Siyavula textbooks: Grade 12 Maths

Collection Editor:
Free High School Science Texts Project

Siyavula textbooks: Grade 12 Maths

Collection Editor:
Free High School Science Texts Project

Authors:
Free High School Science Texts Project
Rory Adams
Mark Horner
Heather Williams

Online:
< http://cnx.org/content/col11242/1.2/ >

C O N N E X I O N S

Rice University, Houston, Texas

Collection structure revised: August 3, 2011
PDF generated: October 29, 2012
For copyright and attribution information for the modules contained in this collection, see p. 157.

Table of Contents

Logarithms[1]

Introduction

In mathematics many ideas are related. We saw that addition and subtraction are related and that multiplication and division are related. Similarly, exponentials and logarithms are related.

Logarithms are commonly refered to as logs, are the "opposite" of exponentials, just as subtraction is the opposite of addition and division is the opposite of multiplication. Logs "undo" exponentials. Technically speaking, logs are the inverses of exponentials. The logarithm of a number x in the base a is defined as the number n such that $a^n = x$.

So, if $a^n = x$, then:

$$log_a(x) = n \quad (1)$$

> ASIDE: When we say "inverse function" we mean that the answer becomes the question and the question becomes the answer. For example, in the equation $a^b = x$ the "question" is "what is a raised to the power b?" The answer is "x." The inverse function would be $log_a x = b$ or "by what power must we raise a to obtain x?" The answer is "b."

The mathematical symbol for logarithm is $log_a(x)$ and it is read "log to the base a of x". For example, $log_{10}(100)$ is "log to the base 10 of 100."

Logarithm Symbols :

Write the following out in words. The first one is done for you.

1. $log_2(4)$ is log to the base 2 of 4
2. $log_{10}(14)$
3. $log_{16}(4)$
4. $log_x(8)$
5. $log_y(x)$

Definition of Logarithms

The logarithm of a number is the value to which the base must be raised to give that number i.e. the exponent. From the first example of the activity $log_2(4)$ means the power of 2 that will give 4. As $2^2 = 4$, we see that

$$log_2(4) = 2 \quad (2)$$

The *exponential-form* is then $2^2 = 4$ and the *logarithmic-form* is $log_2 4 = 2$.

[1]This content is available online at <http://cnx.org/content/m31883/1.4/>.

Definition 1: Logarithms
If $a^n = x$, then: $log_a(x) = n$, where $a > 0$; $a \neq 1$ and $x > 0$.

Applying the definition :

Find the value of:

1. $log_7 343$

$$\begin{array}{r} \text{Reasoning :} \\ 7^3 = 343 \\ \text{therefore, } log_7 343 = 3 \end{array} \quad (3)$$

2. $log_2 8$
3. $log_4 \frac{1}{64}$
4. $log_{10} 1\ 000$

Logarithm Bases

Logarithms, like exponentials, also have a base and $log_2(2)$ is not the same as $log_{10}(2)$.

We generally use the "common" base, 10, or the *natural* base, e.

The number e is an irrational number between 2.71 and 2.72. It comes up surprisingly often in Mathematics, but for now suffice it to say that it is one of the two common bases.

Natural Logarithm

The natural logarithm (symbol ln) is widely used in the sciences. The natural logarithm is to the base e which is approximately 2.71828183.... e is like π and is another example of an irrational number.

While the notation $log_{10}(x)$ and $log_e(x)$ may be used, $log_{10}(x)$ is often written $log(x)$ in Science and $log_e(x)$ is normally written as $ln(x)$ in both Science and Mathematics. So, if you see the log symbol without a base, it means log_{10}.

It is often necessary or convenient to convert a log from one base to another. An engineer might need an approximate solution to a log in a base for which he does not have a table or calculator function, or it may be algebraically convenient to have two logs in the same base.

