a degree), because the primary mirror is so large compared with the wavelength of light. As noticed, diffraction effects are most noticeable when light interacts with objects having sizes on the order of the wavelength of light. However, the effect is still there, and there is a diffraction limit to what is observable. The actual resolution of the Hubble Telescope is not quite as good as that found here. As with all instruments, there are other effects, such as non-uniformities in mirrors or aberrations in lenses that further limit resolution. However, Figure 15.27 gives an indication of the extent of the detail observable with the Hubble because of its size and quality and especially because it is above the Earth's atmosphere.

![Figure 15.27](image1)

(a) The left is a ground-based image. (credit: Ricnun, Wikimedia Commons) (b) The photo on the right was captured by Hubble. (credit: NASA, ESA, and the Hubble Heritage Team (STScI/AURA))

The answer in part (b) indicates that two stars separated by about half a light year can be resolved. The average distance between stars in a galaxy is on the order of 5 light years in the outer parts and about 1 light year near the galactic center. Therefore, the Hubble can resolve most of the individual stars in Andromeda galaxy, even though it lies at such a huge distance that its light takes 2 million years for its light to reach us.

![Figure 15.28](image2)

Figure 15.28 A 305-m-diameter natural bowl at Arecibo in Puerto Rico is lined with reflective material, making it into a radio telescope. It is the largest curved focusing dish in the world. Although \( D \) for Arecibo is much larger than for the Hubble Telescope, it detects much longer wavelength radiation and its diffraction limit is significantly poorer than Hubble's. Arecibo is still very useful, because important information is carried by radio waves that is not carried by visible light. (credit: Tatyana Temirbutulatova, Flickr)

Diffraction is not only a problem for optical instruments but also for the electromagnetic radiation itself. Any beam of light having a finite diameter \( D \) and a wavelength \( \lambda \) exhibits diffraction spreading. The beam spreads out with an angle \( \theta \) given by the equation \( \theta = 1.22\frac{\lambda}{D} \). Take, for example, a laser beam made of rays as parallel as possible (angles between rays as close to \( \theta = 0^\circ \) as possible) instead spreads out at an angle \( \theta = 1.22 \frac{\lambda}{D} \), where \( D \) is the diameter of the beam and \( \lambda \) is its wavelength. This spreading is impossible to observe for a flashlight, because its beam is not very parallel to start with. However, for long-distance transmission of laser beams or microwave signals, diffraction spreading can be significant (see Figure 15.29).

To avoid this, we can increase \( D \). This is done for laser light sent to the Moon to measure its distance from the Earth. The laser beam is expanded through a telescope to make \( D \) much larger and \( \theta \) smaller.
Figure 15.29 The beam produced by this microwave transmission antenna will spread out at a minimum angle \( \theta = 1.22 \frac{\lambda}{D} \) due to diffraction. It is impossible to produce a near-parallel beam, because the beam has a limited diameter.

In most biology laboratories, resolution is presented when the use of the microscope is introduced. The ability of a lens to produce sharp images of two closely spaced point objects is called resolution. The smaller the distance \( x \) by which two objects can be separated and still be seen as distinct, the greater the resolution. The resolving power of a lens is defined as that distance \( x \). An expression for resolving power is obtained from the Rayleigh criterion. In Figure 15.30(a) we have two point objects separated by a distance \( x \). According to the Rayleigh criterion, resolution is possible when the minimum angular separation is

\[
\theta = 1.22 \frac{\lambda}{D} = \frac{x}{d},
\]  

(15.29)

where \( d \) is the distance between the specimen and the objective lens, and we have used the small angle approximation (i.e., we have assumed that \( x \) is much smaller than \( d \)), so that \( \tan \theta \approx \sin \theta \approx \theta \).

Therefore, the resolving power is

\[
x = 1.22 \frac{\lambda d}{D}.
\]  

(15.30)

Another way to look at this is by re-examining the concept of Numerical Aperture (\( NA \)) discussed in Microscopes (https://legacy.cnx.org/content/m42491/latest). There, \( NA \) is a measure of the maximum acceptance angle at which the fiber will take light and still contain it within the fiber. Figure 15.30(b) shows a lens and an object at point \( P \). The \( NA \) here is a measure of the ability of the lens to gather light and resolve fine detail. The angle subtended by the lens at its focus is defined to be \( \theta = 2\alpha \). From the figure and again using the small angle approximation, we can write

\[
\sin \alpha = \frac{D/2}{d} = \frac{D}{2d}.
\]  

(15.31)

The \( NA \) for a lens is \( NA = n \sin \alpha \), where \( n \) is the index of refraction of the medium between the objective lens and the object at point \( P \).

From this definition for \( NA \), we can see that

\[
x = 1.22 \frac{\lambda d}{D} = 1.22 \frac{\lambda}{2 \sin \alpha} = 0.61 \frac{\lambda n}{NA}.
\]  

(15.32)

In a microscope, \( NA \) is important because it relates to the resolving power of a lens. A lens with a large \( NA \) will be able to resolve finer details. Lenses with larger \( NA \) will also be able to collect more light and so give a brighter image. Another way to describe this situation is that the larger the \( NA \), the larger the cone of light that can be brought into the lens, and so more of the diffraction modes will be collected. Thus the microscope has more information to form a clear image, and so its resolving power will be higher.
One of the consequences of diffraction is that the focal point of a beam has a finite width and intensity distribution. Consider focusing when only considering geometric optics, shown in Figure 15.31(a). The focal point is infinitely small with a huge intensity and the capacity to incinerate most samples irrespective of the $NA$ of the objective lens. For wave optics, due to diffraction, the focal point spreads to become a focal spot (see Figure 15.31(b)) with the size of the spot decreasing with increasing $NA$. Consequently, the intensity in the focal spot increases with increasing $NA$. The higher the $NA$, the greater the chances of photodegrading the specimen. However, the spot never becomes a true point.

15.7 Thin Film Interference

The bright colors seen in an oil slick floating on water or in a sunlit soap bubble are caused by interference. The brightest colors are those that interfere constructively. This interference is between light reflected from different surfaces of a thin film; thus, the effect is known as thin film interference. As noticed before, interference effects are most prominent when light interacts with
something having a size similar to its wavelength. A thin film is one having a thickness \( t \) smaller than a few times the wavelength of light, \( \lambda \). Since color is associated indirectly with \( \lambda \) and since all interference depends in some way on the ratio of \( \lambda \) to the size of the object involved, we should expect to see different colors for different thicknesses of a film, as in Figure 15.32.

![Figure 15.32 These soap bubbles exhibit brilliant colors when exposed to sunlight. (credit: Scott Robinson, Flickr)](https://legacy.cnx.org/content/col11951/1.1)

What causes thin film interference? Figure 15.33 shows how light reflected from the top and bottom surfaces of a film can interfere. Incident light is only partially reflected from the top surface of the film (ray 1). The remainder enters the film and is itself partially reflected from the bottom surface. Part of the light reflected from the bottom surface can emerge from the top of the film (ray 2) and interfere with light reflected from the top (ray 1). Since the ray that enters the film travels a greater distance, it may be in or out of phase with the ray reflected from the top. However, consider for a moment, again, the bubbles in Figure 15.32. The bubbles are darkest where they are thinnest. Furthermore, if you observe a soap bubble carefully, you will note it gets dark at the point where it breaks. For very thin films, the difference in path lengths of ray 1 and ray 2 in Figure 15.33 is negligible; so why should they interfere destructively and not constructively? The answer is that a phase change can occur upon reflection. The rule is as follows:

**When light reflects from a medium having an index of refraction greater than that of the medium in which it is traveling, a 180° phase change (or a \( \lambda / 2 \) shift) occurs.**

![Figure 15.33 Light striking a thin film is partially reflected (ray 1) and partially refracted at the top surface. The refracted ray is partially reflected at the bottom surface and emerges as ray 2. These rays will interfere in a way that depends on the thickness of the film and the indices of refraction of the various media.](https://legacy.cnx.org/content/col11951/1.1)

If the film in Figure 15.33 is a soap bubble (essentially water with air on both sides), then there is a \( \lambda / 2 \) shift for ray 1 and none for ray 2. Thus, when the film is very thin, the path length difference between the two rays is negligible, they are exactly out of phase, and destructive interference will occur at all wavelengths and so the soap bubble will be dark here.
The thickness of the film relative to the wavelength of light is the other crucial factor in thin film interference. Ray 2 in Figure 15.33 travels a greater distance than ray 1. For light incident perpendicular to the surface, ray 2 travels a distance approximately $2t$ farther than ray 1. When this distance is an integral or half-integral multiple of the wavelength in the medium ($\lambda_n = \lambda / n$, where $\lambda$ is the wavelength in vacuum and $n$ is the index of refraction), constructive or destructive interference occurs, depending also on whether there is a phase change in either ray.

**Example 15.6 Calculating Non-reflective Lens Coating Using Thin Film Interference**

Sophisticated cameras use a series of several lenses. Light can reflect from the surfaces of these various lenses and degrade image clarity. To limit these reflections, lenses are coated with a thin layer of magnesium fluoride that causes destructive thin film interference. What is the thinnest this film can be, if its index of refraction is 1.38 and it is designed to limit the reflection of 550-nm light, normally the most intense visible wavelength? The index of refraction of glass is 1.52.

**Strategy**

Refer to Figure 15.33 and use $n_1 = 1.00$ for air, $n_2 = 1.38$, and $n_3 = 1.52$. Both ray 1 and ray 2 will have a $\lambda / 2$ shift upon reflection. Thus, to obtain destructive interference, ray 2 will need to travel a half wavelength farther than ray 1. For rays incident perpendicularly, the path length difference is $2t$.

**Solution**

To obtain destructive interference here,

$$2t = \frac{\lambda_{n_2}}{2},$$

where $\lambda_{n_2}$ is the wavelength in the film and is given by $\lambda_{n_2} = \frac{\lambda}{n_2}$.

Thus,

$$2t = \frac{\lambda}{n_2}.$$  \hspace{1cm} (15.34)

Solving for $t$ and entering known values yields

$$t = \frac{\lambda / n_2}{4} = \frac{(550 \text{ nm}) / 1.38}{4} = 99.6 \text{ nm}.$$  \hspace{1cm} (15.35)

**Discussion**

Films such as the one in this example are most effective in producing destructive interference when the thinnest layer is used, since light over a broader range of incident angles will be reduced in intensity. These films are called non-reflective coatings; this is only an approximately correct description, though, since other wavelengths will only be partially cancelled. Non-reflective coatings are used in car windows and sunglasses.

Thin film interference is most constructive or most destructive when the path length difference for the two rays is an integral or half-integral wavelength, respectively. That is, for rays incident perpendicularly, $2t = \lambda_n$, $2\lambda_n$, $3\lambda_n$, … or $2t = \lambda_n / 2$, $3\lambda_n / 2$, $5\lambda_n / 2$, … . To know whether interference is constructive or destructive, you must also determine if there is a phase change upon reflection. Thin film interference thus depends on film thickness, the wavelength of light, and the refractive indices. For white light incident on a film that varies in thickness, you will observe rainbow colors of constructive interference for various wavelengths as the thickness varies.

**Example 15.7 Soap Bubbles: More Than One Thickness can be Constructive**

(a) What are the three smallest thicknesses of a soap bubble that produce constructive interference for red light with a wavelength of 650 nm? The index of refraction of soap is taken to be the same as that of water. (b) What three smallest thicknesses will give destructive interference?

**Strategy and Concept**

Use Figure 15.33 to visualize the bubble. Note that $n_1 = n_3 = 1.00$ for air, and $n_2 = 1.333$ for soap (equivalent to water). There is a $\lambda / 2$ shift for ray 1 reflected from the top surface of the bubble, and no shift for ray 2 reflected from the bottom surface. To get constructive interference, then, the path length difference ($2t$) must be a half-integral multiple of the wavelength—the first three being $\lambda_n / 2$, $3\lambda_n / 2$, and $5\lambda_n / 2$. To get destructive interference, the path length difference
must be an integral multiple of the wavelength—the first three being $0, \lambda_n, \text{ and } 2\lambda_n$.

**Solution for (a)**

Constructive interference occurs here when

$$2t_c = \frac{\lambda_n}{2}, \frac{3\lambda_n}{2}, \frac{5\lambda_n}{2}, \ldots.$$  \hspace{1cm} (15.36)

The smallest constructive thickness $t_c$ thus is

$$t_c = \frac{\lambda_n}{4} = \frac{2\lambda}{4} = \frac{\lambda}{2} = \frac{(650 \text{ nm}) / 1.333}{4} = 122 \text{ nm}.$$ \hspace{1cm} (15.37)

The next thickness that gives constructive interference is $t'_c = 3\lambda_n / 4$, so that

$$t'_c = 366 \text{ nm}.$$ \hspace{1cm} (15.38)

Finally, the third thickness producing constructive interference is $t''_c \leq 5\lambda_n / 4$, so that

$$t''_c = 610 \text{ nm}.$$ \hspace{1cm} (15.39)

**Solution for (b)**

For destructive interference, the path length difference here is an integral multiple of the wavelength. The first occurs for zero thickness, since there is a phase change at the top surface. That is,

$$t_d = 0.$$ \hspace{1cm} (15.40)

The first non-zero thickness producing destructive interference is

$$2t'_d = \lambda_n.$$ \hspace{1cm} (15.41)

Substituting known values gives

$$t'_d = \frac{\lambda}{2} = \frac{2\lambda}{2} = \frac{(650 \text{ nm}) / 1.333}{2} = 244 \text{ nm}.$$ \hspace{1cm} (15.42)

Finally, the third destructive thickness is $2t''_d = 2\lambda_n$, so that

$$t''_d = \frac{\lambda}{n} = \frac{650 \text{ nm}}{1.333} = 488 \text{ nm}.$$ \hspace{1cm} (15.43)

**Discussion**

If the bubble was illuminated with pure red light, we would see bright and dark bands at very uniform increases in thickness. First would be a dark band at 0 thickness, then bright at 122 nm thickness, then dark at 244 nm, bright at 366 nm, dark at 488 nm, and bright at 610 nm. If the bubble varied smoothly in thickness, like a smooth wedge, then the bands would be evenly spaced.

Another example of thin film interference can be seen when microscope slides are separated (see Figure 15.34). The slides are very flat, so that the wedge of air between them increases in thickness very uniformly. A phase change occurs at the second surface but not the first, and so there is a dark band where the slides touch. The rainbow colors of constructive interference repeat, going from violet to red again and again as the distance between the slides increases. As the layer of air increases, the bands become more difficult to see, because slight changes in incident angle have greater effects on path length differences. If pure-wavelength light instead of white light is used, then bright and dark bands are obtained rather than repeating rainbow colors.
An important application of thin film interference is found in the manufacturing of optical instruments. A lens or mirror can be compared with a master as it is being ground, allowing it to be shaped to an accuracy of less than a wavelength over its entire surface. Figure 15.35 illustrates the phenomenon called Newton’s rings, which occurs when the plane surfaces of two lenses are placed together. (The circular bands are called Newton’s rings because Isaac Newton described them and their use in detail. Newton did not discover them; Robert Hooke did, and Newton did not believe they were due to the wave character of light.) Each successive ring of a given color indicates an increase of only one wavelength in the distance between the lens and the blank, so that great precision can be obtained. Once the lens is perfect, there will be no rings.

The wings of certain moths and butterflies have nearly iridescent colors due to thin film interference. In addition to pigmentation, the wing’s color is affected greatly by constructive interference of certain wavelengths reflected from its film-coated surface. Car manufacturers are offering special paint jobs that use thin film interference to produce colors that change with angle. This expensive option is based on variation of thin film path length differences with angle. Security features on credit cards, banknotes, driving licenses and similar items prone to forgery use thin film interference, diffraction gratings, or holograms. Australia led the way with dollar bills printed on polymer with a diffraction grating security feature making the currency difficult to forge. Other countries such as New Zealand and Taiwan are using similar technologies, while the United States currency includes a thin film interference effect.

Making Connections: Take-Home Experiment—Thin Film Interference

One feature of thin film interference and diffraction gratings is that the pattern shifts as you change the angle at which you look or move your head. Find examples of thin film interference and gratings around you. Explain how the patterns change for each specific example. Find examples where the thickness changes giving rise to changing colors. If you can find two microscope slides, then try observing the effect shown in Figure 15.34. Try separating one end of the two slides with a hair or maybe a thin piece of paper and observe the effect.

Problem-Solving Strategies for Wave Optics

**Step 1.** Examine the situation to determine that interference is involved. Identify whether slits or thin film interference are considered in the problem.

**Step 2.** If slits are involved, note that diffraction gratings and double slits produce very similar interference patterns, but that gratings have narrower (sharper) maxima. Single slit patterns are characterized by a large central maximum and smaller maxima to the sides.

**Step 3.** If thin film interference is involved, take note of the path length difference between the two rays that interfere. Be certain to use the wavelength in the medium involved, since it differs from the wavelength in vacuum. Note also that there is an additional $\lambda/2$ phase shift when light reflects from a medium with a greater index of refraction.

**Step 4.** Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful. Draw a
diagram of the situation. Labeling the diagram is useful.

**Step 5.** Make a list of what is given or can be inferred from the problem as stated (identify the knowns).

**Step 6.** Solve the appropriate equation for the quantity to be determined (the unknown), and enter the knowns. Slits, gratings, and the Rayleigh limit involve equations.

**Step 7.** For thin film interference, you will have constructive interference for a total shift that is an integral number of wavelengths. You will have destructive interference for a total shift of a half-integral number of wavelengths. Always keep in mind that crest to crest is constructive whereas crest to trough is destructive.

**Step 8.** Check to see if the answer is reasonable: Does it make sense? Angles in interference patterns cannot be greater than 90°, for example.

### 15.8 Polarization

Polaroid sunglasses are familiar to most of us. They have a special ability to cut the glare of light reflected from water or glass (see Figure 15.36). Polaroids have this ability because of a wave characteristic of light called polarization. What is polarization? How is it produced? What are some of its uses? The answers to these questions are related to the wave character of light.

**Figure 15.36** These two photographs of a river show the effect of a polarizing filter in reducing glare in light reflected from the surface of water. Part (b) of this figure was taken with a polarizing filter and part (a) was not. As a result, the reflection of clouds and sky observed in part (a) is not observed in part (b). Polarizing sunglasses are particularly useful on snow and water. (credit: Amithshs, Wikimedia Commons)

Light is one type of electromagnetic (EM) wave. As noted earlier, EM waves are transverse waves consisting of varying electric and magnetic fields that oscillate perpendicular to the direction of propagation (see Figure 15.37). There are specific directions for the oscillations of the electric and magnetic fields. Polarization is the attribute that a wave’s oscillations have a definite direction relative to the direction of propagation of the wave. (This is not the same type of polarization as that discussed for the separation of charges.) Waves having such a direction are said to be polarized. For an EM wave, we define the direction of polarization to be the direction parallel to the electric field. Thus we can think of the electric field arrows as showing the direction of polarization, as in Figure 15.37.

**Figure 15.37** An EM wave, such as light, is a transverse wave. The electric and magnetic fields are perpendicular to the direction of propagation.

