Partial Fractions*

OpenStax

Abstract

In this section, you will:

- Decompose P(x)/Q(x) , where Q(x) has only nonrepeated linear factors.
- Decompose P(x)/Q(x) , where Q(x) has repeated linear factors.
- Decompose P(x)/Q(x) , where Q(x) has a nonrepeated irreducible quadratic factor.
- Decompose P(x)/Q(x) , where Q(x) has a repeated irreducible quadratic factor.

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized—the decomposition of rational expressions.

Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

1 Decomposing $\frac{P(x)}{Q(x)}$ Where *Q(x)* Has Only Nonrepeated Linear Factors

Recall the algebra regarding adding and subtracting rational expressions. These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at **partial fraction decomposition**, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified **rational expression** to the original expressions, called the **partial fractions**.

For example, suppose we add the following fractions:

$$\frac{2}{x-3}+\frac{-1}{x+2} \tag{1}$$

We would first need to find a common denominator, $(x+2)(x-3)$.

Next, we would write each expression with this common denominator and find the sum of the terms.

$$\begin{aligned} &\tfrac{2}{x-3}\left(\tfrac{x+2}{x+2}\right)+\tfrac{-1}{x+2}\left(\tfrac{x-3}{x-3}\right)= \\ &\tfrac{2x+4-x+3}{(x+2)(x-3)}=\tfrac{x+7}{x^2-x-6} \end{aligned} \tag{2}$$

Partial fraction **decomposition** is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.

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$$\underbrace{\frac{x+7}{x^2-x-6}}_{\text{Simplified sum}} = \underbrace{\frac{2}{x-3}+\frac{-1}{x+2}}_{\text{Partial fraction decomposition}} \tag{3}$$

We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of $x^2 - x - 6$ are $(x-3)(x+2)$, the denominators of the decomposed rational expression. So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.

A GENERAL NOTE LABEL: The **partial fraction decomposition** of $\frac{P(x)}{Q(x)}$ when $Q(x)$ has nonrepeated linear factors and the degree of $P(x)$ is less than the degree of $Q(x)$ is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)}. \tag{4}$$

HOW TO FEATURE: **Given a rational expression with distinct linear factors in the denominator, decompose it.**

1. Use a variable for the original numerators, usually $A, B,$ or C, depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use A_n for each numerator

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{5}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 1

Decomposing a Rational Function with Distinct Linear Factors

Decompose the given **rational expression** with distinct linear factors.

$$\frac{3x}{(x+2)(x-1)} \tag{6}$$

Solution

We will separate the denominator factors and give each numerator a symbolic label, like A, B ,or C.

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)} \tag{7}$$

Multiply both sides of the equation by the common denominator to eliminate the fractions:

$$(x+2)(x-1)\left[\frac{3x}{(x+2)(x-1)}\right] = \overline{)\,(x+2)}\,(x-1)\left[\frac{A}{\overline{)\,(x+2)}}\right] + (x+2)\,\overline{)\,(x-1)}\left[\frac{B}{\overline{)\,(x-1)}}\right] \tag{8}$$

The resulting equation is

$$3x = A(x-1) + B(x+2) \tag{9}$$

Expand the right side of the equation and collect like terms.

$$\begin{aligned} 3x &= Ax - A + Bx + 2B \\ 3x &= (A+B)x - A + 2B \end{aligned} \tag{10}$$

Set up a system of equations associating corresponding coefficients.

$$\begin{aligned} 3 &= \quad A + B \\ 0 &= -A + 2B \end{aligned} \tag{11}$$

Add the two equations and solve for B.

$$\begin{aligned} 3 &= \quad A + B \\ 0 &= -A + 2B \\ &\quad \Psi \\ 3 &= \quad 0 \ + 3B \\ 1 &= B \end{aligned} \tag{12}$$

Substitute $B = 1$ into one of the original equations in the system.

$$\begin{aligned} 3 &= A + 1 \\ 2 &= A \end{aligned} \tag{13}$$

Thus, the partial fraction decomposition is

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)} \tag{14}$$

Another method to use to solve for A or B is by considering the equation that resulted from eliminating the fractions and substituting a value for x that will make either the A- or B-term equal 0. If we let $x = 1$, the
$A-$ term becomes 0 and we can simply solve for B.