Logarithms can be changed from one base to another, by using the change of base formula:

$$log_a x = \frac{log_b x}{log_b a} \quad (4)$$

where b is any base you find convenient. Normally a and b are known, therefore $log_b a$ is normally a known, if irrational, number.

For example, change $log_2 12$ in base 10 is:

$$log_2 12 = \frac{log_{10} 12}{log_{10} 2} \quad (5)$$

Change of Base : Change the following to the indicated base:

1. $log_2(4)$ to base 8
2. $log_{10}(14)$ to base 2
3. $log_{16}(4)$ to base 10
4. $log_x(8)$ to base y

5. $log_y(x)$ to base x

Khan academy video on logarithms - 1

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/mQTWzLpCcW0&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 1

Laws of Logarithms

Just as for the exponents, logarithms have some laws which make working with them easier. These laws are based on the exponential laws and are summarised first and then explained in detail.

$$\begin{array}{rcl} log_a(1) & = & 0 \\ log_a(a) & = & 1 \\ log_a(x \cdot y) & = & log_a(x) + log_a(y) \\ log_a\left(\frac{x}{y}\right) & = & log_a(x) - log_a(y) \\ log_a\left(x^b\right) & = & blog_a(x) \\ log_a\left(\sqrt[b]{x}\right) & = & \frac{log_a(x)}{b} \end{array} \quad (6)$$

Logarithm Law 1: $log_a 1 = 0$

$$\begin{array}{rrcl} \text{Since} & a^0 & = & 1 \\ \text{Then,} & log_a(1) & = & log_a\left(a^0\right) \\ & & = & 0 \quad \text{by definition of logarithm} \end{array} \quad (7)$$

For example,

$$log_2 1 = 0 \quad (8)$$

and

$$log_{25} 1 = 0 \quad (9)$$

Logarithm Law 1: $log_a 1 = 0$:

Simplify the following:

1. $log_2(1) + 5$
2. $log_{10}(1) \times 100$

3. $3 \times log_{16}(1)$
4. $log_x(1) + 2xy$
5. $\frac{log_y(1)}{x}$

Logarithm Law 2: $log_a(a) = 1$

$$\begin{array}{rcl} \text{Since} \quad a^1 & = & a \\ \text{Then,} \quad log_a(a) & = & log_a(a^1) \\ & = & 1 \quad \text{by definition of logarithm} \end{array} \tag{10}$$

For example,

$$log_2 2 = 1 \tag{11}$$

and

$$log_{25} 25 = 1 \tag{12}$$

Logarithm Law 2: $log_a(a) = 1$:

Simplify the following:

1. $log_2(2) + 5$
2. $log_{10}(10) \times 100$
3. $3 \times log_{16}(16)$
4. $log_x(x) + 2xy$
5. $\frac{log_y(y)}{x}$

TIP: Useful to know and remember

When the base is 10, we do not need to state it. From the work done up to now, it is also useful to summarise the following facts:

1. $log 1 = 0$
2. $log 10 = 1$
3. $log 100 = 2$
4. $log 1000 = 3$

Logarithm Law 3: $log_a(x \cdot y) = log_a(x) + log_a(y)$

The derivation of this law is a bit trickier than the first two. Firstly, we need to relate x and y to the base a. So, assume that $x = a^m$ and $y = a^n$. Then from Equation (1), we have that:

$$\begin{array}{rcl} log_a(x) & = & m \\ \text{and} \quad log_a(y) & = & n \end{array} \tag{13}$$

This means that we can write:

$$\begin{array}{rcl} log_a(x\cdot y) & = & log_a(a^m\cdot a^n) \\ & = & log_a(a^{m+n}) \quad \text{Exponential laws} \\ & = & log_a\left(a^{log_a(x)+log_a(y)}\right) \\ & = & log_a(x)+log_a(y) \end{array} \quad (14)$$

For example, show that $log(10\cdot 100)=log10+log100$. Start with calculating the left hand side:

$$\begin{array}{rcl} log(10\cdot 100) & = & log(1000) \\ & = & log\left(10^3\right) \\ & = & 3 \end{array} \quad (15)$$

The right hand side:

$$\begin{array}{rcl} log10+log100 & = & 1+2 \\ & = & 3 \end{array} \quad (16)$$

Both sides are equal. Therefore, $log(10\cdot 100)=log10+log100$.