To examine this further, consider the transverse waves in the ropes shown in Figure 15.38. The oscillations in one rope are in a vertical plane and are said to be vertically polarized. Those in the other rope are in a horizontal plane and are horizontally polarized. If a vertical slit is placed on the first rope, the waves pass through. However, a vertical slit blocks the horizontally polarized waves. For EM waves, the direction of the electric field is analogous to the disturbances on the ropes.
The transverse oscillations in one rope are in a vertical plane, and those in the other rope are in a horizontal plane. The first is said to be vertically polarized, and the other is said to be horizontally polarized. Vertical slits pass vertically polarized waves and block horizontally polarized waves.

The Sun and many other light sources produce waves that are randomly polarized (see Figure 15.39). Such light is said to be unpolarized because it is composed of many waves with all possible directions of polarization. Polaroid materials, invented by the founder of Polaroid Corporation, Edwin Land, act as a polarizing slit for light, allowing only polarization in one direction to pass through. Polarizing filters are composed of long molecules aligned in one direction. Thinking of the molecules as many slits, analogous to those for the oscillating ropes, we can understand why only light with a specific polarization can get through. The axis of a polarizing filter is the direction along which the filter passes the electric field of an EM wave (see Figure 15.40).

Figure 15.38 The transverse oscillations in one rope are in a vertical plane, and those in the other rope are in a horizontal plane. The first is said to be vertically polarized, and the other is said to be horizontally polarized. Vertical slits pass vertically polarized waves and block horizontally polarized waves.

Figure 15.39 The slender arrow represents a ray of unpolarized light. The bold arrows represent the direction of polarization of the individual waves composing the ray. Since the light is unpolarized, the arrows point in all directions.

Figure 15.40 A polarizing filter has a polarization axis that acts as a slit passing through electric fields parallel to its direction. The direction of polarization of an EM wave is defined to be the direction of its electric field.

Figure 15.41 shows the effect of two polarizing filters on originally unpolarized light. The first filter polarizes the light along its axis. When the axes of the first and second filters are aligned (parallel), then all of the polarized light passed by the first filter is also passed by the second. If the second polarizing filter is rotated, only the component of the light parallel to the second filter’s axis is passed. When the axes are perpendicular, no light is passed by the second.

Only the component of the EM wave parallel to the axis of a filter is passed. Let us call the angle between the direction of polarization and the axis of a filter \( \theta \). If the electric field has an amplitude \( E \), then the transmitted part of the wave has an amplitude \( E \cos \theta \) (see Figure 15.42). Since the intensity of a wave is proportional to its amplitude squared, the intensity \( I \) of the transmitted wave is related to the incident wave by

\[
I = I_0 \cos^2 \theta, \tag{15.44}
\]

where \( I_0 \) is the intensity of the polarized wave before passing through the filter. (The above equation is known as Malus’s law.)
Figure 15.41 The effect of rotating two polarizing filters, where the first polarizes the light. (a) All of the polarized light is passed by the second polarizing filter, because its axis is parallel to the first. (b) As the second is rotated, only part of the light is passed. (c) When the second is perpendicular to the first, no light is passed. (d) In this photograph, a polarizing filter is placed above two others. Its axis is perpendicular to the filter on the right (dark area) and parallel to the filter on the left (lighter area). (credit: P.P. Urone)

Figure 15.42 A polarizing filter transmits only the component of the wave parallel to its axis, $E \cos \theta$, reducing the intensity of any light not polarized parallel to its axis.

**Example 15.8 Calculating Intensity Reduction by a Polarizing Filter**

What angle is needed between the direction of polarized light and the axis of a polarizing filter to reduce its intensity by 90.0%?

**Strategy**
When the intensity is reduced by 90.0%, it is 10.0% or 0.100 times its original value. That is, $I = 0.100I_0$. Using this information, the equation $I = I_0 \cos^2 \theta$ can be used to solve for the needed angle.

**Solution**
Solving the equation $I = I_0 \cos^2 \theta$ for $\cos \theta$ and substituting with the relationship between $I$ and $I_0$ gives

$$\cos \theta = \sqrt{\frac{I}{I_0}} = \sqrt{0.100} = 0.3162. \quad (15.45)$$

Solving for $\theta$ yields

$$\theta = \cos^{-1} 0.3162 = 71.6^\circ. \quad (15.46)$$

**Discussion**
A fairly large angle between the direction of polarization and the filter axis is needed to reduce the intensity to 10.0% of its original value. This seems reasonable based on experimenting with polarizing films. It is interesting that, at an angle of 45°, the intensity is reduced to 50% of its original value (as you will show in this section’s Problems & Exercises). Note that
71.6° is 18.4° from reducing the intensity to zero, and that at an angle of 18.4° the intensity is reduced to 90.0% of its original value (as you will also show in Problems & Exercises), giving evidence of symmetry.

Polarization by Reflection

By now you can probably guess that Polaroid sunglasses cut the glare in reflected light because that light is polarized. You can check this for yourself by holding Polaroid sunglasses in front of you and rotating them while looking at light reflected from water or glass. As you rotate the sunglasses, you will notice the light gets bright and dim, but not completely black. This implies the reflected light is partially polarized and cannot be completely blocked by a polarizing filter.

Figure 15.43 illustrates what happens when unpolarized light is reflected from a surface. Vertically polarized light is preferentially refracted at the surface, so that the reflected light is left more horizontally polarized. The reasons for this phenomenon are beyond the scope of this text, but a convenient mnemonic for remembering this is to imagine the polarization direction to be like an arrow. Vertical polarization would be like an arrow perpendicular to the surface and would be more likely to stick and not be reflected. Horizontal polarization is like an arrow bouncing on its side and would be more likely to be reflected. Sunglasses with vertical axes would then block more reflected light than unpolarized light from other sources.

Figure 15.43 Polarization by reflection. Unpolarized light has equal amounts of vertical and horizontal polarization. After interaction with a surface, the vertical components are preferentially absorbed or refracted, leaving the reflected light more horizontally polarized. This is akin to arrows striking on their sides bouncing off, whereas arrows striking on their tips go into the surface.

Since the part of the light that is not reflected is refracted, the amount of polarization depends on the indices of refraction of the media involved. It can be shown that reflected light is completely polarized at a angle of reflection \( \theta_b \), given by

\[
\tan \theta_b = \frac{n_2}{n_1},
\]

where \( n_1 \) is the medium in which the incident and reflected light travel and \( n_2 \) is the index of refraction of the medium that forms the interface that reflects the light. This equation is known as Brewster’s law, and \( \theta_b \) is known as Brewster’s angle, named after the 19th-century Scottish physicist who discovered them.

Things Great and Small: Atomic Explanation of Polarizing Filters

Polarizing filters have a polarization axis that acts as a slit. This slit passes electromagnetic waves (often visible light) that have an electric field parallel to the axis. This is accomplished with long molecules aligned perpendicular to the axis as shown in Figure 15.44.
Long molecules are aligned perpendicular to the axis of a polarizing filter. The component of the electric field in an EM wave perpendicular to these molecules passes through the filter, while the component parallel to the molecules is absorbed.

Figure 15.45 illustrates how the component of the electric field parallel to the long molecules is absorbed. An electromagnetic wave is composed of oscillating electric and magnetic fields. The electric field is strong compared with the magnetic field and is more effective in exerting force on charges in the molecules. The most affected charged particles are the electrons in the molecules, since electron masses are small. If the electron is forced to oscillate, it can absorb energy from the EM wave. This reduces the fields in the wave and, hence, reduces its intensity. In long molecules, electrons can more easily oscillate parallel to the molecule than in the perpendicular direction. The electrons are bound to the molecule and are more restricted in their movement perpendicular to the molecule. Thus, the electrons can absorb EM waves that have a component of their electric field parallel to the molecule. The electrons are much less responsive to electric fields perpendicular to the molecule and will allow those fields to pass. Thus the axis of the polarizing filter is perpendicular to the length of the molecule.

Example 15.9 Calculating Polarization by Reflection

(a) At what angle will light traveling in air be completely polarized horizontally when reflected from water? (b) From glass?

Strategy
All we need to solve these problems are the indices of refraction. Air has $n_1 = 1.00$, water has $n_2 = 1.333$, and crown glass has $n'_2 = 1.520$. The equation $\tan \theta_b = \frac{n_2}{n_1}$ can be directly applied to find $\theta_b$ in each case.

**Solution for (a)**

Putting the known quantities into the equation

$$\tan \theta_b = \frac{n_2}{n_1}$$

(15.48)

Gives

$$\tan \theta_b = \frac{n_2}{n_1} = 1.333 = 1.333.$$  

(15.49)

Solving for the angle $\theta_b$ yields

$$\theta_b = \tan^{-1} 1.333 = 53.1^\circ.$$  

(15.50)

**Solution for (b)**

Similarly, for crown glass and air,

$$\tan \theta'_b = \frac{n'_2}{n_1} = \frac{1.520}{1.00} = 1.52.$$  

(15.51)

Thus,

$$\theta'_b = \tan^{-1} 1.52 = 56.7^\circ.$$  

(15.52)

**Discussion**

Light reflected at these angles could be completely blocked by a good polarizing filter held with its axis vertical. Brewster’s angle for water and air are similar to those for glass and air, so that sunglasses are equally effective for light reflected from either water or glass under similar circumstances. Light not reflected is refracted into these media. So at an incident angle equal to Brewster’s angle, the refracted light will be slightly polarized vertically. It will not be completely polarized vertically, because only a small fraction of the incident light is reflected, and so a significant amount of horizontally polarized light is refracted.

**Polarization by Scattering**

If you hold your Polaroid sunglasses in front of you and rotate them while looking at blue sky, you will see the sky get bright and dim. This is a clear indication that light scattered by air is partially polarized. Figure 15.46 helps illustrate how this happens. Since light is a transverse EM wave, it vibrates the electrons of air molecules perpendicular to the direction it is traveling. The electrons then radiate like small antennae. Since they are oscillating perpendicular to the direction of the light ray, they produce EM radiation that is polarized perpendicular to the direction of the ray. When viewing the light along a line perpendicular to the original ray, as in Figure 15.46, there can be no polarization in the scattered light parallel to the original ray, because that would require the original ray to be a longitudinal wave. Along other directions, a component of the other polarization can be projected along the line of sight, and the scattered light will only be partially polarized. Furthermore, multiple scattering can bring light to your eyes from other directions and can contain different polarizations.
Figure 15.46 Polarization by scattering. Unpolarized light scattering from air molecules shakes their electrons perpendicular to the direction of the original ray. The scattered light therefore has a polarization perpendicular to the original direction and none parallel to the original direction.

Photographs of the sky can be darkened by polarizing filters, a trick used by many photographers to make clouds brighter by contrast. Scattering from other particles, such as smoke or dust, can also polarize light. Detecting polarization in scattered EM waves can be a useful analytical tool in determining the scattering source.

There is a range of optical effects used in sunglasses. Besides being Polaroid, other sunglasses have colored pigments embedded in them, while others use non-reflective or even reflective coatings. A recent development is photochromic lenses, which darken in the sunlight and become clear indoors. Photochromic lenses are embedded with organic microcrystalline molecules that change their properties when exposed to UV in sunlight, but become clear in artificial lighting with no UV.

**Take-Home Experiment: Polarization**

Find Polaroid sunglasses and rotate one while holding the other still and look at different surfaces and objects. Explain your observations. What is the difference in angle from when you see a maximum intensity to when you see a minimum intensity? Find a reflective glass surface and do the same. At what angle does the glass need to be oriented to give minimum glare?

**Liquid Crystals and Other Polarization Effects in Materials**

While you are undoubtedly aware of liquid crystal displays (LCDs) found in watches, calculators, computer screens, cellphones, flat screen televisions, and other myriad places, you may not be aware that they are based on polarization. Liquid crystals are so named because their molecules can be aligned even though they are in a liquid. Liquid crystals have the property that they can rotate the polarization of light passing through them by 90°. Furthermore, this property can be turned off by the application of a voltage, as illustrated in Figure 15.47. It is possible to manipulate this characteristic quickly and in small well-defined regions to create the contrast patterns we see in so many LCD devices.

In flat screen LCD televisions, there is a large light at the back of the TV. The light travels to the front screen through millions of tiny units called pixels (picture elements). One of these is shown in Figure 15.47 (a) and (b). Each unit has three cells, with red, blue, or green filters, each controlled independently. When the voltage across a liquid crystal is switched off, the liquid crystal passes the light through the particular filter. One can vary the picture contrast by varying the strength of the voltage applied to the liquid crystal.
Many crystals and solutions rotate the plane of polarization of light passing through them. Such substances are said to be \textbf{optically active}. Examples include sugar water, insulin, and collagen (see Figure 15.48). In addition to depending on the type of substance, the amount and direction of rotation depends on a number of factors. Among these is the concentration of the substance, the distance the light travels through it, and the wavelength of light. Optical activity is due to the asymmetric shape of molecules in the substance, such as being helical. Measurements of the rotation of polarized light passing through substances can thus be used to measure concentrations, a standard technique for sugars. It can also give information on the shapes of molecules, such as proteins, and factors that affect their shapes, such as temperature and pH.

Glass and plastic become optically active when stressed; the greater the stress, the greater the effect. Optical stress analysis on complicated shapes can be performed by making plastic models of them and observing them through crossed filters, as seen in
Figure 15.49. It is apparent that the effect depends on wavelength as well as stress. The wavelength dependence is sometimes also used for artistic purposes.

Another interesting phenomenon associated with polarized light is the ability of some crystals to split an unpolarized beam of light into two. Such crystals are said to be birefringent (see Figure 15.50). Each of the separated rays has a specific polarization. One behaves normally and is called the ordinary ray, whereas the other does not obey Snell's law and is called the extraordinary ray. Birefringent crystals can be used to produce polarized beams from unpolarized light. Some birefringent materials preferentially absorb one of the polarizations. These materials are called dichroic and can produce polarization by this preferential absorption. This is fundamentally how polarizing filters and other polarizers work. The interested reader is invited to further pursue the numerous properties of materials related to polarization.

15.9 *Extended Topic* Microscopy Enhanced by the Wave Characteristics of Light

Physics research underpins the advancement of developments in microscopy. As we gain knowledge of the wave nature of electromagnetic waves and methods to analyze and interpret signals, new microscopes that enable us to “see” more are being developed. It is the evolution and newer generation of microscopes that are described in this section.

The use of microscopes (microscopy) to observe small details is limited by the wave nature of light. Owing to the fact that light diffracts significantly around small objects, it becomes impossible to observe details significantly smaller than the wavelength of light. One rule of thumb has it that all details smaller than about $\lambda$ are difficult to observe. Radar, for example, can detect the size of an aircraft, but not its individual rivets, since the wavelength of most radar is several centimeters or greater. Similarly, visible light cannot detect individual atoms, since atoms are about 0.1 nm in size and visible wavelengths range from 380 to 760 nm. Ironically, special techniques used to obtain the best possible resolution with microscopes take advantage of the same wave characteristics of light that ultimately limit the detail.

Making Connections: Waves

All attempts to observe the size and shape of objects are limited by the wavelength of the probe. Sonar and medical ultrasound are limited by the wavelength of sound they employ. We shall see that this is also true in electron microscopy, since electrons have a wavelength. Heisenberg’s uncertainty principle asserts that this limit is fundamental and inescapable, as we shall see in quantum mechanics.

The most obvious method of obtaining better detail is to utilize shorter wavelengths. Ultraviolet (UV) microscopes have been constructed with special lenses that transmit UV rays and utilize photographic or electronic techniques to record images. The shorter UV wavelengths allow somewhat greater detail to be observed, but drawbacks, such as the hazard of UV to living tissue and the need for special detection devices and lenses (which tend to be dispersive in the UV), severely limit the use of UV
microscopes. Elsewhere, we will explore practical uses of very short wavelength EM waves, such as x rays, and other short-wavelength probes, such as electrons in electron microscopes, to detect small details.

Another difficulty in microscopy is the fact that many microscopic objects do not absorb much of the light passing through them. The lack of contrast makes image interpretation very difficult. **Contrast** is the difference in intensity between objects and the background on which they are observed. Stains (such as dyes, fluorophores, etc.) are commonly employed to enhance contrast, but these tend to be application specific. More general wave interference techniques can be used to produce contrast. **Figure 15.51** shows the passage of light through a sample. Since the indices of refraction differ, the number of wavelengths in the paths differs. Light emerging from the object is thus out of phase with light from the background and will interfere differently, producing enhanced contrast, especially if the light is coherent and monochromatic—as in laser light.

![Figure 15.51](image1)

**Figure 15.51** Light rays passing through a sample under a microscope will emerge with different phases depending on their paths. The object shown has a greater index of refraction than the background, and so the wavelength decreases as the ray passes through it. Superimposing these rays produces interference that varies with path, enhancing contrast between the object and background.

**Interference microscopes** enhance contrast between objects and background by superimposing a reference beam of light upon the light emerging from the sample. Since light from the background and objects differ in phase, there will be different amounts of constructive and destructive interference, producing the desired contrast in final intensity. **Figure 15.52** shows schematically how this is done. Parallel rays of light from a source are split into two beams by a half-silvered mirror. These beams are called the object and reference beams. Each beam passes through identical optical elements, except that the object beam passes through the object we wish to observe microscopically. The light beams are recombined by another half-silvered mirror and interfere. Since the light rays passing through different parts of the object have different phases, interference will be significantly different and, hence, have greater contrast between them.

![Figure 15.52](image2)

**Figure 15.52** An interference microscope utilizes interference between the reference and object beam to enhance contrast. The two beams are split by a half-silvered mirror; the object beam is sent through the object, and the reference beam is sent through otherwise identical optical elements. The beams are recombined by another half-silvered mirror and interfere. Since the light rays passing through different parts of the object have different phases, interference will be significantly different and, hence, have greater contrast between them.

Another type of microscope utilizing wave interference and differences in phases to enhance contrast is called the **phase-contrast microscope**. While its principle is the same as the interference microscope, the phase-contrast microscope is simpler to use and construct. Its impact (and the principle upon which it is based) was so important that its developer, the Dutch physicist Frits Zernike (1888–1966), was awarded the Nobel Prize in 1953. **Figure 15.53** shows the basic construction of a phase-contrast microscope. Phase differences between light passing through the object and background are produced by passing the rays through different parts of a phase plate (so called because it shifts the phase of the light passing through it). These two light rays
are superimposed in the image plane, producing contrast due to their interference.

A polarization microscope also enhances contrast by utilizing a wave characteristic of light. Polarization microscopes are useful for objects that are optically active or birefringent, particularly if those characteristics vary from place to place in the object. Polarized light is sent through the object and then observed through a polarizing filter that is perpendicular to the original polarization direction. Nearly transparent objects can then appear with strong color and in high contrast. Many polarization effects are wavelength dependent, producing color in the processed image. Contrast results from the action of the polarizing filter in passing only components parallel to its axis.