$$\begin{aligned} 3x &= A(x-1) + B(x+2) \\ 3(1) &= A[(1)-1] + B[(1)+2] \\ 3 &= 0 + 3B \\ 1 &= B \end{aligned} \tag{15}$$

Next, either substitute $B = 1$ into the equation and solve for A, or make the B-term 0 by substituting $x = -2$ into the equation.

$$\begin{aligned} 3x &= A(x-1) + B(x+2) \\ 3(-2) &= A[(-2)-1] + B[(-2)+2] \\ -6 &= -3A + 0 \\ \frac{-6}{-3} &= A \\ 2 &= A \end{aligned} \tag{16}$$

We obtain the same values for A and B using either method, so the decompositions are the same using either method.

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)} \tag{17}$$

Although this method is not seen very often in textbooks, we present it here as an alternative that may make some partial fraction decompositions easier. It is known as the **Heaviside method**, named after Charles Heaviside, a pioneer in the study of electronics.

TRY IT FEATURE:

Exercise 2 ***(Solution on p. 16.)***

Find the partial fraction decomposition of the following expression.

$$\frac{x}{(x-3)(x-2)} \tag{18}$$

2 Decomposing $\frac{P(x)}{Q(x)}$ Where *Q(x)* Has Repeated Linear Factors

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors. We must remember that we account for repeated factors by writing each factor in increasing powers.

A GENERAL NOTE LABEL: The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated linear factor occurring n times and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n} \tag{19}$$

Write the denominator powers in increasing order.

HOW TO FEATURE: **Given a rational expression with repeated linear factors, decompose it.**

1. Use a variable like A, B, or C for the numerators and account for increasing powers of the denominators.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \ . \ . \ . + \frac{A_n}{(ax+b)^n} \tag{20}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 2

Decomposing with Repeated Linear Factors

Decompose the given rational expression with repeated linear factors.

$$\frac{-x^2+2x+4}{x^3-4x^2+4x} \tag{21}$$

Solution

The denominator factors are $x(x-2)^2$. To allow for the repeated factor of $(x-2)$, the decomposition will include three denominators: $x, (x-2)$, and $(x-2)^2$. Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} \tag{22}$$

Next, we multiply both sides by the common denominator.

$$\begin{aligned} x(x-2)^2\left[\tfrac{-x^2+2x+4}{x(x-2)^2}\right] &= \left[\tfrac{A}{x} + \tfrac{B}{(x-2)} + \tfrac{C}{(x-2)^2}\right]x(x-2)^2 \\ -x^2+2x+4 &= A(x-2)^2 + Bx(x-2) + Cx \end{aligned} \tag{23}$$

On the right side of the equation, we expand and collect like terms.

$$\begin{aligned} -x^2+2x+4 &= A\left(x^2-4x+4\right) + B\left(x^2-2x\right) + Cx \\ &= Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx \\ &= (A+B)x^2 + (-4A-2B+C)x + 4A \end{aligned} \tag{24}$$

Next, we compare the coefficients of both sides. This will give the system of equations in three variables:

$$-x^2+2x+4 = (A+B)x^2 + (-4A-2B+C)x + 4A \tag{25}$$

$$\begin{aligned} A+B &= -1 \quad (1) \\ -4A-2B+C &= 2 \quad (2) \\ 4A &= 4 \quad (3) \end{aligned} \tag{26}$$

Solving for A, we have

$$\begin{aligned} 4A &= 4 \\ A &= 1 \end{aligned} \tag{27}$$

Substitute $A = 1$ into equation (1).

$$\begin{aligned} A + B &= -1 \\ (1) + B &= -1 \\ B &= -2 \end{aligned} \tag{28}$$

Then, to solve for C, substitute the values for A and B into equation (2).

$$\begin{aligned} -4A - 2B + C &= 2 \\ -4(1) - 2(-2) + C &= 2 \\ -4 + 4 + C &= 2 \\ C &= 2 \end{aligned} \tag{29}$$

Thus,

$$\frac{-x^2 + 2x + 4}{x^3 - 4x^2 + 4x} = \frac{1}{x} - \frac{2}{(x-2)} + \frac{2}{(x-2)^2} \tag{30}$$

TRY IT FEATURE:

Exercise 4 ***(Solution on p. 16.)***

Find the partial fraction decomposition of the expression with repeated linear factors.

$$\frac{6x - 11}{(x-1)^2} \tag{31}$$

3 Decomposing $\frac{P(x)}{Q(x)}$, Where $Q(x)$ Has a Nonrepeated Irreducible Quadratic Factor

So far, we have performed partial fraction decomposition with expressions that have had linear factors in the denominator, and we applied numerators $A, B,$ or C representing constants. Now we will look at an example where one of the factors in the denominator is a **quadratic** expression that does not factor. This is referred to as an irreducible quadratic factor. In cases like this, we use a linear numerator such as $Ax+B$, $Bx+C$, etc.