Logarithm Law 3: $log_a(x\cdot y)=log_a(x)+log_a(y)$:

Write as seperate logs:

1. $log_2(8\times 4)$
2. $log_8(10\times 10)$
3. $log_{16}(xy)$
4. $log_z(2xy)$
5. $log_x\left(y^2\right)$

Logarithm Law 4: $log_a\left(\frac{x}{y}\right)=log_a(x)-log_a(y)$

The derivation of this law is identical to the derivation of Logarithm Law 3 and is left as an exercise.

For example, show that $log\left(\frac{10}{100}\right)=log10-log100$. Start with calculating the left hand side:

$$\begin{array}{rcl} log\left(\frac{10}{100}\right) & = & log\left(\frac{1}{10}\right) \\ & = & log\left(10^{-1}\right) \\ & = & -1 \end{array} \quad (17)$$

The right hand side:

$$\begin{array}{rcl} log10-log100 & = & 1-2 \\ & = & -1 \end{array} \quad (18)$$

Both sides are equal. Therefore, $log\left(\frac{10}{100}\right)=log10-log100$.

Logarithm Law 4: $log_a\left(\frac{x}{y}\right) = log_a(x) - log_a(y)$:

Write as seperate logs:

1. $log_2\left(\frac{8}{5}\right)$
2. $log_8\left(\frac{100}{3}\right)$
3. $log_{16}\left(\frac{x}{y}\right)$
4. $log_z\left(\frac{2}{y}\right)$
5. $log_x\left(\frac{y}{2}\right)$

Logarithm Law 5: $log_a\left(x^b\right) = blog_a(x)$

Once again, we need to relate x to the base a. So, we let $x = a^m$. Then,

$$\begin{array}{rcl} log_a\left(x^b\right) & = & log_a\left(\left(a^m\right)^b\right) \\ & = & log_a\left(a^{m\cdot b}\right) \quad \text{(exponential laws} \\ \text{But,}\quad m & = & log_a(x) \quad \text{(Assumption that } x = a^m\text{)} \\ \therefore\quad log_a\left(x^b\right) & = & log_a\left(a^{b\cdot log_a(x)}\right) \\ & = & b\cdot log_a(x) \quad \text{(Definition of logarithm)} \end{array} \tag{19}$$

For example, we can show that $log_2\left(5^3\right) = 3log_2(5)$.

$$\begin{array}{rcl} log_2\left(5^3\right) & = & log_2(5\cdot 5\cdot 5) \\ & = & log_2 5 + log_2 5 + log_2 5 \quad (\because log_a(x\cdot y) = log_a(a^m\cdot a^n)) \\ & = & 3log_2 5 \end{array} \tag{20}$$

Therefore, $log_2\left(5^3\right) = 3log_2(5)$.

Logarithm Law 5: $log_a\left(x^b\right) = blog_a(x)$:

Simplify the following:

1. $log_2\left(8^4\right)$
2. $log_8\left(10^{10}\right)$
3. $log_{16}\left(x^y\right)$
4. $log_z\left(y^x\right)$
5. $log_x\left(y^{2x}\right)$

Logarithm Law 6: $log_a\left(\sqrt[b]{x}\right) = \frac{log_a(x)}{b}$

The derivation of this law is identical to the derivation of Logarithm Law 5 and is left as an exercise.