Apart from the UV microscope, the variations of microscopy discussed so far in this section are available as attachments to fairly standard microscopes or as slight variations. The next level of sophistication is provided by commercial confocal microscopes, which use the extended focal region shown in Figure 15.31(b) to obtain three-dimensional images rather than two-dimensional images. Here, only a single plane or region of focus is identified; out-of-focus regions above and below this plane are subtracted out by a computer so the image quality is much better. This type of microscope makes use of fluorescence, where a laser provides the excitation light. Laser light passing through a tiny aperture called a pinhole forms an extended focal region within the specimen. The reflected light passes through the objective lens to a second pinhole and the photomultiplier detector, see Figure 15.54. The second pinhole is the key here and serves to block much of the light from points that are not at the focal point of the objective lens. The pinhole is conjugate (coupled) to the focal point of the lens. The second pinhole and detector are scanned, allowing reflected light from a small region or section of the extended focal region to be imaged at any one time. The out-of-focus light is excluded. Each image is stored in a computer, and a full scanned image is generated in a short time. Live cell processes can also be imaged at adequate scanning speeds allowing the imaging of three-dimensional microscopic movement. Confocal microscopy enhances images over conventional optical microscopy, especially for thicker specimens, and so has become quite popular.

The next level of sophistication is provided by microscopes attached to instruments that isolate and detect only a small wavelength band of light—monochromators and spectral analyzers. Here, the monochromatic light from a laser is scattered from the specimen. This scattered light shifts up or down as it excites particular energy levels in the sample. The uniqueness of the observed scattered light can give detailed information about the chemical composition of a given spot on the sample with high contrast—like molecular fingerprints. Applications are in materials science, nanotechnology, and the biomedical field. Fine details in biochemical processes over time can even be detected. The ultimate in microscopy is the electron microscope—to be discussed later. Research is being conducted into the development of new prototype microscopes that can become commercially available, providing better diagnostic and research capacities.
axis of a polarizing filter: the direction along which the filter passes the electric field of an EM wave

birefringent: crystals that split an unpolarized beam of light into two beams

Brewster’s angle: $\theta_b = \tan^{-1}\left(\frac{n_2}{n_1}\right)$, where $n_2$ is the index of refraction of the medium from which the light is reflected and $n_1$ is the index of refraction of the medium in which the reflected light travels

Brewster’s law: $\tan \theta_b = \frac{n_2}{n_1}$, where $n_1$ is the medium in which the incident and reflected light travel and $n_2$ is the index of refraction of the medium that forms the interface that reflects the light

coherent: waves are in phase or have a definite phase relationship

confocal microscopes: microscopes that use the extended focal region to obtain three-dimensional images rather than two-dimensional images

constructive interference for a diffraction grating: occurs when the condition $d \sin \theta = m\lambda$ (for $m = 0, 1, -1, 2, -2, \ldots$) is satisfied, where $d$ is the distance between slits in the grating, $\lambda$ is the wavelength of light, and $m$ is the order of the maximum

constructive interference for a double slit: the path length difference must be an integral multiple of the wavelength

contrast: the difference in intensity between objects and the background on which they are observed

destructive interference for a double slit: the path length difference must be a half-integral multiple of the wavelength

destructive interference for a single slit: occurs when $D \sin \theta = m\lambda$, (for $m = 1, -1, 2, -2, 3, \ldots$), where $D$ is the slit width, $\lambda$ is the light’s wavelength, $\theta$ is the angle relative to the original direction of the light, and $m$ is the order of the minimum

diffraction: the bending of a wave around the edges of an opening or an obstacle

diffraction grating: a large number of evenly spaced parallel slits

direction of polarization: the direction parallel to the electric field for EM waves

horizontally polarized: the oscillations are in a horizontal plane

Huygens’s principle: every point on a wavefront is a source of wavelets that spread out in the forward direction at the same
speed as the wave itself. The new wavefront is a line tangent to all of the wavelets

incoherent: waves have random phase relationships

interference microscopes: microscopes that enhance contrast between objects and background by superimposing a
reference beam of light upon the light emerging from the sample

optically active: substances that rotate the plane of polarization of light passing through them

order: the integer \( m \) used in the equations for constructive and destructive interference for a double slit

phase-contrast microscope: microscope utilizing wave interference and differences in phases to enhance contrast

polarization: the attribute that wave oscillations have a definite direction relative to the direction of propagation of the wave

polarization microscope: microscope that enhances contrast by utilizing a wave characteristic of light, useful for objects that
are optically active

polarized: waves having the electric and magnetic field oscillations in a definite direction

Rayleigh criterion: two images are just resolvable when the center of the diffraction pattern of one is directly over the first
minimum of the diffraction pattern of the other

reflected light that is completely polarized: light reflected at the angle of reflection \( \theta_b \), known as Brewster’s angle

thin film interference: interference between light reflected from different surfaces of a thin film

ultraviolet (UV) microscopes: microscopes constructed with special lenses that transmit UV rays and utilize photographic or
electronic techniques to record images

unpolarized: waves that are randomly polarized

vertically polarized: the oscillations are in a vertical plane

wavelength in a medium: \( \lambda_n = \frac{\lambda}{n} \), where \( \lambda \) is the wavelength in vacuum, and \( n \) is the index of refraction of the medium

---

**Section Summary**

15.1 The Wave Aspect of Light: Interference

- Wave optics is the branch of optics that must be used when light interacts with small objects or whenever the wave
  characteristics of light are considered.
- Wave characteristics are those associated with interference and diffraction.
- Visible light is the type of electromagnetic wave to which our eyes respond and has a wavelength in the range of 380 to 760
  nm.
- Like all EM waves, the following relationship is valid in vacuum: \( c = f \lambda \), where \( c = 3 \times 10^8 \text{ m/s} \) is the speed of light, \( f \)
  is the frequency of the electromagnetic wave, and \( \lambda \) is its wavelength in vacuum.
- The wavelength \( \lambda_n \) of light in a medium with index of refraction \( n \) is \( \lambda_n = \frac{\lambda}{n} \). Its frequency is the same as in vacuum.

15.2 Huygens’s Principle: Diffraction

- An accurate technique for determining how and where waves propagate is given by Huygens’s principle: Every point on a
  wavefront is a source of wavelets that spread out in the forward direction at the same speed as the wave itself. The new
  wavefront is a line tangent to all of the wavelets.
- Diffraction is the bending of a wave around the edges of an opening or other obstacle.

15.3 Young’s Double Slit Experiment

- Young’s double slit experiment gave definitive proof of the wave character of light.
- An interference pattern is obtained by the superposition of light from two slits.
- There is constructive interference when \( d \sin \theta = m\lambda \) (for \( m = 0, 1, -1, 2, -2, \ldots \) ), where \( d \) is the distance between
  the slits, \( \theta \) is the angle relative to the incident direction, and \( m \) is the order of the interference.
- There is destructive interference when \( d \sin \theta = (m + \frac{1}{2})\lambda \) (for \( m = 0, 1, -1, 2, -2, \ldots \) ).

15.4 Multiple Slit Diffraction

- A diffraction grating is a large collection of evenly spaced parallel slits that produces an interference pattern similar to but
sharper than that of a double slit.

- There is constructive interference for a diffraction grating when \( d \sin \theta = m\lambda \) (for \( m = 0, 1, -1, 2, -2, \ldots \)), where \( d \) is the distance between slits in the grating, \( \lambda \) is the wavelength of light, and \( m \) is the order of the maximum.

### 15.5 Single Slit Diffraction
- A single slit produces an interference pattern characterized by a broad central maximum with narrower and dimmer maxima to the sides.
- There is destructive interference for a single slit when \( D \sin \theta = m\lambda \) (for \( m = 1, -1, 2, -2, 3, \ldots \)), where \( D \) is the slit width, \( \lambda \) is the light's wavelength, \( \theta \) is the angle relative to the original direction of the light, and \( m \) is the order of the minimum. Note that there is no \( m = 0 \) minimum.

### 15.6 Limits of Resolution: The Rayleigh Criterion
- Diffraction limits resolution.
- For a circular aperture, lens, or mirror, the Rayleigh criterion states that two images are just resolvable when the center of the diffraction pattern of one is directly over the first minimum of the diffraction pattern of the other.
- This occurs for two point objects separated by the angle \( \theta = \frac{1.22\lambda}{D} \), where \( \lambda \) is the wavelength of light (or other electromagnetic radiation) and \( D \) is the diameter of the aperture, lens, mirror, etc. This equation also gives the angular spreading of a source of light having a diameter \( D \).

### 15.7 Thin Film Interference
- Thin film interference occurs between the light reflected from the top and bottom surfaces of a film. In addition to the path length difference, there can be a phase change.
- When light reflects from a medium having an index of refraction greater than that of the medium in which it is traveling, a 180º phase change (or a \( \lambda/2 \) shift) occurs.

### 15.8 Polarization
- Polarization is the attribute that wave oscillations have a definite direction relative to the direction of propagation of the wave.
- EM waves are transverse waves that may be polarized.
- The direction of polarization is defined to be the direction parallel to the electric field of the EM wave.
- Unpolarized light is composed of many rays having random polarization directions.
- Light can be polarized by passing it through a polarizing filter or other polarizing material. The intensity \( I \) of polarized light after passing through a polarizing filter is \( I = I_0 \cos^2 \theta \), where \( I_0 \) is the original intensity and \( \theta \) is the angle between the direction of polarization and the axis of the filter.
- Polarization is also produced by reflection.
- Brewster’s law states that reflected light will be completely polarized at the angle of reflection \( \theta_b \), known as Brewster’s angle, given by a statement known as Brewster’s law: \( \tan \theta_b = \frac{n_2}{n_1} \), where \( n_1 \) is the medium in which the incident and reflected light travel and \( n_2 \) is the index of refraction of the medium that forms the interface that reflects the light.
- Polarization can also be produced by scattering.
- There are a number of types of optically active substances that rotate the direction of polarization of light passing through them.

### 15.9 Extended Topic* Microscopy Enhanced by the Wave Characteristics of Light
- To improve microscope images, various techniques utilizing the wave characteristics of light have been developed. Many of these enhance contrast with interference effects.

#### Conceptual Questions

**15.1 The Wave Aspect of Light: Interference**
1. What type of experimental evidence indicates that light is a wave?
2. Give an example of a wave characteristic of light that is easily observed outside the laboratory.

**15.2 Huygens’s Principle: Diffraction**
3. How do wave effects depend on the size of the object with which the wave interacts? For example, why does sound bend around the corner of a building while light does not?
4. Under what conditions can light be modeled like a ray? Like a wave?
5. Go outside in the sunlight and observe your shadow. It has fuzzy edges even if you do not. Is this a diffraction effect? Explain.
6. Why does the wavelength of light decrease when it passes from vacuum into a medium? State which attributes change and which stay the same and, thus, require the wavelength to decrease.
7. Does Huygens's principle apply to all types of waves?

15.3 Young's Double Slit Experiment
8. Young's double slit experiment breaks a single light beam into two sources. Would the same pattern be obtained for two independent sources of light, such as the headlights of a distant car? Explain.
9. Suppose you use the same double slit to perform Young's double slit experiment in air and then repeat the experiment in water. Do the angles to the same parts of the interference pattern get larger or smaller? Does the color of the light change? Explain.
10. Is it possible to create a situation in which there is only destructive interference? Explain.
11. Figure 15.55 shows the central part of the interference pattern for a pure wavelength of red light projected onto a double slit. The pattern is actually a combination of single slit and double slit interference. Note that the bright spots are evenly spaced. Is this a double slit or single slit characteristic? Note that some of the bright spots are dim on either side of the center. Is this a single slit or double slit characteristic? Which is smaller, the slit width or the separation between slits? Explain your responses.

Figure 15.55 This double slit interference pattern also shows signs of single slit interference. (credit: PASCO)

15.4 Multiple Slit Diffraction
12. What is the advantage of a diffraction grating over a double slit in dispersing light into a spectrum?
13. What are the advantages of a diffraction grating over a prism in dispersing light for spectral analysis?
14. Can the lines in a diffraction grating be too close together to be useful as a spectroscopic tool for visible light? If so, what type of EM radiation would the grating be suitable for? Explain.
15. If a beam of white light passes through a diffraction grating with vertical lines, the light is dispersed into rainbow colors on the right and left. If a glass prism disperses white light to the right into a rainbow, how does the sequence of colors compare with that produced on the right by a diffraction grating?
16. Suppose pure-wavelength light falls on a diffraction grating. What happens to the interference pattern if the same light falls on a grating that has more lines per centimeter? What happens to the interference pattern if a longer-wavelength light falls on the same grating? Explain how these two effects are consistent in terms of the relationship of wavelength to the distance between slits.
17. Suppose a feather appears green but has no green pigment. Explain in terms of diffraction.
18. It is possible that there is no minimum in the interference pattern of a single slit. Explain why. Is the same true of double slits and diffraction gratings?

15.5 Single Slit Diffraction
19. As the width of the slit producing a single-slit diffraction pattern is reduced, how will the diffraction pattern produced change?

15.6 Limits of Resolution: The Rayleigh Criterion
20. A beam of light always spreads out. Why can a beam not be created with parallel rays to prevent spreading? Why can lenses, mirrors, or apertures not be used to correct the spreading?

15.7 Thin Film Interference
21. What effect does increasing the wedge angle have on the spacing of interference fringes? If the wedge angle is too large, fringes are not observed. Why?
22. How is the difference in paths taken by two originally in-phase light waves related to whether they interfere constructively or destructively? How can this be affected by reflection? By refraction?
23. Is there a phase change in the light reflected from either surface of a contact lens floating on a person's tear layer? The index of refraction of the lens is about 1.5, and its top surface is dry.
24. In placing a sample on a microscope slide, a glass cover is placed over a water drop on the glass slide. Light incident from above can reflect from the top and bottom of the glass cover and from the glass slide below the water drop. At which surfaces will there be a phase change in the reflected light?

25. Answer the above question if the fluid between the two pieces of crown glass is carbon disulfide.

26. While contemplating the food value of a slice of ham, you notice a rainbow of color reflected from its moist surface. Explain its origin.

27. An inventor notices that a soap bubble is dark at its thinnest and realizes that destructive interference is taking place for all wavelengths. How could she use this knowledge to make a non-reflective coating for lenses that is effective at all wavelengths? That is, what limits would there be on the index of refraction and thickness of the coating? How might this be impractical?

28. A non-reflective coating like the one described in Example 15.6 works ideally for a single wavelength and for perpendicular incidence. What happens for other wavelengths and other incident directions? Be specific.

29. Why is it much more difficult to see interference fringes for light reflected from a thick piece of glass than from a thin film? Would it be easier if monochromatic light were used?

15.8 Polarization

30. Under what circumstances is the phase of light changed by reflection? Is the phase related to polarization?

31. Can a sound wave in air be polarized? Explain.

32. No light passes through two perfect polarizing filters with perpendicular axes. However, if a third polarizing filter is placed between the original two, some light can pass. Why is this? Under what circumstances does most of the light pass?

33. Explain what happens to the energy carried by light that it is dimmed by passing it through two crossed polarizing filters.

34. When particles scattering light are much smaller than its wavelength, the amount of scattering is proportional to \(1/\lambda^4\). Does this mean there is more scattering for small \(\lambda\) than large \(\lambda\)? How does this relate to the fact that the sky is blue?

35. Using the information given in the preceding question, explain why sunsets are red.

36. When light is reflected at Brewster’s angle from a smooth surface, it is 100% polarized parallel to the surface. Part of the light will be refracted into the surface. Describe how you would do an experiment to determine the polarization of the refracted light. What direction would you expect the polarization to have and would you expect it to be 100%?

15.9 *Extended Topic* Microscopy Enhanced by the Wave Characteristics of Light

37. Explain how microscopes can use wave optics to improve contrast and why this is important.

38. A bright white light under water is collimated and directed upon a prism. What range of colors does one see emerging?
18. Figure 15.56 shows a double slit located a distance $x$ from a screen, with the distance from the center of the screen given by $y$. When the distance $d$ between the slits is relatively large, there will be numerous bright spots, called fringes. Show that, for small angles (where $\sin \theta \approx \theta$, with $\theta$ in radians), the distance between fringes is given by $\Delta y = \frac{x\lambda}{d}$.

![Figure 15.56](image)

**Problems & Exercises**

### 15.1 The Wave Aspect of Light: Interference

1. Show that when light passes from air to water, its wavelength decreases to 0.750 times its original value.
2. Find the range of visible wavelengths of light in crown glass.
3. What is the index of refraction of a material for which the wavelength of light is 0.671 times its value in a vacuum? Identify the likely substance.
4. Analysis of an interference effect in a clear solid shows that the wavelength of light in the solid is 329 nm. Knowing this light comes from a He-Ne laser and has a wavelength of 633 nm in air, is the substance zircon or diamond?
5. What is the ratio of thicknesses of crown glass and water that would contain the same number of wavelengths of light?

### 15.3 Young’s Double Slit Experiment

6. At what angle is the first-order maximum for 450-nm wavelength blue light falling on double slits separated by 0.0500 mm?
7. Calculate the angle for the third-order maximum of 580-nm wavelength yellow light falling on double slits separated by 0.100 mm.
8. What is the separation between two slits for which 610-nm orange light has its first maximum at an angle of 30.0°?
9. Find the distance between two slits that produces the first minimum for 410-nm violet light at an angle of 45.0°.
10. Calculate the wavelength of light that has its third minimum at an angle of 30.0° when falling on double slits separated by 3.00 μm. Explicitly, show how you follow the steps in Problem-Solving Strategies for Wave Optics.
11. What is the wavelength of light falling on double slits separated by 2.00 μm if the third-order maximum is at an angle of 60.0°?
12. At what angle is the fourth-order maximum for the situation in Exercise 15.6?
13. What is the highest-order maximum for 400-nm light falling on double slits separated by 25.0 μm?
14. Find the largest wavelength of light falling on double slits separated by 1.20 μm for which there is a first-order maximum. Is this in the visible part of the spectrum?
15. What is the smallest separation between two slits that will produce a second-order maximum for 720-nm red light?
16. (a) What is the smallest separation between two slits that will produce a second-order maximum for any visible light? (b) For all visible light?
17. (a) If the first-order maximum for pure-wavelength light falling on a double slit is at an angle of 10.0°, at what angle is the second-order maximum? (b) What is the angle of the first minimum? (c) What is the highest-order maximum possible here?

### 15.4 Multiple Slit Diffraction

19. Using the result of the problem above, calculate the distance between fringes for 633-nm light falling on double slits separated by 0.0800 mm, located 3.00 m from a screen as in Figure 15.56.
20. Using the result of the problem two problems prior, find the wavelength of light that produces fringes 7.50 mm apart on a screen 2.00 m from double slits separated by 0.120 mm (see Figure 15.56).

[Problem-Solving Strategies for Wave Optics]
27. (a) What do the four angles in the above problem become if a 5000-line-per-centimeter diffraction grating is used? (b) Using this grating, what would the angles be for the second-order maxima? (c) Discuss the relationship between integral reductions in lines per centimeter and the new angles of various order maxima.

28. What is the maximum number of lines per centimeter a diffraction grating can have and produce a complete first-order spectrum for visible light?

29. The yellow light from a sodium vapor lamp seems to be of pure wavelength, but it produces two first-order maxima at 36.093° and 36.129° when projected on a 10,000 line per centimeter diffraction grating. What are the two wavelengths to an accuracy of 0.1 nm?