A GENERAL NOTE LABEL: The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ such that $Q(x)$ has a nonrepeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$ is written as

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{(a_1x^2+b_1x+c_1)} + \frac{A_2x+B_2}{(a_2x^2+b_2x+c_2)} + \cdots + \frac{A_nx+B_n}{(a_nx^2+b_nx+c_n)} \quad (32)$$

The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: A, B, C, and so on.

HOW TO FEATURE: **Given a rational expression where the factors of the denominator are distinct, irreducible quadratic factors, decompose it.**

1. Use variables such as A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator.

$$\frac{P(x)}{Q(x)} = \frac{A}{ax+b} + \frac{A_1x+B_1}{(a_1x^2+b_1x+c_1)} + \frac{A_2x+B_2}{(a_2x^2+b_2x+c_2)} + \cdots + \frac{A_nx+B_n}{(a_nx^2+b_nx+c_n)} \quad (33)$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 3

Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Contains a Nonrepeated Irreducible Quadratic Factor

Find a partial fraction decomposition of the given expression.

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)} \quad (34)$$

Solution

We have one linear factor and one irreducible quadratic factor in the denominator, so one numerator will be a constant and the other numerator will be a linear expression. Thus,

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+x+2)} \quad (35)$$

We follow the same steps as in previous problems. First, clear the fractions by multiplying both sides of the equation by the common denominator.

$$\begin{aligned} (x+3)\left(x^2+x+2\right)\left[\tfrac{8x^2+12x-20}{(x+3)(x^2+x+2)}\right] &= \left[\tfrac{A}{(x+3)} + \tfrac{Bx+C}{(x^2+x+2)}\right](x+3)\left(x^2+x+2\right) \\ 8x^2+12x-20 &= A\left(x^2+x+2\right) + (Bx+C)(x+3) \end{aligned} \quad (36)$$

Notice we could easily solve for A by choosing a value for x that will make the $Bx+C$ term equal 0. Let $x=-3$ and substitute it into the equation.

$$\begin{aligned} 8x^2+12x-20 &= A\left(x^2+x+2\right) + (Bx+C)(x+3) \\ 8(-3)^2+12(-3)-20 &= A\left((-3)^2+(-3)+2\right) + (B(-3)+C)((-3)+3) \\ 16 &= 8A \\ A &= 2 \end{aligned} \quad (37)$$

Now that we know the value of A, substitute it back into the equation. Then expand the right side and collect like terms.

$$\begin{aligned} 8x^2 + 12x - 20 &= 2\left(x^2 + x + 2\right) + (Bx + C)(x + 3) \\ 8x^2 + 12x - 20 &= 2x^2 + 2x + 4 + Bx^2 + 3B + Cx + 3C \\ 8x^2 + 12x - 20 &= (2 + B)x^2 + (2 + 3B + C)x + (4 + 3C) \end{aligned} \quad (38)$$

Setting the coefficients of terms on the right side equal to the coefficients of terms on the left side gives the system of equations.

$$\begin{aligned} 2 + B &= 8 && (1) \\ 2 + 3B + C &= 12 && (2) \\ 4 + 3C &= -20 && (3) \end{aligned} \quad (39)$$

Solve for B using equation (1) and solve for C using equation (3).

$$\begin{aligned} 2 + B &= 8 \qquad (1) \\ B &= 6 \\ \\ 4 + 3C &= -20 \qquad (3) \\ 3C &= -24 \\ C &= -8 \end{aligned} \quad (40)$$

Thus, the partial fraction decomposition of the expression is

$$\frac{8x^2 + 12x - 20}{(x + 3)\left(x^2 + x + 2\right)} = \frac{2}{(x + 3)} + \frac{6x - 8}{\left(x^2 + x + 2\right)} \quad (41)$$

QA FEATURE: ***Could we have just set up a system of equations to solve Example 3?***