For example, we can show that $log_2\left(\sqrt[3]{5}\right) = \frac{log_2 5}{3}$.

$$\begin{array}{rcl} log_2\left(\sqrt[3]{5}\right) & = & log_2\left(5^{\frac{1}{3}}\right) \\ & = & \frac{1}{3}log_2 5 \quad \left(\because log_a\left(x^b\right) = blog_a(x)\right) \\ & = & \frac{log_2 5}{3} \end{array} \tag{21}$$

Therefore, $log_2\left(\sqrt[3]{5}\right) = \frac{log_2 5}{3}$.

Logarithm Law 6: $log_a\left(\sqrt[b]{x}\right) = \frac{log_a(x)}{b}$ **:**

Simplify the following:

1. $log_2\left(\sqrt[4]{8}\right)$
2. $log_8\left(\sqrt[10]{10}\right)$
3. $log_{16}\left(\sqrt[y]{x}\right)$
4. $log_z\left(\sqrt[x]{y}\right)$
5. $log_x\left(\sqrt[2x]{y}\right)$

TIP: The final answer doesn't have to *look* simple.

Khan academy video on logarithms - 2

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/PupNgv49_WY&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 2

Khan academy video on logarithms - 3

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/TMmxKZaCqe0&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 3

Exercise 1: Simplification of Logs
Simplify, without use of a calculator:

$$3log3 + log125 \qquad (22)$$

Exercise 2: Simplification of Logs
Simplify, without use of a calculator:

$$8^{\frac{2}{3}} + log_2 32 \qquad (23)$$

Exercise 3: Simplify to one log
Write $2log3 + log2 - log5$ as the logarithm of a single number.

TIP: Exponent rule: $\left(x^b\right)^a = x^{ab}$

Solving simple log equations

In grade 10 you solved some exponential equations by trial and error, because you did not know the great power of logarithms yet. Now it is much easier to solve these equations by using logarithms.

For example to solve x in $25^x = 50$ correct to two decimal places you simply apply the following reasoning. If the LHS = RHS then the logarithm of the LHS must be equal to the logarithm of the RHS. By applying Law 5, you will be able to use your calculator to solve for x.

Exercise 4: Solving Log equations
Solve for x: $25^x = 50$ correct to two decimal places.

In general, the exponential equation should be simplified as much as possible. Then the aim is to make the unknown quantity (i.e. x) the subject of the equation.

For example, the equation

$$2^{(x+2)} = 1 \tag{24}$$

is solved by moving all terms with the unknown to one side of the equation and taking all constants to the other side of the equation

$$\begin{array}{rcl} 2^x \cdot 2^2 & = & 1 \\ 2^x & = & \frac{1}{2^2} \end{array} \tag{25}$$

Then, take the logarithm of each side.

$$\begin{array}{rcl} log\left(2^x\right) & = & log\left(\frac{1}{2^2}\right) \\ xlog\left(2\right) & = & -log\left(2^2\right) \\ xlog\left(2\right) & = & -2log\left(2\right) \quad \text{Divide both sides by } log\left(2\right) \\ \therefore \quad x & = & -2 \end{array} \tag{26}$$

Substituting into the original equation, yields

$$2^{-2+2} = 2^0 = 1 \tag{27}$$

Similarly, $9^{(1-2x)} = 3^4$ is solved as follows:

$$\begin{array}{rcl} 9^{(1-2x)} & = & 3^4 \\ 3^{2(1-2x)} & = & 3^4 \\ 3^{2-4x} & = & 3^4 \quad \text{take the logarithm of both sides} \\ log\left(3^{2-4x}\right) & = & log\left(3^4\right) \\ (2-4x)\,log\left(3\right) & = & 4log\left(3\right) \quad \text{divide both sides by } log\left(3\right) \\ 2-4x & = & 4 \\ -4x & = & 2 \\ \therefore x & = & -\frac{1}{2} \end{array} \tag{28}$$

Substituting into the original equation, yields

$$9^{\left(1-2\left(\frac{-1}{2}\right)\right)} = 9^{(1+1)} = 3^{2(2)} = 3^4 \tag{29}$$

Exercise 5: Exponential Equation
Solve for x in $7 \cdot 5^{(3x+3)} = 35$

Exercises

Solve for x:

1. $log_3 x = 2$
2. $10^{log27} = x$
3. $3^{2x-1} = 27^{2x-1}$

Logarithmic applications in the Real World

Logarithms are part of a number of formulae used in the Physical Sciences. There are formulae that deal with earthquakes, with sound, and pH-levels to mention a few. To work out time periods is growth or decay, logs are used to solve the particular equation.