30. What is the spacing between structures in a feather that acts as a reflection grating, given that they produce a first-order maximum for 525-nm light at a 30.0° angle?

31. Structures on a bird feather act like a reflection grating having 8000 lines per centimeter. What is the angle of the first-order maximum for 600-nm light?

32. An opal such as that shown in Figure 15.17 acts like a reflection grating with rows separated by about 8 μm. If the opal is illuminated normally, (a) at what angle will red light be seen and (b) at what angle will blue light be seen?

33. At what angle does a diffraction grating produces a second-order maximum for light having a first-order maximum at 20.0°?

34. Show that a diffraction grating cannot produce a second-order maximum for a given wavelength of light unless the first-order maximum is at an angle less than 30.0°.

35. If a diffraction grating produces a first-order maximum for the shortest wavelength of visible light at 30.0°, at what angle will the first-order maximum be for the longest wavelength of visible light?

36. (a) Find the maximum number of lines per centimeter a diffraction grating can have and produce a maximum for the smallest wavelength of visible light. (b) Would such a grating be useful for ultraviolet spectra? (c) For infrared spectra?

37. (a) Show that a 30,000-line-per-centimeter grating will not produce a maximum for visible light. (b) What is the longest wavelength for which it does produce a first-order maximum? (c) What is the greatest number of lines per centimeter a diffraction grating can have and produce a complete second-order spectrum for visible light?

38. A He–Ne laser beam is reflected from the surface of a CD onto a wall. The brightest spot is the reflected beam at an angle equal to the angle of incidence. However, fringes are also observed. If the wall is 1.50 m from the CD, and the first fringe is 0.600 m from the central maximum, what is the spacing of grooves on the CD?

39. The analysis shown in the figure below also applies to diffraction gratings with lines separated by a distance \( d \).

![Figure 15.57](image)

What is the distance between fringes produced by a diffraction grating having 125 lines per centimeter for 600-nm light, if the screen is 1.50 m away?

40. Unreasonable Results

Red light of wavelength of 700 nm falls on a double slit separated by 400 nm. (a) At what angle is the first-order maximum in the diffraction pattern? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

41. Unreasonable Results

(a) What visible wavelength has its fourth-order maximum at an angle of 25.0° when projected on a 25,000-line-per-centimeter diffraction grating? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

42. Construct Your Own Problem

Consider a spectrometer based on a diffraction grating. Construct a problem in which you calculate the distance between two wavelengths of electromagnetic radiation in your spectrometer. Among the things to be considered are the wavelengths you wish to be able to distinguish, the number of lines per meter on the diffraction grating, and the distance from the grating to the screen or detector. Discuss the practicality of the device in terms of being able to discern between wavelengths of interest.

15.5 Single Slit Diffraction

43. (a) At what angle is the first minimum for 550-nm light falling on a single slit of width 1.00 μm? (b) Will there be a second minimum?

44. (a) Calculate the angle at which a 2.00-μm-wide slit produces its first minimum for 410-nm violet light. (b) Where is the first minimum for 700-nm red light?

45. (a) How wide is a single slit that produces its first minimum for 633-nm light at an angle of 28.0°? (b) At what angle will the second minimum be?
46. (a) What is the width of a single slit that produces its first minimum at 60.0° for 600-nm light? (b) Find the wavelength of light that has its first minimum at 62.0°.

47. Find the wavelength of light that has its third minimum at an angle of 48.6° when it falls on a slit of width 3.00 μm.

48. Calculate the wavelength of light that produces its first minimum at an angle of 36.9° when falling on a single slit of width 1.00 μm.

49. (a) Sodium vapor light averaging 589 nm in wavelength falls on a single slit of width 7.50 μm. At what angle does it produce its second minimum? (b) What is the highest-order minimum produced?

50. (a) Find the angle of the third diffraction minimum for 633-nm light falling on a slit of width 20.0 μm. (b) What is the minimum width of a single slit if the diffraction pattern falls on a screen 1.00 m from the slit? (c) Discuss the ease or difficulty of measuring such a distance.

51. (a) Find the angle between the first minima for the two sodium vapor lines, which have wavelengths of 589.1 and 589.6 nm, when they fall upon a single slit of width 2.00 μm. (b) What is the distance between these minima if the diffraction pattern falls on a screen 1.00 m from the slit? (c) Discuss the ease or difficulty of measuring such a distance.

52. (a) What is the minimum width of a single slit (in multiples of λ) that will produce a first minimum for a wavelength λ? (b) What is its minimum width if it produces 50 minima? (c) 1000 minima?

53. (a) If a single slit produces a first minimum at 14.5°, at what angle is the second-order minimum? (b) What is the angle of the third-order minimum? (c) Is there a fourth-order minimum? (d) Use your answers to illustrate how the angular width of the central maximum is about twice the angular width of the next maximum (which is the angle between the first and second minima).

54. A double slit produces a diffraction pattern that is a combination of single and double slit interference. Find the ratio of the width of the slits to the separation between them, if the first minimum of the single slit pattern falls on the fifth maximum of the double slit pattern. (This will greatly reduce the intensity of the fifth maximum.)

55. Integrated Concepts

A water break at the entrance to a harbor consists of a rock barrier with a 50.0-m-wide opening. Ocean waves of 20.0-m wavelength approach the opening straight on. At what angle to the incident direction are the boats inside the harbor most protected against wave action?

56. Integrated Concepts

An aircraft maintenance technician walks past a tall hangar door that acts like a single slit for sound entering the hangar. Outside the door, on a line perpendicular to the opening in the door, a jet engine makes a 600-Hz sound. At what angle with the door will the technician observe the first minimum in sound intensity if the vertical opening is 0.800 m wide and the speed of sound is 340 m/s?

15.6 Limits of Resolution: The Rayleigh

57. The 300-m-diameter Arecibo radio telescope pictured in Figure 15.28 detects radio waves with a 4.00 cm average wavelength.

(a) What is the angle between two just-resolvable point sources for this telescope?

(b) How close together could these point sources be at the 2 million light year distance of the Andromeda galaxy?

58. Assuming the angular resolution found for the Hubble Telescope in Example 15.5, what is the smallest detail that could be observed on the Moon?

59. Diffraction spreading for a flashlight is insignificant compared with other limitations in its optics, such as spherical aberrations in its mirror. To show this, calculate the minimum angular spreading of a flashlight beam that is originally 5.00 cm in diameter with an average wavelength of 600 nm.

60. (a) What is the minimum angular spread of a 633-nm wavelength He-Ne laser beam that is originally 1.00 mm in diameter?

(b) If this laser is aimed at a mountain cliff 15.0 km away, how big will the illuminated spot be?

(c) How big a spot would be illuminated on the Moon, neglecting atmospheric effects? (This might be done to hit a corner reflector to measure the round-trip time and, hence, distance.) Explicitly show how you follow the steps in Problem-Solving Strategies for Wave Optics.

61. A telescope can be used to enlarge the diameter of a laser beam and limit diffraction spreading. The laser beam is sent through the telescope in opposite the normal direction and can then be projected onto a satellite or the Moon.

(a) If this is done with the Mount Wilson telescope, producing a 2.54-m-diameter beam of 633-nm light, what is the minimum angular spread of the beam (the angle between the beam's maximum and its first zero)?

(b) Neglecting atmospheric effects, what is the size of the spot this beam would make on the Moon, assuming a lunar distance of 3.84×10^8 m?

62. The limit to the eye's acuity is actually related to diffraction by the pupil.

(a) What is the angle between two just-resolvable points of light for a 3.00-mm-diameter pupil, assuming an average wavelength of 550 nm?

(b) Take your result to be the practical limit for the eye. What is the greatest possible distance a car can be from you if you can resolve its two headlights, given they are 1.30 m apart?

(c) What is the distance between two just-resolvable points held at an arm's length (0.800 m) from your eye?

(d) How does your answer to (c) compare to details you normally observe in everyday circumstances?

63. What is the minimum diameter mirror on a telescope that would allow you to see details as small as 5.00 km on the Moon some 384,000 km away? Assume an average wavelength of 550 nm for the light received.

64. You are told not to shoot until you see the whites of their eyes. If the eyes are separated by 6.5 cm and the diameter of your pupil is 5.0 mm, at what distance can you resolve the two eyes using light of wavelength 555 nm?
65. (a) The planet Pluto and its Moon Charon are separated by 19,600 km. Neglecting atmospheric effects, should the 5.08-m-diameter Mount Palomar telescope be able to resolve these bodies when they are 4.50 x 10^9 km from Earth? Assume an average wavelength of 550 nm. 

(b) In actuality, it is just barely possible to discern that Pluto and Charon are separate bodies using an Earth-based telescope. What are the reasons for this?

66. The headlights of a car are 1.3 m apart. What is the maximum distance at which the eye can resolve these two headlights? Take the pupil diameter to be 0.40 cm.

67. When dots are placed on a page from a laser printer, they must be close enough so that you do not see the individual dots of ink. To do this, the separation of the dots must be less than Raleigh’s criterion. Take the pupil of the eye to be 3.0 mm and the distance from the paper to the eye of 35 cm; find the minimum separation of two dots such that they cannot be resolved. How many dots per inch (dpi) does this correspond to?

68. Unreasonable Results

An amateur astronomer wants to build a telescope with a diffraction limit that will allow him to see if there are people on the moons of Jupiter.

(a) What diameter mirror is needed to be able to see 1.00 m detail on a Jovian Moon at a distance of 7.50 x 10^8 km from Earth? The wavelength of light averages 600 nm.

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

69. Construct Your Own Problem

Consider diffraction limits for an electromagnetic wave interacting with a circular object. Construct a problem in which you calculate the limit of angular resolution with a device, using this circular object (such as a lens, mirror, or antenna) to make observations. Also calculate the limit to spatial resolution (such as the size of features observable on the Moon) for observations at a specific distance from the device. Among the things to be considered are the wavelength of electromagnetic radiation used, the size of the circular object, and the distance to the system or phenomenon being observed.

15.7 Thin Film Interference

70. A soap bubble is 100 nm thick and illuminated by white light incident perpendicular to its surface. What wavelength and color of visible light is most constructively reflected, assuming the same index of refraction as water?

71. An oil slick on water is 120 nm thick and illuminated by white light incident perpendicular to its surface. What color does the oil appear (what is the most constructively reflected wavelength), given its index of refraction is 1.40?

72. Calculate the minimum thickness of an oil slick on water that appears blue when illuminated by white light perpendicular to its surface. Take the blue wavelength to be 470 nm and the index of refraction of oil to be 1.40.

73. Find the minimum thickness of a soap bubble that appears red when illuminated by white light perpendicular to its surface. Take the wavelength to be 680 nm, and assume the same index of refraction as water.

74. A film of soapy water (n = 1.33) on top of a plastic cutting board has a thickness of 233 nm. What color is most strongly reflected if it is illuminated perpendicular to its surface?

75. What are the three smallest non-zero thicknesses of soapy water (n = 1.33) on Plexiglas if it appears green (constructively reflecting 520-nm light) when illuminated perpendicularly by white light? Explicitly show how you follow the steps in Problem Solving Strategies for Wave Optics.

76. Suppose you have a lens system that is to be used primarily for 700-nm red light. What is the second thinnest coating of fluorite (magnesium fluoride) that would be non-reflective for this wavelength?

77. (a) As a soap bubble thins it becomes dark, because the path length difference becomes small compared with the wavelength of light and there is a phase shift at the top surface. If it becomes dark when the path length difference is less than one-fourth the wavelength, what is the thickest the bubble can be and appear dark at all visible wavelengths? Assume the same index of refraction as water. (b) Discuss the fragility of the film considering the thickness found.

78. A film of oil on water will appear dark when it is very thin, because the path length difference becomes small compared with the wavelength of light and there is a phase shift at the top surface. If it becomes dark when the path length difference is less than one-fourth the wavelength, what is the thickest the oil can be and appear dark at all visible wavelengths? Oil has an index of refraction of 1.40.

79. Figure 15.34 shows two glass slides illuminated by pure-wavelength light incident perpendicularly. The top slide shows two 7.50-cm-long glass slides illuminated by pure-wavelength light incident perpendicularly. The top slide touches the bottom slide at one end and rests on a 0.100-mm-diameter hair at the other end, forming a wedge of air. (a) How far apart are the dark bands, if the slides are 7.50 cm long and 580-nm light is used? (b) Is there any difference if the slides are made from crown or flint glass? Explain.

80. Figure 15.34 shows two 7.50-cm-long glass slides illuminated by pure 589-nm wavelength light incident perpendicularly. The top slide touches the bottom slide at one end and rests on some debris at the other end, forming a wedge of air. How thick is the debris, if the dark bands are 1.00 mm apart?

81. Repeat Exercise 15.70, but take the light to be incident at a 45º angle.

82. Repeat Exercise 15.71, but take the light to be incident at a 45º angle.

83. Unreasonable Results

To save money on making military aircraft invisible to radar, an inventor decides to coat them with a non-reflective material having an index of refraction of 1.20, which is between that of air and the surface of the plane. This, he reasons, should be much cheaper than designing Stealth bombers. (a) What thickness should the coating be to inhibit the reflection of 4.00-cm wavelength radar? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

15.8 Polarization

84. What angle is needed between the direction of polarized light and the axis of a polarizing filter to cut its intensity in half?
85. The angle between the axes of two polarizing filters is 45.0°. By how much does the second filter reduce the intensity of the light coming through the first?

86. If you have completely polarized light of intensity 150 W/m², what will its intensity be after passing through a polarizing filter with its axis at an 89.0° angle to the light’s polarization direction?

87. What angle would the axis of a polarizing filter need to make with the direction of polarized light of intensity 1.00 kW/m² to reduce the intensity to 10.0 W/m²?

88. At the end of Example 15.8, it was stated that the intensity of polarized light is reduced to 90.0% of its original value by passing through a polarizing filter with its axis at an angle of 18.4° to the direction of polarization. Verify this statement.

89. Show that if you have three polarizing filters, with the second at an angle of 45° to the first and the third at an angle of 90.0° to the first, the intensity of light passed by the first will be reduced to 25.0% of its value. (This is in contrast to having only the first and third, which reduces the intensity to zero, so that placing the second between them increases the intensity of the transmitted light.)

90. Prove that, if \( I \) is the intensity of light transmitted by two polarizing filters with axes at an angle \( \theta \) and \( I' \) is the intensity when the axes are at an angle 90.0°=0, then \( I + I' = I_0 \) the original intensity. (Hint: Use the trigonometric identities \( \cos (90.0°-\theta) = \sin \theta \) and \( \cos^2 \theta + \sin^2 \theta = 1 \).)

91. At what angle will light reflected from diamond be completely polarized?

92. What is Brewster’s angle for light traveling in water that is reflected from crown glass?

93. A scuba diver sees light reflected from the water’s surface. At what angle will this light be completely polarized?

94. At what angle is light inside crown glass completely polarized when reflected from water, as in a fish tank?

95. Light reflected at 55.6° from a window is completely polarized. What is the window’s index of refraction and the likely substance of which it is made?

96. (a) Light reflected at 62.5° from a gemstone in a ring is completely polarized. Can the gem be a diamond? (b) At what angle would the light be completely polarized if the gem was in water?

97. If \( \theta_b \) is Brewster’s angle for light reflected from the top of an interface between two substances, and \( \theta'_b \) is Brewster’s angle for light reflected from below, prove that \( \theta_b + \theta'_b = 90.0° \).

98. Integrated Concepts
If a polarizing filter reduces the intensity of polarized light to 50.0% of its original value, by how much are the electric and magnetic fields reduced?

99. Integrated Concepts
Suppose you put on two pairs of Polaroid sunglasses with their axes at an angle of 15.0°. How much longer will it take the light to deposit a given amount of energy in your eye compared with a single pair of sunglasses? Assume the lenses are clear except for their polarizing characteristics.

100. Integrated Concepts
(a) On a day when the intensity of sunlight is 1.00 kW/m², a circular lens 0.200 m in diameter focuses light onto water in a black aluminum beaker. Two polarizing sheets of plastic are placed in front of the lens with their axes at an angle of 20.0°. Assuming the sunlight is unpolarized and the polarizers are 100% efficient, what is the initial rate of heating of the water in °C/s, assuming it is 80.0% absorbed? The aluminum beaker has a mass of 30.0 grams and contains 250 grams of water. (b) Do the polarizing filters get hot? Explain.
## APPENDIX A | ATOMIC MASSES