Yes, we could have solved it by setting up a system of equations without solving for A first. The expansion on the right would be:

$$\begin{aligned} 8x^2 + 12x - 20 &= Ax^2 + Ax + 2A + Bx^2 + 3B + Cx + 3C \\ 8x^2 + 12x - 20 &= (A + B)x^2 + (A + 3B + C)x + (2A + 3C) \end{aligned} \quad (42)$$

So the system of equations would be:

$$\begin{aligned} A+B&=8\\ A+3B+C&=12\\ 2A+3C&=-20 \end{aligned} \tag{43}$$

Try it feature:

Exercise 6 ***(Solution on p. 16.)***

Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.

$$\frac{5x^2-6x+7}{(x-1)(x^2+1)} \tag{44}$$

4 Decomposing $\frac{P(x)}{Q(x)}$ When *Q(x)* Has a Repeated Irreducible Quadratic Factor

Now that we can decompose a simplified **rational expression** with an irreducible **quadratic** factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.

A general note label: The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{(ax^2+bx+c)^n}=\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n} \tag{45}$$

Write the denominators in increasing powers.

How to feature: **Given a rational expression that has a repeated irreducible factor, decompose it.**

1. Use variables like A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as

$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_n+B_n}{(ax^2+bx+c)^n} \tag{46}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 4

Decomposing a Rational Function with a Repeated Irreducible Quadratic Factor in the Denominator

Decompose the given expression that has a repeated irreducible factor in the denominator.

$$\frac{x^4+x^3+x^2-x+1}{x{(x^2+1)}^2} \tag{47}$$

Solution

The factors of the denominator are $x, \left(x^2+1\right)$, and $\left(x^2+1\right)^2$. Recall that, when a factor in the denominator is a quadratic that includes at least two terms, the numerator must be of the linear form $Ax+B$. So, let's begin the decomposition.

$$\frac{x^4+x^3+x^2-x+1}{x{(x^2+1)}^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{{(x^2+1)}^2} \tag{48}$$

We eliminate the denominators by multiplying each term by $x{\left(x^2+1\right)}^2$. Thus,

$$x^4+x^3+x^2-x+1 = A{\left(x^2+1\right)}^2 + (Bx+C)(x)\left(x^2+1\right) + (Dx+E)(x) \tag{49}$$

Expand the right side.

$$\begin{aligned} x^4+x^3+x^2-x+1 &= A\left(x^4+2x^2+1\right) + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex \\ &= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex \end{aligned} \tag{50}$$

Now we will collect like terms.

$$x^4+x^3+x^2-x+1 = (A+B)x^4 + (C)x^3 + (2A+B+D)x^2 + (C+E)x + A \tag{51}$$

Set up the system of equations matching corresponding coefficients on each side of the equal sign.

$$\begin{aligned} A+B &= 1 \\ C &= 1 \\ 2A+B+D &= 1 \\ C+E &= -1 \\ A &= 1 \end{aligned} \tag{52}$$

We can use substitution from this point. Substitute $A=1$ into the first equation.

$$\begin{aligned} 1+B &= 1 \\ B &= 0 \end{aligned} \tag{53}$$

Substitute $A=1$ and $B=0$ into the third equation.

$$\begin{aligned} 2(1)+0+D &= 1 \\ D &= -1 \end{aligned} \tag{54}$$

Substitute $C=1$ into the fourth equation.

$$\begin{aligned} 1 + E &= -1 \\ E &= -2 \end{aligned} \tag{55}$$

Now we have solved for all of the unknowns on the right side of the equal sign. We have $A = 1$, $B = 0$, $C = 1$, $D = -1$, and $E = -2$. We can write the decomposition as follows:

$$\frac{x^4 + x^3 + x^2 - x + 1}{x(x^2+1)^2} = \frac{1}{x} + \frac{1}{(x^2+1)} - \frac{x+2}{(x^2+1)^2} \tag{56}$$

Try IT Feature label:

Exercise 8 *(Solution on p. 16.)*

Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2} \tag{57}$$

Media Feature Label: Access these online resources for additional instruction and practice with partial fractions.