Exercise 6: Using the growth formula
A city grows 5% every 2 years. How long will it take for the city to triple its size?

Exercise 7: Logs in Compound Interest
I have R12 000 to invest. I need the money to grow to at least R30 000. If it is invested at a compound interest rate of 13% per annum, for how long (in full years) does my investment need to grow ?

Exercises

1. The population of a certain bacteria is expected to grow exponentially at a rate of 15 % every hour. If the initial population is 5 000, how long will it take for the population to reach 100 000 ?
2. Plus Bank is offering a savings account with an interest rate if 10 % per annum compounded monthly. You can afford to save R 300 per month. How long will it take you to save R 20 000 ? (Give your answer in years and months)

End of Chapter Exercises

1. Show that
$$log_a \left(\frac{x}{y}\right) = log_a (x) - log_a (y) \tag{30}$$
2. Show that
$$log_a \left(\sqrt[b]{x}\right) = \frac{log_a (x)}{b} \tag{31}$$
3. Without using a calculator show that:
$$log\frac{75}{16} - 2log\frac{5}{9} + log\frac{32}{243} = log2 \tag{32}$$
4. Given that $5^n = x$ and $n = log_2 y$
   a. Write y in terms of n
   b. Express $log_8 4y$ in terms of n
   c. Express 50^{n+1} in terms of x and y

5. Simplify, without the use of a calculator:
   a. $8^{\frac{2}{3}} + log_2 32$
   b. $log_3 9 - log_5 \sqrt{5}$
   c. $\left(\frac{5}{4^{-1}-9^{-1}}\right)^{\frac{1}{2}} + log_3 9^{2,12}$
6. Simplify to a single number, without use of a calculator:
   a. $log_5 125 + \frac{log32 - log8}{log8}$
   b. $log3 - log0,3$
7. Given: $log_3 6 = a$ and $log_6 5 = b$
   a. Express $log_3 2$ in terms of a.
   b. Hence, or otherwise, find $log_3 10$ in terms of a and b.
8. Given: $pq^k = qp^{-1}$ Prove: $k = 1 - 2log_q p$
9. Evaluate without using a calculator: $(log_7 49)^5 + log_5 \left(\frac{1}{125}\right) - 13\, log_9 1$
10. If $log5 = 0,7$, determine, **without using a calculator**:
    a. $log_2 5$
    b. $10^{-1,4}$
11. Given: $M = log_2 (x+3) + log_2 (x-3)$
    a. Determine the values of x for which M is defined.
    b. Solve for x if $M = 4$.
12. Solve: $\left(x^3\right)^{logx} = 10x^2$ (Answer(s) may be left in surd form, if necessary.)
13. Find the value of $(log_{27} 3)^3$ without the use of a calculator.
14. Simplify By using a calculator: $log_4 8 + 2log_3 \sqrt{27}$
15. Write $log4500$ in terms of a and b if $2 = 10^a$ and $9 = 10^b$.
16. Calculate: $\frac{5^{2006} - 5^{2004} + 24}{5^{2004} + 1}$
17. Solve the following equation for x without the use of a calculator and using the fact that $\sqrt{10} \approx 3,16$:

$$2log(x+1) = \frac{6}{log(x+1)} - 1 \qquad (33)$$

18. Solve the following equation for x: $6^{6x} = 66$ (Give answer correct to 2 decimal places.)

Chapter 1

Sequences and series

1.1 Arithmetic & Geometric Sequences, Recursive Formulae[1]

1.1.1 Introduction

In this chapter we extend the arithmetic and quadratic sequences studied in earlier grades, to geometric sequences. We also look at series, which is the summing of the terms in a sequence.