<table>
<thead>
<tr>
<th>Atomic Number, Z</th>
<th>Name</th>
<th>Atomic Number, A</th>
<th>Symbol</th>
<th>Atomic Mass (u)</th>
<th>Percent Abundance or Decay Mode</th>
<th>Half-life, $t_{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>neutron</td>
<td>1</td>
<td>$n$</td>
<td>1.008665</td>
<td>$\beta^-$</td>
<td>10.37 min</td>
</tr>
<tr>
<td>1</td>
<td>Hydrogen</td>
<td>1</td>
<td>$^1H$</td>
<td>1.007825</td>
<td>99.985%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Deuterium</td>
<td>2</td>
<td>$^2H$</td>
<td>2.014102</td>
<td>0.015%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tritium</td>
<td>3</td>
<td>$^3H$</td>
<td>3.016050</td>
<td>$\beta^-$</td>
<td>12.33 y</td>
</tr>
<tr>
<td>2</td>
<td>Helium</td>
<td>3</td>
<td>$^3He$</td>
<td>3.016030</td>
<td>$1.38 \times 10^{-4}%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>$^4He$</td>
<td>4.002603</td>
<td>$\approx 100%$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Lithium</td>
<td>6</td>
<td>$^6Li$</td>
<td>6.015121</td>
<td>7.5%</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>$^7Li$</td>
<td>7.016003</td>
<td>92.5%</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Beryllium</td>
<td>7</td>
<td>$^7Be$</td>
<td>7.016928</td>
<td>EC</td>
<td>53.29 d</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>$^9Be$</td>
<td>9.012182</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Boron</td>
<td>10</td>
<td>$^{10}B$</td>
<td>10.012937</td>
<td>19.9%</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>$^{11}B$</td>
<td>11.009305</td>
<td>80.1%</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Carbon</td>
<td>11</td>
<td>$^{11}C$</td>
<td>11.011432</td>
<td>EC, $\beta^+$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
<td>$^{12}C$</td>
<td>12.000000</td>
<td>98.90%</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>13</td>
<td>$^{13}C$</td>
<td>13.003355</td>
<td>1.10%</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>14</td>
<td>$^{14}C$</td>
<td>14.003241</td>
<td>$\beta^-$</td>
<td>5730 y</td>
</tr>
<tr>
<td>7</td>
<td>Nitrogen</td>
<td>13</td>
<td>$^{13}N$</td>
<td>13.005738</td>
<td>$\beta^+$</td>
<td>9.96 min</td>
</tr>
<tr>
<td></td>
<td></td>
<td>14</td>
<td>$^{14}N$</td>
<td>14.003074</td>
<td>99.63%</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>15</td>
<td>$^{15}N$</td>
<td>15.000108</td>
<td>0.37%</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Oxygen</td>
<td>15</td>
<td>$^{15}O$</td>
<td>15.003065</td>
<td>EC, $\beta^+$</td>
<td>122 s</td>
</tr>
<tr>
<td></td>
<td></td>
<td>16</td>
<td>$^{16}O$</td>
<td>15.994915</td>
<td>99.76%</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>18</td>
<td>$^{18}O$</td>
<td>17.999160</td>
<td>0.200%</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Fluorine</td>
<td>18</td>
<td>$^{18}F$</td>
<td>18.000937</td>
<td>EC, $\beta^+$</td>
<td>1.83 h</td>
</tr>
<tr>
<td></td>
<td></td>
<td>19</td>
<td>$^{19}F$</td>
<td>18.998403</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Neon</td>
<td>20</td>
<td>$^{20}Ne$</td>
<td>19.992435</td>
<td>90.51%</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>22</td>
<td>$^{22}Ne$</td>
<td>21.991383</td>
<td>9.22%</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Sodium</td>
<td>22</td>
<td>$^{22}Na$</td>
<td>21.994434</td>
<td>$\beta^+$</td>
<td>2.602 y</td>
</tr>
<tr>
<td></td>
<td></td>
<td>23</td>
<td>$^{23}Na$</td>
<td>22.989767</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>Atomic Number, Z</td>
<td>Name</td>
<td>Atomic Mass Number, A</td>
<td>Symbol</td>
<td>Atomic Mass (u)</td>
<td>Percent Abundance or Decay Mode</td>
<td>Half-life, t₁/₂</td>
</tr>
<tr>
<td>-----------------</td>
<td>--------------</td>
<td>-----------------------</td>
<td>--------</td>
<td>-----------------</td>
<td>-------------------------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>24</td>
<td>Magnesium</td>
<td>24</td>
<td>²⁴Na</td>
<td>23.990 961</td>
<td>$\beta^-$</td>
<td>14.96 h</td>
</tr>
<tr>
<td>12</td>
<td>Magnesium</td>
<td>24</td>
<td>²⁴Mg</td>
<td>23.985 042</td>
<td>78.99%</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Aluminum</td>
<td>27</td>
<td>²⁷Al</td>
<td>26.981 539</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>Silicon</td>
<td>28</td>
<td>²⁸Si</td>
<td>27.976 927</td>
<td>92.23%</td>
<td>2.62 h</td>
</tr>
<tr>
<td>31</td>
<td>Silicon</td>
<td>31</td>
<td>³¹Si</td>
<td>30.975 362</td>
<td>$\beta^-$</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Phosphorus</td>
<td>31</td>
<td>³¹P</td>
<td>30.973 762</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>Sulfur</td>
<td>32</td>
<td>³²S</td>
<td>31.972 070</td>
<td>95.02%</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>Sulfur</td>
<td>35</td>
<td>³⁵S</td>
<td>34.969 031</td>
<td>$\beta^-$</td>
<td>87.4 d</td>
</tr>
<tr>
<td>17</td>
<td>Chlorine</td>
<td>35</td>
<td>³⁵Cl</td>
<td>34.968 852</td>
<td>75.77%</td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>Chlorine</td>
<td>37</td>
<td>³⁷Cl</td>
<td>36.965 903</td>
<td>24.23%</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>Argon</td>
<td>40</td>
<td>⁴⁰Ar</td>
<td>39.962 384</td>
<td>99.60%</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>Potassium</td>
<td>39</td>
<td>³⁹K</td>
<td>38.963 707</td>
<td>93.26%</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>Potassium</td>
<td>40</td>
<td>⁴⁰K</td>
<td>39.963 999</td>
<td>0.0117%, EC, $\beta^-$</td>
<td>1.28×10⁹ y</td>
</tr>
<tr>
<td>20</td>
<td>Calcium</td>
<td>40</td>
<td>⁴⁰Ca</td>
<td>39.962 591</td>
<td>96.94%</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>Scandium</td>
<td>45</td>
<td>⁴⁵Sc</td>
<td>44.955 910</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>Titanium</td>
<td>48</td>
<td>⁴⁸Ti</td>
<td>47.947 947</td>
<td>73.8%</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>Vanadium</td>
<td>51</td>
<td>⁵¹V</td>
<td>50.943 962</td>
<td>99.75%</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>Chromium</td>
<td>52</td>
<td>⁵²Cr</td>
<td>51.940 509</td>
<td>83.79%</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>Manganese</td>
<td>55</td>
<td>⁵⁵Mn</td>
<td>54.938 047</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>Iron</td>
<td>56</td>
<td>⁵⁶Fe</td>
<td>55.934 939</td>
<td>91.72%</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>Cobalt</td>
<td>59</td>
<td>⁵⁹Co</td>
<td>58.933 198</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>Cobalt</td>
<td>60</td>
<td>⁶⁰Co</td>
<td>59.933 819</td>
<td>$\beta^-$</td>
<td>5.271 y</td>
</tr>
<tr>
<td>28</td>
<td>Nickel</td>
<td>58</td>
<td>⁵⁸Ni</td>
<td>57.935 346</td>
<td>68.27%</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>Nickel</td>
<td>60</td>
<td>⁶⁰Ni</td>
<td>59.930 788</td>
<td>26.10%</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>Copper</td>
<td>63</td>
<td>⁶³Cu</td>
<td>62.939 598</td>
<td>69.17%</td>
<td></td>
</tr>
<tr>
<td>65</td>
<td>Copper</td>
<td>65</td>
<td>⁶⁵Cu</td>
<td>64.927 793</td>
<td>30.83%</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>Zinc</td>
<td>64</td>
<td>⁶⁴Zn</td>
<td>63.929 145</td>
<td>48.6%</td>
<td></td>
</tr>
<tr>
<td>66</td>
<td>Zinc</td>
<td>66</td>
<td>⁶⁶Zn</td>
<td>65.926 034</td>
<td>27.9%</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>Gallium</td>
<td>69</td>
<td>⁶⁹Ga</td>
<td>68.925 580</td>
<td>60.1%</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>Germanium</td>
<td>72</td>
<td>⁷²Ge</td>
<td>71.922 079</td>
<td>27.4%</td>
<td></td>
</tr>
<tr>
<td>Atomic Number, ( Z )</td>
<td>Name</td>
<td>Atomic Number, ( A )</td>
<td>Symbol</td>
<td>Atomic Mass (u)</td>
<td>Percent Abundance or Decay Mode</td>
<td>Half-life, ( t_{1/2} )</td>
</tr>
<tr>
<td>-----------------</td>
<td>--------------</td>
<td>-------------------------</td>
<td>--------</td>
<td>-----------------</td>
<td>---------------------------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>74</td>
<td>74 Ge</td>
<td>73.921 177</td>
<td>36.5%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>Arsenic</td>
<td>75 As</td>
<td>74.921 594</td>
<td>100%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>Selenium</td>
<td>80 Se</td>
<td>79.916 520</td>
<td>49.7%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>Bromine</td>
<td>79 Br</td>
<td>78.918 336</td>
<td>50.69%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>Krypton</td>
<td>84 Kr</td>
<td>83.911 507</td>
<td>57.0%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>Rubidium</td>
<td>85 Rb</td>
<td>84.911 794</td>
<td>72.17%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>Strontium</td>
<td>86 Sr</td>
<td>85.909 267</td>
<td>9.86%</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>88Sr</td>
<td>87.905 619</td>
<td>82.58%</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>90 Sr</td>
<td>89.907 738</td>
<td>( \beta^- )</td>
<td>28.8 y</td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>Yttrium</td>
<td>89 Y</td>
<td>88.905 849</td>
<td>100%</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>90 Y</td>
<td>89.907 152</td>
<td>( \beta^- )</td>
<td>64.1 h</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>Zirconium</td>
<td>90 Zr</td>
<td>89.904 703</td>
<td>51.45%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>Niobium</td>
<td>93 Nb</td>
<td>92.906 377</td>
<td>100%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>Molybdenum</td>
<td>98 Mo</td>
<td>97.905 406</td>
<td>24.13%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>Technetium</td>
<td>98 Tc</td>
<td>97.907 215</td>
<td>( \beta^- )</td>
<td>4.2x10^6 y</td>
<td></td>
</tr>
<tr>
<td>44</td>
<td>Ruthenium</td>
<td>102 Ru</td>
<td>101.904 348</td>
<td>31.6%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>Rhodium</td>
<td>103 Rh</td>
<td>102.905 500</td>
<td>100%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>46</td>
<td>Palladium</td>
<td>106 Pd</td>
<td>105.903 478</td>
<td>27.33%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>47</td>
<td>Silver</td>
<td>107 Ag</td>
<td>106.905 092</td>
<td>51.84%</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>109 Ag</td>
<td>108.904 757</td>
<td>48.16%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>Cadmium</td>
<td>114 Cd</td>
<td>113.903 357</td>
<td>28.73%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>49</td>
<td>Indium</td>
<td>115 In</td>
<td>114.903 880</td>
<td>95.7%, ( \beta^- )</td>
<td>4.4x10^14 y</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>Tin</td>
<td>120 Sn</td>
<td>119.902 200</td>
<td>32.59%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>51</td>
<td>Antimony</td>
<td>121 Sb</td>
<td>120.903 821</td>
<td>57.3%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>52</td>
<td>Tellurium</td>
<td>130 Te</td>
<td>129.906 229</td>
<td>33.8%, ( \beta^- )</td>
<td>2.5x10^21 y</td>
<td></td>
</tr>
<tr>
<td>53</td>
<td>Iodine</td>
<td>127 I</td>
<td>126.904 473</td>
<td>100%</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>131 I</td>
<td>130.906 114</td>
<td>( \beta^- )</td>
<td>8.040 d</td>
<td></td>
</tr>
<tr>
<td>54</td>
<td>Xenon</td>
<td>132 Xe</td>
<td>131.904 144</td>
<td>26.9%</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>136 Xe</td>
<td>135.907 214</td>
<td>8.9%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>55</td>
<td>Cesium</td>
<td>133 Cs</td>
<td>132.905 429</td>
<td>100%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Atomic Number, Z</td>
<td>Name</td>
<td>Atomic Mass Number, A</td>
<td>Symbol</td>
<td>Atomic Mass (u)</td>
<td>Percent Abundance or Decay Mode</td>
<td>Half-life, t_{1/2}</td>
</tr>
<tr>
<td>-----------------</td>
<td>-------------</td>
<td>-----------------------</td>
<td>--------</td>
<td>-----------------</td>
<td>---------------------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>134</td>
<td>Cs</td>
<td>134</td>
<td>$^{134}$Cs</td>
<td>133.906 696</td>
<td>EC, $\beta^-$</td>
<td>2.06 y</td>
</tr>
<tr>
<td>137</td>
<td>Ba</td>
<td>137</td>
<td>$^{137}$Ba</td>
<td>136.905 812</td>
<td>11.23%</td>
<td></td>
</tr>
<tr>
<td>138</td>
<td>Ba</td>
<td>138</td>
<td>$^{138}$Ba</td>
<td>137.905 232</td>
<td>71.70%</td>
<td></td>
</tr>
<tr>
<td>139</td>
<td>La</td>
<td>139</td>
<td>$^{139}$La</td>
<td>138.906 346</td>
<td>99.91%</td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>Ce</td>
<td>140</td>
<td>$^{140}$Ce</td>
<td>139.905 433</td>
<td>88.48%</td>
<td></td>
</tr>
<tr>
<td>141</td>
<td>Pr</td>
<td>141</td>
<td>$^{141}$Pr</td>
<td>140.907 647</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>142</td>
<td>Nd</td>
<td>142</td>
<td>$^{142}$Nd</td>
<td>141.907 719</td>
<td>27.13%</td>
<td></td>
</tr>
<tr>
<td>145</td>
<td>Pm</td>
<td>145</td>
<td>$^{145}$Pm</td>
<td>144.912 743</td>
<td>EC, $\alpha$</td>
<td>17.7 y</td>
</tr>
<tr>
<td>152</td>
<td>Sm</td>
<td>152</td>
<td>$^{152}$Sm</td>
<td>151.919 729</td>
<td>26.7%</td>
<td></td>
</tr>
<tr>
<td>153</td>
<td>Eu</td>
<td>153</td>
<td>$^{153}$Eu</td>
<td>152.921 225</td>
<td>52.2%</td>
<td></td>
</tr>
<tr>
<td>158</td>
<td>Gd</td>
<td>158</td>
<td>$^{158}$Gd</td>
<td>157.924 099</td>
<td>24.84%</td>
<td></td>
</tr>
<tr>
<td>159</td>
<td>Tb</td>
<td>159</td>
<td>$^{159}$Tb</td>
<td>158.925 342</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>164</td>
<td>Dy</td>
<td>164</td>
<td>$^{164}$Dy</td>
<td>163.929 171</td>
<td>28.2%</td>
<td></td>
</tr>
<tr>
<td>165</td>
<td>Ho</td>
<td>165</td>
<td>$^{165}$Ho</td>
<td>164.930 319</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>166</td>
<td>Er</td>
<td>166</td>
<td>$^{166}$Er</td>
<td>165.930 290</td>
<td>33.6%</td>
<td></td>
</tr>
<tr>
<td>169</td>
<td>Tm</td>
<td>169</td>
<td>$^{169}$Tm</td>
<td>168.934 212</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>174</td>
<td>Yb</td>
<td>174</td>
<td>$^{174}$Yb</td>
<td>173.938 859</td>
<td>31.8%</td>
<td></td>
</tr>
<tr>
<td>175</td>
<td>Lu</td>
<td>175</td>
<td>$^{175}$Lu</td>
<td>174.940 770</td>
<td>97.41%</td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>Hf</td>
<td>180</td>
<td>$^{180}$Hf</td>
<td>179.946 545</td>
<td>35.10%</td>
<td></td>
</tr>
<tr>
<td>181</td>
<td>Ta</td>
<td>181</td>
<td>$^{181}$Ta</td>
<td>180.947 992</td>
<td>99.98%</td>
<td></td>
</tr>
<tr>
<td>184</td>
<td>W</td>
<td>184</td>
<td>$^{184}$W</td>
<td>183.950 928</td>
<td>30.67%</td>
<td></td>
</tr>
<tr>
<td>187</td>
<td>Re</td>
<td>187</td>
<td>$^{187}$Re</td>
<td>186.955 744</td>
<td>62.6%, $\beta^-$</td>
<td>$4.6\times10^{10}$ y</td>
</tr>
<tr>
<td>191</td>
<td>Os</td>
<td>191</td>
<td>$^{191}$Os</td>
<td>190.960 920</td>
<td>$\beta^-$</td>
<td>15.4 d</td>
</tr>
<tr>
<td>192</td>
<td>Os</td>
<td>192</td>
<td>$^{192}$Os</td>
<td>191.961 467</td>
<td>41.0%</td>
<td></td>
</tr>
<tr>
<td>191</td>
<td>Ir</td>
<td>191</td>
<td>$^{191}$Ir</td>
<td>190.960 584</td>
<td>37.3%</td>
<td></td>
</tr>
<tr>
<td>193</td>
<td>Ir</td>
<td>193</td>
<td>$^{193}$Ir</td>
<td>192.962 917</td>
<td>62.7%</td>
<td></td>
</tr>
<tr>
<td>195</td>
<td>Pt</td>
<td>195</td>
<td>$^{195}$Pt</td>
<td>194.964 766</td>
<td>33.8%</td>
<td></td>
</tr>
<tr>
<td>197</td>
<td>Au</td>
<td>197</td>
<td>$^{197}$Au</td>
<td>196.966 543</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>198</td>
<td>Au</td>
<td>198</td>
<td>$^{198}$Au</td>
<td>197.968 217</td>
<td>$\beta^-$</td>
<td>2.696 d</td>
</tr>
<tr>
<td>199</td>
<td>Hg</td>
<td>199</td>
<td>$^{199}$Hg</td>
<td>198.968 253</td>
<td>16.87%</td>
<td></td>
</tr>
<tr>
<td>Atomic Number, Z</td>
<td>Name</td>
<td>Atomic Mass Number, A</td>
<td>Symbol</td>
<td>Atomic Mass (u)</td>
<td>Percent Abundance or Decay Mode</td>
<td>Half-life, t_{1/2}</td>
</tr>
<tr>
<td>-----------------</td>
<td>-------------</td>
<td>-----------------------</td>
<td>--------</td>
<td>----------------</td>
<td>-------------------------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>81</td>
<td>Thallium</td>
<td>205</td>
<td>Tl</td>
<td>204.974 401</td>
<td>70.48%</td>
<td></td>
</tr>
<tr>
<td>82</td>
<td>Lead</td>
<td>206</td>
<td>Pb</td>
<td>205.974 440</td>
<td>24.1%</td>
<td></td>
</tr>
<tr>
<td>207</td>
<td>Pb</td>
<td></td>
<td></td>
<td>206.975 872</td>
<td>22.1%</td>
<td></td>
</tr>
<tr>
<td>208</td>
<td>Pb</td>
<td></td>
<td></td>
<td>207.976 627</td>
<td>52.4%</td>
<td></td>
</tr>
<tr>
<td>210</td>
<td>Pb</td>
<td>210</td>
<td>Pb</td>
<td>209.984 163</td>
<td>$\alpha, \beta^-$</td>
<td>22.3 y</td>
</tr>
<tr>
<td>211</td>
<td>Pb</td>
<td>211</td>
<td>Pb</td>
<td>210.988 735</td>
<td>$\beta^-$</td>
<td>36.1 min</td>
</tr>
<tr>
<td>212</td>
<td>Pb</td>
<td>212</td>
<td>Pb</td>
<td>211.991 871</td>
<td>$\beta^-$</td>
<td>10.64 h</td>
</tr>
<tr>
<td>83</td>
<td>Bismuth</td>
<td>209</td>
<td>Bi</td>
<td>208.980 374</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>84</td>
<td>Polonium</td>
<td>210</td>
<td>Po</td>
<td>209.982 848</td>
<td>$\alpha$</td>
<td>138.38 d</td>
</tr>
<tr>
<td>85</td>
<td>Astatine</td>
<td>218</td>
<td>At</td>
<td>218.008 684</td>
<td>$\alpha, \beta^-$</td>
<td></td>
</tr>
<tr>
<td>86</td>
<td>Radon</td>
<td>222</td>
<td>Rn</td>
<td>222.017 570</td>
<td>$\alpha$</td>
<td>3.82 d</td>
</tr>
<tr>
<td>87</td>
<td>Francium</td>
<td>223</td>
<td>Fr</td>
<td>223.019 733</td>
<td>$\alpha, \beta^-$</td>
<td>21.8 min</td>
</tr>
<tr>
<td>88</td>
<td>Radium</td>
<td>226</td>
<td>Ra</td>
<td>226.025 402</td>
<td>$\alpha$</td>
<td>$1.60 \times 10^3$ y</td>
</tr>
<tr>
<td>89</td>
<td>Actinium</td>
<td>227</td>
<td>Ac</td>
<td>227.027 750</td>
<td>$\alpha, \beta^-$</td>
<td>21.8 y</td>
</tr>
<tr>
<td>90</td>
<td>Thorium</td>
<td>228</td>
<td>Th</td>
<td>228.028 715</td>
<td>$\alpha$</td>
<td>1.91 y</td>
</tr>
<tr>
<td>232</td>
<td>Th</td>
<td>232</td>
<td>Th</td>
<td>232.038 054</td>
<td>100%, $\alpha$</td>
<td>$1.41 \times 10^{10}$ y</td>
</tr>
<tr>
<td>91</td>
<td>Protactinium</td>
<td>231</td>
<td>Pa</td>
<td>231.035 880</td>
<td>$\alpha$</td>
<td>$3.28 \times 10^4$ y</td>
</tr>
<tr>
<td>92</td>
<td>Uranium</td>
<td>233</td>
<td>U</td>
<td>233.039 628</td>
<td>$\alpha$</td>
<td>$1.59 \times 10^3$ y</td>
</tr>
<tr>
<td>235</td>
<td>U</td>
<td>235</td>
<td>U</td>
<td>235.043 924</td>
<td>0.720%, $\alpha$</td>
<td>$7.04 \times 10^8$ y</td>
</tr>
<tr>
<td>236</td>
<td>U</td>
<td>236</td>
<td>U</td>
<td>236.045 562</td>
<td>$\alpha$</td>
<td>$2.34 \times 10^7$ y</td>
</tr>
<tr>
<td>238</td>
<td>U</td>
<td>238</td>
<td>U</td>
<td>238.050 784</td>
<td>99.2745%, $\alpha$</td>
<td>$4.47 \times 10^9$ y</td>
</tr>
<tr>
<td>239</td>
<td>U</td>
<td>239</td>
<td>U</td>
<td>239.054 289</td>
<td>$\beta^-$</td>
<td>23.5 min</td>
</tr>
<tr>
<td>93</td>
<td>Neptunium</td>
<td>239</td>
<td>Np</td>
<td>239.052 933</td>
<td>$\beta^-$</td>
<td>2.355 d</td>
</tr>
<tr>
<td>94</td>
<td>Plutonium</td>
<td>239</td>
<td>Pu</td>
<td>239.052 157</td>
<td>$\alpha$</td>
<td>$2.41 \times 10^4$ y</td>
</tr>
<tr>
<td>95</td>
<td>Americium</td>
<td>243</td>
<td>Am</td>
<td>243.061 375</td>
<td>$\alpha$, fission</td>
<td>$7.37 \times 10^3$ y</td>
</tr>
<tr>
<td>96</td>
<td>Curium</td>
<td>245</td>
<td>Cm</td>
<td>245.065 483</td>
<td>$\alpha$</td>
<td>$8.50 \times 10^3$ y</td>
</tr>
<tr>
<td>97</td>
<td>Berkelium</td>
<td>247</td>
<td>Bk</td>
<td>247.070 300</td>
<td>$\alpha$</td>
<td>$1.38 \times 10^3$ y</td>
</tr>
<tr>
<td>Atomic Number, Z</td>
<td>Name</td>
<td>Atomic Mass Number, A</td>
<td>Symbol</td>
<td>Atomic Mass (u)</td>
<td>Percent Abundance or Decay Mode</td>
<td>Half-life, t_{1/2}</td>
</tr>
<tr>
<td>-----------------</td>
<td>--------------</td>
<td>-----------------------</td>
<td>--------</td>
<td>----------------</td>
<td>---------------------------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>98</td>
<td>Californium</td>
<td>249</td>
<td>²⁴⁹Cf</td>
<td>249.074 844</td>
<td>α</td>
<td>351 y</td>
</tr>
<tr>
<td>99</td>
<td>Einsteinium</td>
<td>254</td>
<td>²⁵⁴Es</td>
<td>254.088 019</td>
<td>α, β⁻</td>
<td>276 d</td>
</tr>
<tr>
<td>100</td>
<td>Fermium</td>
<td>253</td>
<td>²⁵³Fm</td>
<td>253.085 173</td>
<td>EC, α</td>
<td>3.00 d</td>
</tr>
<tr>
<td>101</td>
<td>Mendelevium</td>
<td>255</td>
<td>²⁵⁵Md</td>
<td>255.091 081</td>
<td>EC, α</td>
<td>27 min</td>
</tr>
<tr>
<td>102</td>
<td>Nobelium</td>
<td>255</td>
<td>²⁵⁵No</td>
<td>255.093 260</td>
<td>EC, α</td>
<td>3.1 min</td>
</tr>
<tr>
<td>103</td>
<td>Lawrencium</td>
<td>257</td>
<td>²⁵⁷Lr</td>
<td>257.099 480</td>
<td>EC, α</td>
<td>0.646 s</td>
</tr>
<tr>
<td>104</td>
<td>Rutherfordium</td>
<td>261</td>
<td>²⁶¹Rf</td>
<td>261.108 690</td>
<td>α</td>
<td>1.08 min</td>
</tr>
<tr>
<td>105</td>
<td>Dubnium</td>
<td>262</td>
<td>²⁶²Db</td>
<td>262.113 760</td>
<td>α, fission</td>
<td>34 s</td>
</tr>
<tr>
<td>106</td>
<td>Seaborgium</td>
<td>263</td>
<td>²⁶³Sg</td>
<td>263.11 86</td>
<td>α, fission</td>
<td>0.8 s</td>
</tr>
<tr>
<td>107</td>
<td>Bohrium</td>
<td>262</td>
<td>²⁶²Bh</td>
<td>262.123 1</td>
<td>α</td>
<td>0.102 s</td>
</tr>
<tr>
<td>108</td>
<td>Hassium</td>
<td>264</td>
<td>²⁶⁴Hs</td>
<td>264.128 5</td>
<td>α</td>
<td>0.08 ms</td>
</tr>
<tr>
<td>109</td>
<td>Meitnerium</td>
<td>266</td>
<td>²⁶⁶Mt</td>
<td>266.137 8</td>
<td>α</td>
<td>3.4 ms</td>
</tr>
</tbody>
</table>
### APPENDIX B | USEFUL INFORMATION