- Partial Fraction Decomposition[1]
- Partial Fraction Decomposition With Repeated Linear Factors[2]
- Partial Fraction Decomposition With Linear and Quadratic Factors[3]

5 Key Concepts

- Decompose $\frac{P(x)}{Q(x)}$ by writing the partial fractions as $\frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2}$. Solve by clearing the fractions, expanding the right side, collecting like terms, and setting corresponding coefficients equal to each other, then setting up and solving a system of equations. See Example 1.
- The decomposition of $\frac{P(x)}{Q(x)}$ with repeated linear factors must account for the factors of the denominator in increasing powers. See Example 2.
- The decomposition of $\frac{P(x)}{Q(x)}$ with a nonrepeated irreducible quadratic factor needs a linear numerator over the quadratic factor, as in $\frac{A}{x} + \frac{Bx+C}{(ax^2+bx+c)}$. See Example 3.
- In the decomposition of $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a repeated irreducible quadratic factor, when the irreducible quadratic factors are repeated, powers of the denominator factors must be represented in increasing powers as

$$\frac{Ax+B}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}. \tag{58}$$

See Example 4.

[1] http://openstaxcollege.org/l/partdecomp
[2] http://openstaxcollege.org/l/partdecomprlf
[3] http://openstaxcollege.org/l/partdecomlqu

6 Section Exercises

6.1 Verbal

Exercise 9 *(Solution on p. 16.)*

Can any quotient of polynomials be decomposed into at least two partial fractions? If so, explain why, and if not, give an example of such a fraction

Exercise 10

Can you explain why a partial fraction decomposition is unique? (Hint: Think about it as a system of equations.)

Exercise 11 *(Solution on p. 16.)*

Can you explain how to verify a partial fraction decomposition graphically?

Exercise 12

You are unsure if you correctly decomposed the partial fraction correctly. Explain how you could double-check your answer.

Exercise 13 *(Solution on p. 16.)*

Once you have a system of equations generated by the partial fraction decomposition, can you explain another method to solve it? For example if you had $\frac{7x+13}{3x^2+8x+15} = \frac{A}{x+1} + \frac{B}{3x+5}$, we eventually simplify to $7x + 13 = A\,(3x + 5) + B\,(x + 1)$. Explain how you could intelligently choose an x-value that will eliminate either A or B and solve for A and B.

6.2 Algebraic

For the following exercises, find the decomposition of the partial fraction for the nonrepeating linear factors.

Exercise 14

$\frac{5x+16}{x^2+10x+24}$

Exercise 15 *(Solution on p. 16.)*

$\frac{3x-79}{x^2-5x-24}$

Exercise 16

$\frac{-x-24}{x^2-2x-24}$

Exercise 17 *(Solution on p. 16.)*

$\frac{10x+47}{x^2+7x+10}$

Exercise 18

$\frac{x}{6x^2+25x+25}$

Exercise 19 *(Solution on p. 16.)*

$\frac{32x-11}{20x^2-13x+2}$

Exercise 20

$\frac{x+1}{x^2+7x+10}$

Exercise 21 *(Solution on p. 16.)*

$\frac{5x}{x^2-9}$

Exercise 22

$\frac{10x}{x^2-25}$

Exercise 23 *(Solution on p. 16.)*

$\frac{6x}{x^2-4}$

Exercise 24

$\frac{2x-3}{x^2-6x+5}$

Exercise 25 *(Solution on p. 16.)*

$\frac{4x-1}{x^2-x-6}$

Exercise 26

$\frac{4x+3}{x^2+8x+15}$

Exercise 27 *(Solution on p. 16.)*

$\frac{3x-1}{x^2-5x+6}$

For the following exercises, find the decomposition of the partial fraction for the repeating linear factors.

Exercise 28

$\frac{-5x-19}{(x+4)^2}$

Exercise 29 *(Solution on p. 16.)*

$\frac{x}{(x-2)^2}$

Exercise 30

$\frac{7x+14}{(x+3)^2}$

Exercise 31 *(Solution on p. 16.)*

$\frac{-24x-27}{(4x+5)^2}$

Exercise 32

$\frac{-24x-27}{(6x-7)^2}$

Exercise 33 *(Solution on p. 16.)*

$\frac{5-x}{(x-7)^2}$

Exercise 34

$\frac{5x+14}{2x^2+12x+18}$

Exercise 35 *(Solution on p. 16.)*

$\frac{5x^2+20x+8}{2x(x+1)^2}$

Exercise 36

$\frac{4x^2+55x+25}{5x(3x+5)^2}$

Exercise 37 *(Solution on p. 16.)*

$\frac{54x^3+127x^2+80x+16}{2x^2(3x+2)^2}$

Exercise 38

$\frac{x^3-5x^2+12x+144}{x^2(x^2+12x+36)}$

For the following exercises, find the decomposition of the partial fraction for the irreducible nonrepeating quadratic factor.