1.1.2 Arithmetic Sequences

The simplest type of numerical sequence is an *arithmetic sequence*.

Definition 1.1: Arithmetic Sequence
An *arithmetic* (or *linear*) *sequence* is a sequence of numbers in which each new term is calculated by **adding** a constant value to the previous term

For example, $1, 2, 3, 4, 5, 6, ...$ is an arithmetic sequence because you add 1 to the current term to get the next term:

first term:	1
second term:	2=1+1
third term:	3=2+1
$\vdots$	
n^{th} term:	$n = (n-1) + 1$

Table 1.1

1.1.2.1 Common Difference :

Find the constant value that is added to get the following sequences and write out the next 5 terms.

1. $2, 6, 10, 14, 18, 22, ...$
2. $-5, -3, -1, 1, 3, ...$
3. $1, 4, 7, 10, 13, 16, ...$
4. $-1, 10, 21, 32, 43, 54, ...$
5. $3, 0, -3, -6, -9, -12, ...$

[1] This content is available online at <http://cnx.org/content/m39302/1.1/>.

1.1.2.2 General Equation for the n^{th}-term of an Arithmetic Sequence

More formally, the number we start out with is called a_1 (the first term), and the difference between each successive term is denoted d, called the *common difference*.

The general arithmetic sequence looks like:

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= a_1 + d \\ a_3 &= a_2 + d = (a_1 + d) + d = a_1 + 2d \\ a_4 &= a_3 + d = (a_1 + 2d) + d = a_1 + 3d \\ ... & \\ a_n &= a_1 + d \cdot (n-1) \end{aligned} \tag{1.1}$$

Thus, the equation for the n^{th}-term will be:

$$a_n = a_1 + d \cdot (n-1) \tag{1.2}$$

Given a_1 and the common difference, d, the entire set of numbers belonging to an arithmetic sequence can be generated.

Definition 1.2: Arithmetic Sequence

An *arithmetic* (or *linear*) *sequence* is a sequence of numbers in which each new term is calculated by adding a constant value to the previous term:

$$a_n = a_{n-1} + d \tag{1.3}$$

where

- a_n represents the new term, the n^{th}-term, that is calculated;
- a_{n-1} represents the previous term, the $(n-1)^{th}$-term;
- d represents some constant.

TIP: Test for Arithmetic Sequences

A simple test for an arithmetic sequence is to check that the difference between consecutive terms is constant:

$$a_2 - a_1 = a_3 - a_2 = a_n - a_{n-1} = d \tag{1.4}$$

This is quite an important equation, and is the definitive test for an arithmetic sequence. If this condition does not hold, the sequence is not an arithmetic sequence.

1.1.2.2.1 Plotting a graph of terms in an arithmetic sequence

Plotting a graph of the terms of sequence sometimes helps in determining the type of sequence involved. For an arithmetic sequence, plotting a_n vs. n results in:

Figure 1.1

1.1.3 Geometric Sequences

Definition 1.3: Geometric Sequences
A geometric sequence is a sequence in which every number in the sequence is equal to the previous number in the sequence, **multiplied** by a constant number.

This means that the *ratio* between consecutive numbers in the geometric sequence is a constant. We will explain what we mean by ratio after looking at the following example.

1.1.3.1 Example - A Flu Epidemic

1.1.3.1.1 What is influenza?

Influenza (commonly called “the flu”) is caused by the influenza virus, which infects the respiratory tract (nose, throat, lungs). It can cause mild to severe illness that most of us get during winter time. The main way that the influenza virus is spread is from person to person in respiratory droplets of coughs and sneezes. (This is called “droplet spread”.) This can happen when droplets from a cough or sneeze of an infected person are propelled (generally, up to a metre) through the air and deposited on the mouth or nose of people nearby. It is good practise to cover your mouth when you cough or sneeze so as not to infect others around you when you have the flu.