This appendix is broken into several tables.

- **Table B1**, Important Constants
- **Table B2**, Submicroscopic Masses
- **Table B3**, Solar System Data
- **Table B4**, Metric Prefixes for Powers of Ten and Their Symbols
- **Table B5**, The Greek Alphabet
- **Table B6**, SI units
- **Table B7**, Selected British Units
- **Table B8**, Other Units
- **Table B9**, Useful Formulae

#### Table B1 Important Constants

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Best Value</th>
<th>Approximate Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>Speed of light in vacuum</td>
<td>$2.99792458 \times 10^8 \text{ m/s}$</td>
<td>$3.00 \times 10^8 \text{ m/s}$</td>
</tr>
<tr>
<td>$G$</td>
<td>Gravitational constant</td>
<td>$6.67408(31) \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$</td>
<td>$6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$</td>
</tr>
<tr>
<td>$N_A$</td>
<td>Avogadro’s number</td>
<td>$6.02214076 \times 10^{23}$</td>
<td>$6.02 \times 10^{23}$</td>
</tr>
<tr>
<td>$k$</td>
<td>Boltzmann’s constant</td>
<td>$1.380649 \times 10^{-23} \text{ J/K}$</td>
<td>$1.38 \times 10^{-23} \text{ J/K}$</td>
</tr>
<tr>
<td>$R$</td>
<td>Gas constant</td>
<td>$8.3144621(75) \text{ J/mol} \cdot \text{K}$</td>
<td>$8.31 \text{ J/mol} \cdot \text{K} = 1.99 \text{ cal/mol} \cdot \text{K} = 0.0821 \text{ atm} \cdot \text{L/mol} \cdot \text{K}$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Stefan-Boltzmann constant</td>
<td>$5.670373(21) \times 10^{-8} \text{ W/}\text{m}^2 \cdot \text{K}$</td>
<td>$5.67 \times 10^{-8} \text{ W/}\text{m}^2 \cdot \text{K}$</td>
</tr>
<tr>
<td>$k$</td>
<td>Coulomb force constant</td>
<td>$8.987551788 \ldots \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$</td>
<td>$8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$</td>
</tr>
<tr>
<td>$q_e$</td>
<td>Charge on electron</td>
<td>$-1.602176634 \times 10^{-19} \text{ C}$</td>
<td>$-1.60 \times 10^{-19} \text{ C}$</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>Permittivity of free space</td>
<td>$8.854187817 \ldots \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$</td>
<td>$8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>Permeability of free space</td>
<td>$4\pi \times 10^{-7} \text{ T} \cdot \text{m}/\text{A}$</td>
<td>$1.26 \times 10^{-6} \text{ T} \cdot \text{m}/\text{A}$</td>
</tr>
<tr>
<td>$h$</td>
<td>Planck’s constant</td>
<td>$6.62607015 \times 10^{-34} \text{ J} \cdot \text{s}$</td>
<td>$6.63 \times 10^{-34} \text{ J} \cdot \text{s}$</td>
</tr>
</tbody>
</table>

#### Table B2 Submicroscopic Masses

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Best Value</th>
<th>Approximate Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_e$</td>
<td>Electron mass</td>
<td>$9.10938291(40) \times 10^{-31} \text{ kg}$</td>
<td>$9.11 \times 10^{-31} \text{ kg}$</td>
</tr>
<tr>
<td>$m_p$</td>
<td>Proton mass</td>
<td>$1.672621777(74) \times 10^{-27} \text{ kg}$</td>
<td>$1.6726 \times 10^{-27} \text{ kg}$</td>
</tr>
<tr>
<td>$m_n$</td>
<td>Neutron mass</td>
<td>$1.674927351(74) \times 10^{-27} \text{ kg}$</td>
<td>$1.6749 \times 10^{-27} \text{ kg}$</td>
</tr>
<tr>
<td>$u$</td>
<td>Atomic mass unit</td>
<td>$1.660538921(73) \times 10^{-27} \text{ kg}$</td>
<td>$1.6605 \times 10^{-27} \text{ kg}$</td>
</tr>
</tbody>
</table>

#### Table B3 Solar System Data

| Sun | mass | $1.99 \times 10^{30} \text{ kg}$ |

---

1. Stated values are according to the National Institute of Standards and Technology Reference on Constants, Units, and Uncertainty, [www.physics.nist.gov/cuu](http://www.physics.nist.gov/cuu) (accessed May 18, 2012). Values in parentheses are the uncertainties in the last digits. Numbers without uncertainties are exact as defined.

2. Stated values are according to the National Institute of Standards and Technology Reference on Constants, Units, and Uncertainty, [www.physics.nist.gov/cuu](http://www.physics.nist.gov/cuu) (accessed May 18, 2012). Values in parentheses are the uncertainties in the last digits. Numbers without uncertainties are exact as defined.
Earth

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average radius</td>
<td>6.96\times10^8\text{m}</td>
</tr>
<tr>
<td>Earth-sun distance (average)</td>
<td>1.496\times10^{11}\text{m}</td>
</tr>
<tr>
<td>Mass</td>
<td>5.9736\times10^{24}\text{kg}</td>
</tr>
<tr>
<td>Average radius</td>
<td>6.376\times10^6\text{m}</td>
</tr>
<tr>
<td>Orbital period</td>
<td>3.16\times10^7\text{s}</td>
</tr>
</tbody>
</table>

Moon

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average radius</td>
<td>6.376\times10^6\text{m}</td>
</tr>
<tr>
<td>Mass</td>
<td>7.35\times10^{22}\text{kg}</td>
</tr>
<tr>
<td>Average radius</td>
<td>1.74\times10^6\text{m}</td>
</tr>
<tr>
<td>Orbital period (average)</td>
<td>2.36\times10^6\text{s}</td>
</tr>
<tr>
<td>Earth-moon distance (average)</td>
<td>3.84\times10^8\text{m}</td>
</tr>
</tbody>
</table>

Table B4 Metric Prefixes for Powers of Ten and Their Symbols

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Symbol</th>
<th>Value</th>
<th>Prefix</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>tera</td>
<td>T</td>
<td>10^{12}</td>
<td>deci</td>
<td>d</td>
<td>10^{-1}</td>
</tr>
<tr>
<td>giga</td>
<td>G</td>
<td>10^{9}</td>
<td>centi</td>
<td>c</td>
<td>10^{-2}</td>
</tr>
<tr>
<td>mega</td>
<td>M</td>
<td>10^{6}</td>
<td>milli</td>
<td>m</td>
<td>10^{-3}</td>
</tr>
<tr>
<td>kilo</td>
<td>k</td>
<td>10^{3}</td>
<td>micro</td>
<td>\mu</td>
<td>10^{-6}</td>
</tr>
<tr>
<td>hecto</td>
<td>h</td>
<td>10^{2}</td>
<td>nano</td>
<td>n</td>
<td>10^{-9}</td>
</tr>
<tr>
<td>deka</td>
<td>da</td>
<td>10^{1}</td>
<td>pico</td>
<td>p</td>
<td>10^{-12}</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>10^{0}(=1)</td>
<td>femto</td>
<td>f</td>
<td>10^{-15}</td>
</tr>
</tbody>
</table>

Table B5 The Greek Alphabet

<table>
<thead>
<tr>
<th>Letter</th>
<th>Symbol</th>
<th>Lowercase</th>
<th>Alpha</th>
<th>Α</th>
<th>α</th>
<th>Eta</th>
<th>Η</th>
<th>η</th>
<th>Nu</th>
<th>Ν</th>
<th>ν</th>
<th>Tau</th>
<th>Τ</th>
<th>τ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta</td>
<td>Β</td>
<td>β</td>
<td>Theta</td>
<td>Θ</td>
<td>θ</td>
<td>Xi</td>
<td>Ξ</td>
<td>ε</td>
<td>Omicron</td>
<td>Ώ</td>
<td>ο</td>
<td>Phi</td>
<td>Φ</td>
<td>ϕ</td>
</tr>
<tr>
<td>Gamma</td>
<td>Γ</td>
<td>γ</td>
<td>Iota</td>
<td>Ι</td>
<td>i</td>
<td>Omicron</td>
<td>Ο</td>
<td>o</td>
<td>Phi</td>
<td>Ε</td>
<td>ε</td>
<td>Chi</td>
<td>Χ</td>
<td>χ</td>
</tr>
<tr>
<td>Delta</td>
<td>Δ</td>
<td>δ</td>
<td>Kappa</td>
<td>Κ</td>
<td>κ</td>
<td>Pi</td>
<td>Π</td>
<td>π</td>
<td>Chi</td>
<td>Κ</td>
<td>κ</td>
<td>Rho</td>
<td>Ρ</td>
<td>ρ</td>
</tr>
<tr>
<td>Epsilon</td>
<td>Ε</td>
<td>ε</td>
<td>Lambda</td>
<td>Λ</td>
<td>λ</td>
<td>Rho</td>
<td>Ρ</td>
<td>ρ</td>
<td>Psi</td>
<td>Ψ</td>
<td>ψ</td>
<td>Omega</td>
<td>Ω</td>
<td>ω</td>
</tr>
</tbody>
</table>

Table B6 SI Units

<table>
<thead>
<tr>
<th>Entity</th>
<th>Abbreviation</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fundamental units</td>
<td>m</td>
<td>meter</td>
</tr>
<tr>
<td>Mass</td>
<td>kg</td>
<td>kilogram</td>
</tr>
<tr>
<td>Time</td>
<td>s</td>
<td>second</td>
</tr>
<tr>
<td>Current</td>
<td>A</td>
<td>ampere</td>
</tr>
<tr>
<td>Supplementary unit</td>
<td>rad</td>
<td>radian</td>
</tr>
<tr>
<td>Derived units</td>
<td>N = kg \cdot m^2/s^2</td>
<td>newton</td>
</tr>
<tr>
<td>Energy</td>
<td>J = kg \cdot m^2/s^2</td>
<td>joule</td>
</tr>
<tr>
<td>Entity</td>
<td>Abbreviation</td>
<td>Name</td>
</tr>
<tr>
<td>------------------------------------</td>
<td>--------------</td>
<td>----------</td>
</tr>
<tr>
<td>Power</td>
<td>W = J/s</td>
<td>watt</td>
</tr>
<tr>
<td>Pressure</td>
<td>Pa = N/m²</td>
<td>pascal</td>
</tr>
<tr>
<td>Frequency</td>
<td>Hz = 1/s</td>
<td>hertz</td>
</tr>
<tr>
<td>Electronic potential</td>
<td>V = J/C</td>
<td>volt</td>
</tr>
<tr>
<td>Capacitance</td>
<td>F = C/V</td>
<td>farad</td>
</tr>
<tr>
<td>Charge</td>
<td>C = s·A</td>
<td>coulomb</td>
</tr>
<tr>
<td>Resistance</td>
<td>Ω = V/A</td>
<td>ohm</td>
</tr>
<tr>
<td>Magnetic field</td>
<td>T = N/(A·m)</td>
<td>tesla</td>
</tr>
<tr>
<td>Nuclear decay rate</td>
<td>Bq = 1/s</td>
<td>becquerel</td>
</tr>
</tbody>
</table>

**Table B7 Selected British Units**

<table>
<thead>
<tr>
<th>Length</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 inch (in.)</td>
<td>2.54 cm (exactly)</td>
</tr>
<tr>
<td>1 foot (ft)</td>
<td>0.3048 m</td>
</tr>
<tr>
<td>1 mile (mi)</td>
<td>1.609 km</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Force</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 pound (lb)</td>
<td>4.448 N</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Energy</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 British thermal unit (Btu)</td>
<td>1.055×10³ J</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Power</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 horsepower (hp)</td>
<td>746 W</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pressure</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 lb/in²</td>
<td>6.895×10³ Pa</td>
</tr>
</tbody>
</table>