Exercise 39 *(Solution on p. 16.)*

$\frac{4x^2+6x+11}{(x+2)(x^2+x+3)}$

Exercise 40

$\frac{4x^2+9x+23}{(x-1)(x^2+6x+11)}$

Exercise 41 *(Solution on p. 16.)*

$\frac{-2x^2+10x+4}{(x-1)(x^2+3x+8)}$

Exercise 42

$\frac{x^2+3x+1}{(x+1)(x^2+5x-2)}$

Exercise 43 *(Solution on p. 16.)*

$\frac{4x^2+17x-1}{(x+3)(x^2+6x+1)}$

Exercise 44

$\frac{4x^2}{(x+5)(x^2+7x-5)}$

Exercise 45 *(Solution on p. 16.)*

$\frac{4x^2+5x+3}{x^3-1}$

Exercise 46

$\frac{-5x^2+18x-4}{x^3+8}$

Exercise 47 *(Solution on p. 17.)*

$\frac{3x^2-7x+33}{x^3+27}$

Exercise 48

$\frac{x^2+2x+40}{x^3-125}$

Exercise 49 *(Solution on p. 17.)*

$\frac{4x^2+4x+12}{8x^3-27}$

Exercise 50

$\frac{-50x^2+5x-3}{125x^3-1}$

Exercise 51 *(Solution on p. 17.)*

$\frac{-2x^3-30x^2+36x+216}{x^4+216x}$

For the following exercises, find the decomposition of the partial fraction for the irreducible repeating quadratic factor.

Exercise 52

$\frac{3x^3+2x^2+14x+15}{(x^2+4)^2}$

Exercise 53 *(Solution on p. 17.)*

$\frac{x^3+6x^2+5x+9}{(x^2+1)^2}$

Exercise 54

$\frac{x^3-x^2+x-1}{(x^2-3)^2}$

Exercise 55 *(Solution on p. 17.)*

$\frac{x^2+5x+5}{(x+2)^2}$

Exercise 56

$\frac{x^3+2x^2+4x}{(x^2+2x+9)^2}$

Exercise 57 *(Solution on p. 17.)*

$\frac{x^2+25}{(x^2+3x+25)^2}$

Exercise 58

$\frac{2x^3+11x+7x+70}{(2x^2+x+14)^2}$

Exercise 59 *(Solution on p. 17.)*

$\frac{5x+2}{x(x^2+4)^2}$

Exercise 60

$\frac{x^4+x^3+8x^2+6x+36}{x(x^2+6)^2}$

Exercise 61 *(Solution on p. 17.)*

$\frac{2x-9}{(x^2-x)^2}$

Exercise 62

$\frac{5x^3-2x+1}{(x^2+2x)^2}$

6.3 Extensions

For the following exercises, find the partial fraction expansion.

Exercise 63 *(Solution on p. 17.)*

$\frac{x^2+4}{(x+1)^3}$

Exercise 64

$\frac{x^3-4x^2+5x+4}{(x-2)^3}$

For the following exercises, perform the operation and then find the partial fraction decomposition.