Assume that you have the flu virus, and you forgot to cover your mouth when two friends came to visit while you were sick in bed. They leave, and the next day they also have the flu. Let’s assume that they in turn spread the virus to two of their friends by the same droplet spread the following day. Assuming this pattern continues and each sick person infects 2 other friends, we can represent these events in the following manner:

Figure 1.2: Each person infects two more people with the flu virus.

Again we can tabulate the events and formulate an equation for the general case:

Day, n	**Number of newly-infected people**
1	$2 = 2$
2	$4 = 2 \times 2 = 2 \times 2^1$
3	$8 = 2 \times 4 = 2 \times 2 \times 2 = 2 \times 2^2$
4	$16 = 2 \times 8 = 2 \times 2 \times 2 \times 2 = 2 \times 2^3$
5	$32 = 2 \times 16 = 2 \times 2 \times 2 \times 2 \times 2 = 2 \times 2^4$
$\vdots$	$\vdots$
n	$= 2 \times 2 \times 2 \times 2 \times ... \times 2 = 2 \times 2^{n-1}$

Table 1.2

The above table represents the number of **newly-infected** people after n days since you first infected your 2 friends.

You sneeze and the virus is carried over to 2 people who start the chain ($a_1 = 2$). The next day, each one then infects 2 of their friends. Now 4 people are newly-infected. Each of them infects 2 people the third day, and 8 people are infected, and so on. These events can be written as a geometric sequence:

$$2; 4; 8; 16; 32; ... \tag{1.5}$$

Note the common factor (2) between the events. Recall from the linear arithmetic sequence how the common difference between terms were established. In the geometric sequence we can determine the *common ratio*, r, by

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = r \tag{1.6}$$

Or, more general,

$$\frac{a_n}{a_{n-1}} = r \tag{1.7}$$

1.1.3.1.2 Common Factor of Geometric Sequence :

Determine the common factor for the following geometric sequences:

1. $5, 10, 20, 40, 80, ...$
2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$
3. $7, 28, 112, 448, ...$
4. $2, 6, 18, 54, ...$
5. $-3, 30, -300, 3000, ...$

1.1.3.2 General Equation for the n^{th}-term of a Geometric Sequence

From the above example we know $a_1 = 2$ and $r = 2$, and we have seen from the table that the n^{th}-term is given by $a_n = 2 \times 2^{n-1}$. Thus, in general,

$$a_n = a_1 \cdot r^{n-1} \tag{1.8}$$

where a_1 is the first term and r is called the *common ratio*.

So, if we want to know how many people are newly-infected after 10 days, we need to work out a_{10}:

$$\begin{aligned} a_n &= a_1 \cdot r^{n-1} \\ a_{10} &= 2 \times 2^{10-1} \\ &= 2 \times 2^9 \\ &= 2 \times 512 \\ &= 1024 \end{aligned} \tag{1.9}$$

That is, after 10 days, there are 1 024 newly-infected people.

Or, how many days would pass before 16 384 people become newly infected with the flu virus?

$$\begin{aligned} a_n &= a_1 \cdot r^{n-1} \\ 16\ 384 &= 2 \times 2^{n-1} \\ 16\ 384 \div 2 &= 2^{n-1} \\ 8\ 192 &= 2^{n-1} \\ 2^{13} &= 2^{n-1} \\ 13 &= n-1 \\ n &= 14 \end{aligned} \quad (1.10)$$

That is, 14 days pass before 16 384 people are newly-infected.