**Table B8 Other Units**

<table>
<thead>
<tr>
<th>Length</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 light year (ly)</td>
<td>9.46×10¹⁵ m</td>
</tr>
<tr>
<td>1 astronomical unit (au)</td>
<td>1.50×10¹¹ m</td>
</tr>
<tr>
<td>1 nautical mile</td>
<td>1.852 km</td>
</tr>
<tr>
<td>1 angstrom (Å)</td>
<td>10⁻¹⁰ m</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Area</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 acre (ac)</td>
<td>4.05×10³ m²</td>
</tr>
<tr>
<td>1 square foot (ft²)</td>
<td>9.29×10⁻² m²</td>
</tr>
<tr>
<td>1 barn (b)</td>
<td>10⁻²⁸ m²</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volume</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 liter (L)</td>
<td>10⁻³ m³</td>
</tr>
<tr>
<td>1 U.S. gallon (gal)</td>
<td>3.785×10⁻³ m³</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mass</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 solar mass</td>
<td>1.99×10³⁰ kg</td>
</tr>
<tr>
<td>1 metric ton</td>
<td>10³ kg</td>
</tr>
<tr>
<td>1 atomic mass unit (u)</td>
<td>1.6605×10⁻²⁷ kg</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year (y)</td>
<td>3.16×10⁷ s</td>
</tr>
<tr>
<td><strong>Speed</strong></td>
<td>1 day ((d)) = 86,400 s</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>1 mile per hour (mph) = 1.609 km/h</td>
</tr>
<tr>
<td></td>
<td>1 nautical mile per hour (naut) = 1.852 km/h</td>
</tr>
<tr>
<td><strong>Angle</strong></td>
<td>1 degree ((^\circ)) = 1.745 × 10^{-2} rad</td>
</tr>
<tr>
<td></td>
<td>1 minute of arc (') = 1 / 60 degree</td>
</tr>
<tr>
<td></td>
<td>1 second of arc ('') = 1 / 60 minute of arc</td>
</tr>
<tr>
<td></td>
<td>1 grad = 1.571 × 10^{-2} rad</td>
</tr>
<tr>
<td><strong>Energy</strong></td>
<td>1 kiloton TNT (kT) = 4.2 × 10^{12} J</td>
</tr>
<tr>
<td></td>
<td>1 kilowatt hour (kW · h) = 3.60 × 10^{6} J</td>
</tr>
<tr>
<td></td>
<td>1 food calorie (kcal) = 4186 J</td>
</tr>
<tr>
<td></td>
<td>1 calorie (cal) = 4.186 J</td>
</tr>
<tr>
<td></td>
<td>1 electron volt (eV) = 1.60 × 10^{-19} J</td>
</tr>
<tr>
<td><strong>Pressure</strong></td>
<td>1 atmosphere (atm) = 1.013 × 10^{5} Pa</td>
</tr>
<tr>
<td></td>
<td>1 millimeter of mercury (mm Hg) = 133.3 Pa</td>
</tr>
<tr>
<td></td>
<td>1 torricelli (torr) = 1 mm Hg = 133.3 Pa</td>
</tr>
<tr>
<td><strong>Nuclear decay rate</strong></td>
<td>1 curie (Ci) = 3.70 × 10^{10} Bq</td>
</tr>
</tbody>
</table>

**Table B9 Useful Formulae**

| **Circumference of a circle with radius** \(r\) or diameter \(d\) | \(C = 2\pi r = \pi d\) |
| **Area of a circle with radius** \(r\) or diameter \(d\) | \(A = \pi r^2 = \pi d^2 / 4\) |
| **Area of a sphere with radius** \(r\) | \(A = 4\pi r^2\) |
| **Volume of a sphere with radius** \(r\) | \(V = (4/3)(\pi r^3)\) |
APPENDIX C | GLOSSARY OF KEY SYMBOLS AND NOTATION