Exercise 65 *(Solution on p. 17.)*

$\frac{7}{x+8}+\frac{5}{x-2}-\frac{x-1}{x^2-6x-16}$

Exercise 66

$\frac{1}{x-4}-\frac{3}{x+6}-\frac{2x+7}{x^2+2x-24}$

Exercise 67 *(Solution on p. 17.)*

$\frac{2x}{x^2-16}-\frac{1-2x}{x^2+6x+8}-\frac{x-5}{x^2-4x}$

Solutions to Exercises in this Module

Solution to Exercise (p. 4)
$\frac{3}{x-3} - \frac{2}{x-2}$
Solution to Exercise (p. 6)
$\frac{6}{x-1} - \frac{5}{(x-1)^2}$
Solution to Exercise (p. 9)
$\frac{3}{x-1} + \frac{2x-4}{x^2+1}$
Solution to Exercise (p. 11)
$\frac{x-2}{x^2-2x+3} + \frac{2x+1}{(x^2-2x+3)^2}$
Solution to Exercise (p. 12)
No, a quotient of polynomials can only be decomposed if the denominator can be factored. For example, $\frac{1}{x^2+1}$ cannot be decomposed because the denominator cannot be factored.
Solution to Exercise (p. 12)
Graph both sides and ensure they are equal.
Solution to Exercise (p. 12)
If we choose $x = -1$, then the B-term disappears, letting us immediately know that $A = 3$. We could alternatively plug in $x = -\frac{5}{3}$, giving us a B-value of -2.
Solution to Exercise (p. 12)
$\frac{8}{x+3} - \frac{5}{x-8}$
Solution to Exercise (p. 12)
$\frac{1}{x+5} + \frac{9}{x+2}$
Solution to Exercise (p. 12)
$\frac{3}{5x-2} + \frac{4}{4x-1}$
Solution to Exercise (p. 12)
$\frac{5}{2(x+3)} + \frac{5}{2(x-3)}$
Solution to Exercise (p. 12)
$\frac{3}{x+2} + \frac{3}{x-2}$
Solution to Exercise (p. 12)
$\frac{9}{5(x+2)} + \frac{11}{5(x-3)}$
Solution to Exercise (p. 13)
$\frac{8}{x-3} - \frac{5}{x-2}$
Solution to Exercise (p. 13)
$\frac{1}{x-2} + \frac{2}{(x-2)^2}$
Solution to Exercise (p. 13)
$-\frac{6}{4x+5} + \frac{3}{(4x+5)^2}$
Solution to Exercise (p. 13)
$-\frac{1}{x-7} - \frac{2}{(x-7)^2}$
Solution to Exercise (p. 13)
$\frac{4}{x} - \frac{3}{2(x+1)} + \frac{7}{2(x+1)^2}$
Solution to Exercise (p. 13)
$\frac{4}{x} + \frac{2}{x^2} - \frac{3}{3x+2} + \frac{7}{2(3x+2)^2}$
Solution to Exercise (p. 13)
$\frac{x+1}{x^2+x+3} + \frac{3}{x+2}$
Solution to Exercise (p. 13)
$\frac{4-3x}{x^2+3x+8} + \frac{1}{x-1}$
Solution to Exercise (p. 13)
$\frac{2x-1}{x^2+6x+1} + \frac{2}{x+3}$

Solution to Exercise (p. 14)
$\frac{1}{x^2+x+1}+\frac{4}{x-1}$
Solution to Exercise (p. 14)
$\frac{2}{x^2-3x+9}+\frac{3}{x+3}$
Solution to Exercise (p. 14)
$-\frac{1}{4x^2+6x+9}+\frac{1}{2x-3}$
Solution to Exercise (p. 14)
$\frac{1}{x}+\frac{1}{x+6}-\frac{4x}{x^2-6x+36}$
Solution to Exercise (p. 14)
$\frac{x+6}{x^2+1}+\frac{4x+3}{(x^2+1)^2}$
Solution to Exercise (p. 14)
$\frac{x+1}{x+2}+\frac{2x+3}{(x+2)^2}$
Solution to Exercise (p. 14)
$\frac{1}{x^2+3x+25}-\frac{3x}{(x^2+3x+25)^2}$
Solution to Exercise (p. 14)
$\frac{1}{8x}-\frac{x}{8(x^2+4)}+\frac{10-x}{2(x^2+4)^2}$
Solution to Exercise (p. 14)
$-\frac{16}{x}-\frac{9}{x^2}+\frac{16}{x-1}-\frac{7}{(x-1)^2}$
Solution to Exercise (p. 15)
$\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{5}{(x+1)^3}$
Solution to Exercise (p. 15)
$\frac{5}{x-2}-\frac{3}{10(x+2)}+\frac{7}{x+8}-\frac{7}{10(x-8)}$
Solution to Exercise (p. 15)
$-\frac{5}{4x}-\frac{5}{2(x+2)}+\frac{11}{2(x+4)}+\frac{5}{4(x+4)}$

Glossary

Definition 1: partial fractions
the individual fractions that make up the sum or difference of a rational expression before combining them into a simplified rational expression

Definition 2: partial fraction decomposition
the process of returning a simplified rational expression to its original form, a sum or difference of simpler rational expressions