In this glossary, key symbols and notation are briefly defined.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>any symbol</td>
<td>average (indicated by a bar over a symbol—e.g., ( \bar{v} ) is average velocity)</td>
</tr>
<tr>
<td>°C</td>
<td>Celsius degree</td>
</tr>
<tr>
<td>°F</td>
<td>Fahrenheit degree</td>
</tr>
<tr>
<td>//</td>
<td>parallel</td>
</tr>
<tr>
<td>⊥</td>
<td>perpendicular</td>
</tr>
<tr>
<td>( \propto )</td>
<td>proportional to</td>
</tr>
<tr>
<td>±</td>
<td>plus or minus</td>
</tr>
<tr>
<td>0</td>
<td>zero as a subscript denotes an initial value</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>alpha rays</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>angular acceleration</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>temperature coefficient(s) of resistivity</td>
</tr>
<tr>
<td>( \beta )</td>
<td>beta rays</td>
</tr>
<tr>
<td>( \beta )</td>
<td>sound level</td>
</tr>
<tr>
<td>( \beta )</td>
<td>volume coefficient of expansion</td>
</tr>
<tr>
<td>( \beta^- )</td>
<td>electron emitted in nuclear beta decay</td>
</tr>
<tr>
<td>( \beta^+ )</td>
<td>positron decay</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>gamma rays</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>surface tension</td>
</tr>
<tr>
<td>( \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} )</td>
<td>a constant used in relativity</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>change in whatever quantity follows</td>
</tr>
<tr>
<td>( \delta )</td>
<td>uncertainty in whatever quantity follows</td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>change in energy between the initial and final orbits of an electron in an atom</td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>uncertainty in energy</td>
</tr>
<tr>
<td>( \Delta m )</td>
<td>difference in mass between initial and final products</td>
</tr>
<tr>
<td>( \Delta N )</td>
<td>number of decays that occur</td>
</tr>
<tr>
<td>( \Delta p )</td>
<td>change in momentum</td>
</tr>
<tr>
<td>( \Delta p )</td>
<td>uncertainty in momentum</td>
</tr>
<tr>
<td>( \Delta PE_g )</td>
<td>change in gravitational potential energy</td>
</tr>
<tr>
<td>( \Delta \theta )</td>
<td>rotation angle</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$\Delta s$</td>
<td>distance traveled along a circular path</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>uncertainty in time</td>
</tr>
<tr>
<td>$\Delta t_0$</td>
<td>proper time as measured by an observer at rest relative to the process</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>potential difference</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>uncertainty in position</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>permittivity of free space</td>
</tr>
<tr>
<td>$\eta$</td>
<td>viscosity</td>
</tr>
<tr>
<td>$\theta$</td>
<td>angle between the force vector and the displacement vector</td>
</tr>
<tr>
<td>$\theta$</td>
<td>angle between two lines</td>
</tr>
<tr>
<td>$\theta$</td>
<td>contact angle</td>
</tr>
<tr>
<td>$\theta$</td>
<td>direction of the resultant</td>
</tr>
<tr>
<td>$\theta_b$</td>
<td>Brewster's angle</td>
</tr>
<tr>
<td>$\theta_c$</td>
<td>critical angle</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>dielectric constant</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>decay constant of a nuclide</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>wavelength</td>
</tr>
<tr>
<td>$\lambda_n$</td>
<td>wavelength in a medium</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>permeability of free space</td>
</tr>
<tr>
<td>$\mu_k$</td>
<td>coefficient of kinetic friction</td>
</tr>
<tr>
<td>$\mu_s$</td>
<td>coefficient of static friction</td>
</tr>
<tr>
<td>$\nu_e$</td>
<td>electron neutrino</td>
</tr>
<tr>
<td>$\pi^+$</td>
<td>positive pion</td>
</tr>
<tr>
<td>$\pi^-$</td>
<td>negative pion</td>
</tr>
<tr>
<td>$\pi^0$</td>
<td>neutral pion</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>critical density, the density needed to just halt universal expansion</td>
</tr>
<tr>
<td>$\rho_f$</td>
<td>fluid density</td>
</tr>
<tr>
<td>$\rho_{\text{obj}}$</td>
<td>average density of an object</td>
</tr>
<tr>
<td>$\rho / \rho_w$</td>
<td>specific gravity</td>
</tr>
<tr>
<td>$\tau$</td>
<td>characteristic time constant for a resistance and inductance (RL) or resistance and capacitance (RC) circuit</td>
</tr>
<tr>
<td>$\tau$</td>
<td>characteristic time for a resistor and capacitor (RC) circuit</td>
</tr>
<tr>
<td>$\tau$</td>
<td>torque</td>
</tr>
<tr>
<td>$\Upsilon$</td>
<td>upsilon meson</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>magnetic flux</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>ϕ</td>
<td>phase angle</td>
</tr>
<tr>
<td>Ω</td>
<td>ohm (unit)</td>
</tr>
<tr>
<td>ω</td>
<td>angular velocity</td>
</tr>
<tr>
<td>A</td>
<td>ampere (current unit)</td>
</tr>
<tr>
<td>A</td>
<td>area</td>
</tr>
<tr>
<td>A</td>
<td>cross-sectional area</td>
</tr>
<tr>
<td>A</td>
<td>total number of nucleons</td>
</tr>
<tr>
<td>a</td>
<td>acceleration</td>
</tr>
<tr>
<td>a_B</td>
<td>Bohr radius</td>
</tr>
<tr>
<td>a_c</td>
<td>centripetal acceleration</td>
</tr>
<tr>
<td>a_t</td>
<td>tangential acceleration</td>
</tr>
<tr>
<td>AC</td>
<td>alternating current</td>
</tr>
<tr>
<td>AM</td>
<td>amplitude modulation</td>
</tr>
<tr>
<td>atm</td>
<td>atmosphere</td>
</tr>
<tr>
<td>B</td>
<td>baryon number</td>
</tr>
<tr>
<td>B</td>
<td>blue quark color</td>
</tr>
<tr>
<td>B^−</td>
<td>antiblue (yellow) antiquark color</td>
</tr>
<tr>
<td>b</td>
<td>quark flavor bottom or beauty</td>
</tr>
<tr>
<td>B</td>
<td>bulk modulus</td>
</tr>
<tr>
<td>B</td>
<td>magnetic field strength</td>
</tr>
<tr>
<td>B_{int}</td>
<td>electron's intrinsic magnetic field</td>
</tr>
<tr>
<td>B_{orb}</td>
<td>orbital magnetic field</td>
</tr>
<tr>
<td>BE</td>
<td>binding energy of a nucleus—it is the energy required to completely disassemble it into separate protons and neutrons</td>
</tr>
<tr>
<td>BE/A</td>
<td>binding energy per nucleon</td>
</tr>
<tr>
<td>Bq</td>
<td>becquerel—one decay per second</td>
</tr>
<tr>
<td>C</td>
<td>capacitance (amount of charge stored per volt)</td>
</tr>
<tr>
<td>C</td>
<td>coulomb (a fundamental SI unit of charge)</td>
</tr>
<tr>
<td>C_p</td>
<td>total capacitance in parallel</td>
</tr>
<tr>
<td>C_s</td>
<td>total capacitance in series</td>
</tr>
<tr>
<td>CG</td>
<td>center of gravity</td>
</tr>
<tr>
<td>CM</td>
<td>center of mass</td>
</tr>
<tr>
<td>c</td>
<td>quark flavor charm</td>
</tr>
<tr>
<td>c</td>
<td>specific heat</td>
</tr>
<tr>
<td>c</td>
<td>speed of light</td>
</tr>
<tr>
<td>Cal</td>
<td>kilocalorie</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>cal</td>
<td>calorie</td>
</tr>
<tr>
<td>$COP_{hp}$</td>
<td>heat pump's coefficient of performance</td>
</tr>
<tr>
<td>$COP_{ref}$</td>
<td>coefficient of performance for refrigerators and air conditioners</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>cosine</td>
</tr>
<tr>
<td>$\cot \theta$</td>
<td>cotangent</td>
</tr>
<tr>
<td>$\csc \theta$</td>
<td>cosecant</td>
</tr>
<tr>
<td>$D$</td>
<td>diffusion constant</td>
</tr>
<tr>
<td>$d$</td>
<td>displacement</td>
</tr>
<tr>
<td>$d$</td>
<td>quark flavor down</td>
</tr>
<tr>
<td>dB</td>
<td>decibel</td>
</tr>
<tr>
<td>$d_i$</td>
<td>distance of an image from the center of a lens</td>
</tr>
<tr>
<td>$d_o$</td>
<td>distance of an object from the center of a lens</td>
</tr>
<tr>
<td>DC</td>
<td>direct current</td>
</tr>
<tr>
<td>$E$</td>
<td>electric field strength</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>emf (voltage) or Hall electromotive force</td>
</tr>
<tr>
<td>emf</td>
<td>electromotive force</td>
</tr>
<tr>
<td>$E$</td>
<td>energy of a single photon</td>
</tr>
<tr>
<td>$E$</td>
<td>nuclear reaction energy</td>
</tr>
<tr>
<td>$E$</td>
<td>relativistic total energy</td>
</tr>
<tr>
<td>$E$</td>
<td>total energy</td>
</tr>
<tr>
<td>$E_0$</td>
<td>ground state energy for hydrogen</td>
</tr>
<tr>
<td>$E_0$</td>
<td>rest energy</td>
</tr>
<tr>
<td>EC</td>
<td>electron capture</td>
</tr>
<tr>
<td>$E_{cap}$</td>
<td>energy stored in a capacitor</td>
</tr>
<tr>
<td>$Eff$</td>
<td>efficiency—the useful work output divided by the energy input</td>
</tr>
<tr>
<td>$Eff_C$</td>
<td>Carnot efficiency</td>
</tr>
<tr>
<td>$E_{in}$</td>
<td>energy consumed (food digested in humans)</td>
</tr>
<tr>
<td>$E_{ind}$</td>
<td>energy stored in an inductor</td>
</tr>
<tr>
<td>$E_{out}$</td>
<td>energy output</td>
</tr>
<tr>
<td>$e$</td>
<td>emissivity of an object</td>
</tr>
<tr>
<td>$e^+$</td>
<td>antielectron or positron</td>
</tr>
<tr>
<td>eV</td>
<td>electron volt</td>
</tr>
<tr>
<td>F</td>
<td>farad (unit of capacitance, a coulomb per volt)</td>
</tr>
<tr>
<td>F</td>
<td>focal point of a lens</td>
</tr>
<tr>
<td>F</td>
<td>force</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$F$</td>
<td>magnitude of a force</td>
</tr>
<tr>
<td>$F$</td>
<td>restoring force</td>
</tr>
<tr>
<td>$F_B$</td>
<td>buoyant force</td>
</tr>
<tr>
<td>$F_c$</td>
<td>centripetal force</td>
</tr>
<tr>
<td>$F_i$</td>
<td>force input</td>
</tr>
<tr>
<td>$F_{\text{net}}$</td>
<td>net force</td>
</tr>
<tr>
<td>$F_o$</td>
<td>force output</td>
</tr>
<tr>
<td>$F_{\text{FM}}$</td>
<td>frequency modulation</td>
</tr>
<tr>
<td>$f$</td>
<td>focal length</td>
</tr>
<tr>
<td>$f$</td>
<td>frequency</td>
</tr>
<tr>
<td>$f_0$</td>
<td>resonant frequency of a resistance, inductance, and capacitance ($RLC$) series circuit</td>
</tr>
<tr>
<td>$f_0$</td>
<td>threshold frequency for a particular material (photoelectric effect)</td>
</tr>
<tr>
<td>$f_1$</td>
<td>fundamental</td>
</tr>
<tr>
<td>$f_2$</td>
<td>first overtone</td>
</tr>
<tr>
<td>$f_3$</td>
<td>second overtone</td>
</tr>
<tr>
<td>$f_B$</td>
<td>beat frequency</td>
</tr>
<tr>
<td>$f_k$</td>
<td>magnitude of kinetic friction</td>
</tr>
<tr>
<td>$f_s$</td>
<td>magnitude of static friction</td>
</tr>
<tr>
<td>$G$</td>
<td>gravitational constant</td>
</tr>
<tr>
<td>$G^-$</td>
<td>green quark color</td>
</tr>
<tr>
<td>$\bar{G}$</td>
<td>antigreen (magenta) antiquark color</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration due to gravity</td>
</tr>
<tr>
<td>$g$</td>
<td>gluons (carrier particles for strong nuclear force)</td>
</tr>
<tr>
<td>$h$</td>
<td>change in vertical position</td>
</tr>
<tr>
<td>$h$</td>
<td>height above some reference point</td>
</tr>
<tr>
<td>$h$</td>
<td>maximum height of a projectile</td>
</tr>
<tr>
<td>$h$</td>
<td>Planck’s constant</td>
</tr>
<tr>
<td>$hf$</td>
<td>photon energy</td>
</tr>
<tr>
<td>$h_i$</td>
<td>height of the image</td>
</tr>
<tr>
<td>$h_o$</td>
<td>height of the object</td>
</tr>
<tr>
<td>$I$</td>
<td>electric current</td>
</tr>
<tr>
<td>$I$</td>
<td>intensity</td>
</tr>
<tr>
<td>$I$</td>
<td>intensity of a transmitted wave</td>
</tr>
<tr>
<td>$I$</td>
<td>moment of inertia (also called rotational inertia)</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$I_0$</td>
<td>intensity of a polarized wave before passing through a filter</td>
</tr>
<tr>
<td>$I_{\text{ave}}$</td>
<td>average intensity for a continuous sinusoidal electromagnetic wave</td>
</tr>
<tr>
<td>$I_{\text{rms}}$</td>
<td>average current</td>
</tr>
<tr>
<td>$J$</td>
<td>joule</td>
</tr>
<tr>
<td>$J/\Psi$</td>
<td>Joules/psi meson</td>
</tr>
<tr>
<td>$K$</td>
<td>kelvin</td>
</tr>
<tr>
<td>$k$</td>
<td>Boltzmann constant</td>
</tr>
<tr>
<td>$k$</td>
<td>force constant of a spring</td>
</tr>
<tr>
<td>$K_{\alpha}$</td>
<td>x rays created when an electron falls into an $n = 1$ shell vacancy from the $n = 3$ shell</td>
</tr>
<tr>
<td>$K_{\beta}$</td>
<td>x rays created when an electron falls into an $n = 2$ shell vacancy from the $n = 3$ shell</td>
</tr>
<tr>
<td>kcal</td>
<td>kilocalorie</td>
</tr>
<tr>
<td>KE</td>
<td>translational kinetic energy</td>
</tr>
<tr>
<td>KE + PE</td>
<td>mechanical energy</td>
</tr>
<tr>
<td>KE$_e$</td>
<td>kinetic energy of an ejected electron</td>
</tr>
<tr>
<td>KE$_{\text{rel}}$</td>
<td>relativistic kinetic energy</td>
</tr>
<tr>
<td>KE$_{\text{rot}}$</td>
<td>rotational kinetic energy</td>
</tr>
<tr>
<td>KE</td>
<td>thermal energy</td>
</tr>
<tr>
<td>kg</td>
<td>kilogram (a fundamental SI unit of mass)</td>
</tr>
<tr>
<td>$L$</td>
<td>angular momentum</td>
</tr>
<tr>
<td>$L$</td>
<td>magnitude of angular momentum</td>
</tr>
<tr>
<td>$L$</td>
<td>self-inductance</td>
</tr>
<tr>
<td>$L^\ell$</td>
<td>angular momentum quantum number</td>
</tr>
<tr>
<td>$L_{\alpha}$</td>
<td>x rays created when an electron falls into an $n = 2$ shell from the $n = 3$ shell</td>
</tr>
<tr>
<td>$L_e$</td>
<td>electron total family number</td>
</tr>
<tr>
<td>$L_{\mu}$</td>
<td>muon family total number</td>
</tr>
<tr>
<td>$L_{\tau}$</td>
<td>tau family total number</td>
</tr>
<tr>
<td>$L_f$</td>
<td>heat of fusion</td>
</tr>
<tr>
<td>$L_f$ and $L_v$</td>
<td>latent heat coefficients</td>
</tr>
<tr>
<td>$L_{\text{orb}}$</td>
<td>orbital angular momentum</td>
</tr>
<tr>
<td>$L_s$</td>
<td>heat of sublimation</td>
</tr>
<tr>
<td>$L_v$</td>
<td>heat of vaporization</td>
</tr>
<tr>
<td>$L_z$</td>
<td>z-component of the angular momentum</td>
</tr>
<tr>
<td>$M$</td>
<td>angular magnification</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$M$</td>
<td>mutual inductance</td>
</tr>
<tr>
<td>$m$</td>
<td>indicates metastable state</td>
</tr>
<tr>
<td>$m$</td>
<td>magnification</td>
</tr>
<tr>
<td>$m$</td>
<td>mass</td>
</tr>
<tr>
<td>$m$</td>
<td>mass of an object as measured by a person at rest relative to the object</td>
</tr>
<tr>
<td>$m$</td>
<td>meter (a fundamental SI unit of length)</td>
</tr>
<tr>
<td>$m$</td>
<td>order of interference</td>
</tr>
<tr>
<td>$m$</td>
<td>overall magnification (product of the individual magnifications)</td>
</tr>
<tr>
<td>$m(A \chi)$</td>
<td>atomic mass of a nuclide</td>
</tr>
<tr>
<td>MA</td>
<td>mechanical advantage</td>
</tr>
<tr>
<td>$m_e$</td>
<td>magnification of the eyepiece</td>
</tr>
<tr>
<td>$m_e$</td>
<td>mass of the electron</td>
</tr>
<tr>
<td>$m_e$</td>
<td>angular momentum projection quantum number</td>
</tr>
<tr>
<td>$m_n$</td>
<td>mass of a neutron</td>
</tr>
<tr>
<td>$m_o$</td>
<td>magnification of the objective lens</td>
</tr>
<tr>
<td>mol</td>
<td>mole</td>
</tr>
<tr>
<td>$m_p$</td>
<td>mass of a proton</td>
</tr>
<tr>
<td>$m_s$</td>
<td>spin projection quantum number</td>
</tr>
<tr>
<td>$N$</td>
<td>magnitude of the normal force</td>
</tr>
<tr>
<td>$N$</td>
<td>newton</td>
</tr>
<tr>
<td>$N$</td>
<td>normal force</td>
</tr>
<tr>
<td>$N$</td>
<td>number of neutrons</td>
</tr>
<tr>
<td>$n$</td>
<td>index of refraction</td>
</tr>
<tr>
<td>$n$</td>
<td>number of free charges per unit volume</td>
</tr>
<tr>
<td>$N_A$</td>
<td>Avogadro's number</td>
</tr>
<tr>
<td>$N_r$</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>$N \cdot m$</td>
<td>newton-meter (work-energy unit)</td>
</tr>
<tr>
<td>$N \cdot m$</td>
<td>newtons times meters (SI unit of torque)</td>
</tr>
<tr>
<td>OE</td>
<td>other energy</td>
</tr>
<tr>
<td>$P$</td>
<td>power</td>
</tr>
<tr>
<td>$P$</td>
<td>power of a lens</td>
</tr>
<tr>
<td>$P$</td>
<td>pressure</td>
</tr>
<tr>
<td>$P$</td>
<td>momentum</td>
</tr>
<tr>
<td>$p$</td>
<td>momentum magnitude</td>
</tr>
<tr>
<td>$p$</td>
<td>relativistic momentum</td>
</tr>
<tr>
<td>$P_{tot}$</td>
<td>total momentum</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$P_{\text{tot}}$</td>
<td>total momentum some time later</td>
</tr>
<tr>
<td>$P_{\text{abs}}$</td>
<td>absolute pressure</td>
</tr>
<tr>
<td>$P_{\text{atm}}$</td>
<td>atmospheric pressure</td>
</tr>
<tr>
<td>$P_{\text{atm}}$</td>
<td>standard atmospheric pressure</td>
</tr>
<tr>
<td>PE</td>
<td>potential energy</td>
</tr>
<tr>
<td>$\text{PE}_{\text{el}}$</td>
<td>elastic potential energy</td>
</tr>
<tr>
<td>$\text{PE}_{\text{elec}}$</td>
<td>electric potential energy</td>
</tr>
<tr>
<td>$\text{PE}_{\text{s}}$</td>
<td>potential energy of a spring</td>
</tr>
<tr>
<td>$P_{\text{g}}$</td>
<td>gauge pressure</td>
</tr>
<tr>
<td>$P_{\text{in}}$</td>
<td>power consumption or input</td>
</tr>
<tr>
<td>$P_{\text{out}}$</td>
<td>useful power output going into useful work or a desired, form of energy</td>
</tr>
<tr>
<td>$Q$</td>
<td>latent heat</td>
</tr>
<tr>
<td>$Q$</td>
<td>net heat transferred into a system</td>
</tr>
<tr>
<td>$Q$</td>
<td>flow rate—volume per unit time flowing past a point</td>
</tr>
<tr>
<td>$+Q$</td>
<td>positive charge</td>
</tr>
<tr>
<td>$-Q$</td>
<td>negative charge</td>
</tr>
<tr>
<td>$q$</td>
<td>electron charge</td>
</tr>
<tr>
<td>$q_p$</td>
<td>charge of a proton</td>
</tr>
<tr>
<td>$q$</td>
<td>test charge</td>
</tr>
<tr>
<td>QF</td>
<td>quality factor</td>
</tr>
<tr>
<td>$R$</td>
<td>activity, the rate of decay</td>
</tr>
<tr>
<td>$R$</td>
<td>radius of curvature of a spherical mirror</td>
</tr>
<tr>
<td>$R$</td>
<td>red quark color</td>
</tr>
<tr>
<td>$R$</td>
<td>antired (cyan) quark color</td>
</tr>
<tr>
<td>$R$</td>
<td>resistance</td>
</tr>
<tr>
<td>$R$</td>
<td>resultant or total displacement</td>
</tr>
<tr>
<td>$R$</td>
<td>Rydberg constant</td>
</tr>
<tr>
<td>$R$</td>
<td>universal gas constant</td>
</tr>
<tr>
<td>$r$</td>
<td>distance from pivot point to the point where a force is applied</td>
</tr>
<tr>
<td>$r$</td>
<td>internal resistance</td>
</tr>
<tr>
<td>$r_{\perp}$</td>
<td>perpendicular lever arm</td>
</tr>
<tr>
<td>$r$</td>
<td>radius of a nucleus</td>
</tr>
<tr>
<td>$r$</td>
<td>radius of curvature</td>
</tr>
<tr>
<td>$r$</td>
<td>resistivity</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>-------</td>
<td>------------</td>
</tr>
<tr>
<td>r or rad</td>
<td>radiation dose unit</td>
</tr>
<tr>
<td>rem</td>
<td>roentgen equivalent man</td>
</tr>
<tr>
<td>rad</td>
<td>radian</td>
</tr>
<tr>
<td>RBE</td>
<td>relative biological effectiveness</td>
</tr>
<tr>
<td>RC</td>
<td>resistor and capacitor circuit</td>
</tr>
<tr>
<td>rms</td>
<td>root mean square</td>
</tr>
<tr>
<td>r_n</td>
<td>radius of the nth H-atom orbit</td>
</tr>
<tr>
<td>R_p</td>
<td>total resistance of a parallel connection</td>
</tr>
<tr>
<td>R_s</td>
<td>total resistance of a series connection</td>
</tr>
<tr>
<td>R_s</td>
<td>Schwarzschild radius</td>
</tr>
<tr>
<td>S</td>
<td>entropy</td>
</tr>
<tr>
<td>S</td>
<td>intrinsic spin (intrinsic angular momentum)</td>
</tr>
<tr>
<td>S</td>
<td>magnitude of the intrinsic (internal) spin angular momentum</td>
</tr>
<tr>
<td>S</td>
<td>shear modulus</td>
</tr>
<tr>
<td>S</td>
<td>strangeness quantum number</td>
</tr>
<tr>
<td>s</td>
<td>quark flavor strange</td>
</tr>
<tr>
<td>s</td>
<td>second (fundamental SI unit of time)</td>
</tr>
<tr>
<td>s</td>
<td>spin quantum number</td>
</tr>
<tr>
<td>s</td>
<td>total displacement</td>
</tr>
<tr>
<td>sec θ</td>
<td>secant</td>
</tr>
<tr>
<td>sin θ</td>
<td>sine</td>
</tr>
<tr>
<td>s_c</td>
<td>z-component of spin angular momentum</td>
</tr>
<tr>
<td>T</td>
<td>period—time to complete one oscillation</td>
</tr>
<tr>
<td>T</td>
<td>temperature</td>
</tr>
<tr>
<td>T_c</td>
<td>critical temperature—temperature below which a material becomes a superconductor</td>
</tr>
<tr>
<td>T</td>
<td>tension</td>
</tr>
<tr>
<td>T</td>
<td>tesla (magnetic field strength B)</td>
</tr>
<tr>
<td>t</td>
<td>quark flavor top or truth</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>t_{1/2}</td>
<td>half-life—the time in which half of the original nuclei decay</td>
</tr>
<tr>
<td>tan θ</td>
<td>tangent</td>
</tr>
<tr>
<td>U</td>
<td>internal energy</td>
</tr>
<tr>
<td>u</td>
<td>quark flavor up</td>
</tr>
<tr>
<td>u</td>
<td>unified atomic mass unit</td>
</tr>
<tr>
<td>u</td>
<td>velocity of an object relative to an observer</td>
</tr>
<tr>
<td>u'</td>
<td>velocity relative to another observer</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>( V )</td>
<td>electric potential</td>
</tr>
<tr>
<td>( V )</td>
<td>terminal voltage</td>
</tr>
<tr>
<td>( V )</td>
<td>volt (unit)</td>
</tr>
<tr>
<td>( V )</td>
<td>volume</td>
</tr>
<tr>
<td>( v )</td>
<td>relative velocity between two observers</td>
</tr>
<tr>
<td>( v )</td>
<td>speed of light in a material</td>
</tr>
<tr>
<td>( v )</td>
<td>velocity</td>
</tr>
<tr>
<td>( \bar{v} )</td>
<td>average fluid velocity</td>
</tr>
<tr>
<td>( V_B - V_A )</td>
<td>change in potential</td>
</tr>
<tr>
<td>( v_d )</td>
<td>drift velocity</td>
</tr>
<tr>
<td>( V_P )</td>
<td>transformer input voltage</td>
</tr>
<tr>
<td>( V_{\text{rms}} )</td>
<td>rms voltage</td>
</tr>
<tr>
<td>( V_s )</td>
<td>transformer output voltage</td>
</tr>
<tr>
<td>( v_{\text{tot}} )</td>
<td>total velocity</td>
</tr>
<tr>
<td>( v_w )</td>
<td>propagation speed of sound or other wave</td>
</tr>
<tr>
<td>( v_w )</td>
<td>wave velocity</td>
</tr>
<tr>
<td>( W )</td>
<td>work</td>
</tr>
<tr>
<td>( W )</td>
<td>net work done by a system</td>
</tr>
<tr>
<td>( W )</td>
<td>watt</td>
</tr>
<tr>
<td>( w )</td>
<td>weight</td>
</tr>
<tr>
<td>( w_f )</td>
<td>weight of the fluid displaced by an object</td>
</tr>
<tr>
<td>( W_C )</td>
<td>total work done by all conservative forces</td>
</tr>
<tr>
<td>( W_{\text{nc}} )</td>
<td>total work done by all nonconservative forces</td>
</tr>
<tr>
<td>( W_{\text{out}} )</td>
<td>useful work output</td>
</tr>
<tr>
<td>( X )</td>
<td>amplitude</td>
</tr>
<tr>
<td>( X )</td>
<td>symbol for an element</td>
</tr>
<tr>
<td>( Z ) ( X_N )</td>
<td>notation for a particular nuclide</td>
</tr>
<tr>
<td>( x )</td>
<td>deformation or displacement from equilibrium</td>
</tr>
<tr>
<td>( x )</td>
<td>displacement of a spring from its undeformed position</td>
</tr>
<tr>
<td>( x )</td>
<td>horizontal axis</td>
</tr>
<tr>
<td>( X_C )</td>
<td>capacitive reactance</td>
</tr>
<tr>
<td>( X_L )</td>
<td>inductive reactance</td>
</tr>
<tr>
<td>( x_{\text{rms}} )</td>
<td>root mean square diffusion distance</td>
</tr>
<tr>
<td>( y )</td>
<td>vertical axis</td>
</tr>
<tr>
<td>( Y )</td>
<td>elastic modulus or Young's modulus</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------</td>
</tr>
<tr>
<td>Z</td>
<td>atomic number (number of protons in a nucleus)</td>
</tr>
<tr>
<td>Z</td>
<td>impedance</td>
</tr>
</tbody>
</table>
ink-jet printer, 463
Instantaneous acceleration, 43
Instantaneous speed, 37
Instantaneous velocity, 37
insulator, 463
insulators, 442
intensity, 423, 426, 528, 530
Internal kinetic energy, 296, 298
Internal equilibrium, 326, 339
ionosphere, 455, 463
isolated system, 294, 308
J
joule, 244, 276
K
kilogram, 14, 26
kilowatt-hour, 276
kilowatt-hours, 269
kinematics of rotational motion, 352, 380
kinetic energy, 246, 276
kinetic friction, 178, 196
L
laser printer, 463
Laser printers, 459
law, 9, 26
law of conservation of angular momentum, 372, 380
law of conservation of charge, 440, 463
law of conservation of energy, 262, 276
law of inertia, 136, 166, 168
Linear momentum, 288
linear momentum, 308
longitudinal wave, 417, 426
M
magnetic field, 513, 530
Magnetic field lines, 511
magnetic field lines, 530
magnetic field strength, 530
magnitude, 94
magnitude (of a vector), 121
magnitude of kinetic friction, 196
magnitude of kinetic friction \( f_k \), 179
magnitude of static friction, 196
magnitude of static friction \( f_s \), 178
mass, 136, 166
Mass, 166
maximum field strength, 528, 530
Maxwell's equations, 510, 530
mechanical advantage, 330, 339
mechanical energy, 256, 276, 501
Mechanical energy, 480
metabolic rate, 270, 276
meter, 14, 26
method of adding percents, 22, 26
metric system, 15, 27
Microgravity, 223
microgravity, 229
Microwaves, 520
microwaves, 530
model, 9, 27, 38, 76
Modern physics, 12
moment of inertia, 357, 358, 380
motion, 106, 121
N
natural frequency, 413, 426
net external force, 137, 166
net work, 276
neutral equilibrium, 326, 339
Newton, 138
newton-meters, 244
Newton's first law of motion, 135, 136, 166, 168
Newton's second law of motion, 136, 166
Newton's third law of motion, 142, 167, 166
Newton's universal law of gravitation, 218, 229
Nodes, 420
nodes, 426
non-inertial frame of reference, 215, 229
nonconservative force, 257, 276
normal force, 146, 166
Nuclear energy, 263
nuclear energy, 276
Q
optically active, 571, 576
order, 546, 576
order of magnitude, 15, 27
oscillate, 392, 426, 530
overdamped, 426
overtones, 420, 426
P
parallel plate capacitor, 488, 501
percent uncertainty, 22, 27
perfectly inelastic collision, 298, 308
period, 396, 426
periodic motion, 396, 426
perpendicular lever arm, 320, 339
phase-contrast microscope, 573, 576
photocathode, 458, 463
physical quantity, 13, 27
Physics, 6
physics, 27
pH, 204, 229
point charge, 447, 463
point masses, 302, 308
polar molecule, 453, 463, 493, 501
polarization, 442, 463, 576
Polarization, 564
polarization microscope, 574, 576
polarized, 454, 463, 564, 576
position, 32, 77
potential difference, 477
potential difference (or voltage), 501
potential energy, 254, 255, 276
potential energy of a spring, 254, 276
power, 266, 276
precision, 20, 27
projectile, 106, 121
Projectile motion, 106
projectile motion, 121
proton, 463
protons, 438
Q
Quantum mechanics, 12
quantum mechanics, 27
quark, 308
quarks, 295
R
Radar, 520
radar, 530
radius, 205, 229
radiant energy, 263, 276
radio waves, 510, 517, 530
radius of curvature, 204, 230
range, 112, 121
Rayleigh criterion, 555, 576
reflected light, 567
reflected light that is completely polarized, 576
reflective velocity, 118
relative velocity, 121
Relativity, 12
relativity, 27, 118, 121
Renewable forms of energy, 273
renewable forms of energy, 276
resonance, 413, 426
resonant, 514, 530
resonate, 413, 426
restoring force, 392, 426
resultant, 94, 121
resultant vector, 93, 121
right-hand rule, 378, 380
RLC circuit, 530
rotation angle, 204, 230
rotational inertia, 357, 380
rotational kinetic energy, 361, 362, 380
S
scalar, 35, 77, 99, 121, 481, 501
scientific method, 10, 27
screening, 453, 463
second, 14, 27
second law of motion, 289, 308
shear deformation, 193, 196
SI unit of torque, 321
SI units, 13, 27
SI units of torque, 339
significant figures, 23, 27
Simple Harmonic Motion, 398
simple harmonic motion, 426
simple harmonic oscillator, 398, 426
simple pendulum, 402, 426
slope, 69, 77
speed of light, 530
stable equilibrium, 324, 339
standing wave, 419, 514, 530
static electricity, 436, 463
static equilibrium, 318, 328, 339, 339
static friction, 178, 196
Stokes' law, 187, 196
strain, 192, 196
stress, 192, 196
superposition, 418, 426
system, 136, 166
T
tail, 93, 121
tangential acceleration, 350, 380
Television, 519
tensile strength, 188
tensile strength, 196
tension, 149, 166

test charge, 447, 463

theory, 9, 27

thermal agitation, 520, 531

thermal energy, 257, 263, 276

thin film interference, 559, 576

thrust, 143, 167, 166

time, 36, 77

Torque, 320

torque, 339, 357, 380

trajectory, 106, 121

transverse wave, 417, 426, 514, 531

TV, 531

U

ultra high frequency, 519

ultra-high frequency (UHF), 531

ultracentrifuge, 210, 230

Ultraviolet (UV) microscopes, 572

ultraviolet (UV) microscopes, 576

ultraviolet radiation (UV), 523, 531

uncertainty, 21, 27

under damping, 426

underdamped, 410

uniform circular motion, 204, 230

units, 13, 27

unpolarized, 565, 576

unstable equilibrium, 324, 339

useful work, 271, 276

V

Van de Graaff generator, 463

Van de Graaff generators, 457

vector, 35, 77, 92, 121, 463, 481, 501

vector addition, 114, 121, 450, 463

vectors, 91, 449

velocity, 114, 121

vertically polarized, 564, 576

very high frequency, 519

very high frequency (VHF), 531

Visible light, 522

visible light, 531

voltage, 477

W

watt, 267, 276

wave, 415, 426

wave velocity, 415, 426

wavelength, 416, 426, 513, 531

wavelength in a medium, 541, 576

waves, 392

weight, 138, 166

Weight, 146

work, 242, 276

work-energy theorem, 246, 276, 362, 381, 380

X

X-ray, 526, 531

xerography, 458, 463

Y

y-intercept, 69, 77
Index
ABOUT CONNEXIONS

Since 1999, Connexions has been pioneering a global system where anyone can create course materials and make them fully accessible and easily reusable free of charge. We are a Web-based authoring, teaching and learning environment open to anyone interested in education, including students, teachers, professors and lifelong learners. We connect ideas and facilitate educational communities. Connexions's modular, interactive courses are in use worldwide by universities, community colleges, K-12 schools, distance learners, and lifelong learners. Connexions materials are in many languages, including English, Spanish, Chinese, Japanese, Italian, Vietnamese, French, Portuguese, and Thai.