1.1.3.2.1 General Equation of Geometric Sequence :

Determine the formula for the following geometric sequences:

1. $5, 10, 20, 40, 80, ...$
2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$
3. $7, 28, 112, 448, ...$
4. $2, 6, 18, 54, ...$
5. $-3, 30, -300, 3000, ...$

1.1.3.3 Exercises

1. What is the important characteristic of an arithmetic sequence?
2. Write down how you would go about finding the formula for the n^{th} term of an arithmetic sequence?
3. A single square is made from 4 matchsticks. Two squares in a row needs 7 matchsticks and 3 squares in a row needs 10 matchsticks. Determine:
   a. the first term
   b. the common difference
   c. the formula for the general term
   d. how many matchsticks are in a row of 25 squares

Figure 1.3

4. $5; x; y$ is an arithmetic sequence and $x; y; 81$ is a geometric sequence. All terms in the sequences are integers. Calculate the values of x and y.

1.1.4 Recursive Formulae for Sequences

When discussing arithmetic and quadratic sequences, we noticed that the difference between two consecutive terms in the sequence could be written in a general way.

For an arithmetic sequence, where a new term is calculated by taking the previous term and adding a constant value, d:

$$a_n = a_{n-1} + d \tag{1.11}$$

The above equation is an example of a *recursive equation* since we can calculate the n^{th}-term only by considering the previous term in the sequence. Compare this with equation (1.2),

$$a_n = a_1 + d \cdot (n-1) \tag{1.12}$$

where one can directly calculate the n^{th}-term of an arithmetic sequence without knowing previous terms.

For quadratic sequences, we noticed the difference between consecutive terms is given by (1.13):

$$a_n - a_{n-1} = D \cdot (n-2) + d \tag{1.13}$$

Therefore, we re-write the equation as

$$a_n = a_{n-1} + D \cdot (n-2) + d \tag{1.14}$$

which is then a recursive equation for a quadratic sequence with common second difference, D.

Using (1.7), the recursive equation for a geometric sequence is:

$$a_n = r \cdot a_{n-1} \tag{1.15}$$

Recursive equations are extremely powerful: you can work out every term in the series just by knowing previous terms. As you can see from the examples above, working out a_n using the previous term a_{n-1} can be a much simpler computation than working out a_n from scratch using a general formula. This means that using a recursive formula when using a computer to work out a sequence would mean the computer would finish its calculations significantly quicker.

1.1.4.1 Recursive Formula :

Write the first 5 terms of the following sequences, given their recursive formulae:

1. $a_n = 2a_{n-1} + 3, a_1 = 1$
2. $a_n = a_{n-1}, a_1 = 11$
3. $a_n = 2a_{n-1}^2, a_1 = 2$

1.1.4.2 The Fibonacci Sequence

Consider the following sequence:

$$0; 1; 1; 2; 3; 5; 8; 13; 21; 34; ... \tag{1.16}$$

The above sequence is called the *Fibonacci sequence*. Each new term is calculated by adding the previous two terms. Hence, we can write down the recursive equation:

$$a_n = a_{n-1} + a_{n-2} \tag{1.17}$$

1.2 Sigma notation, Finite & Infinite Series[2]

1.2.1 Series

In this section we simply work on the concept of *adding* up the numbers belonging to arithmetic and geometric sequences. We call the sum of *any* sequence of numbers a *series*.

1.2.1.1 Some Basics

If we add up the terms of a sequence, we obtain what is called a *series*. If we only sum a finite amount of terms, we get a *finite series*. We use the symbol S_n to mean the sum of the first n terms of a sequence $\{a_1; a_2; a_3; ...; a_n\}$:

$$S_n = a_1 + a_2 + a_3 + ... + a_n \tag{1.18}$$

For example, if we have the following sequence of numbers

$$1; 4; 9; 25; 36; 49; ... \tag{1.19}$$

and we wish to find the sum of the first 4 terms, then we write

$$S_4 = 1 + 4 + 9 + 25 = 39 \tag{1.20}$$

The above is an example of a finite series since we are only summing 4 terms.

If we sum infinitely many terms of a sequence, we get an *infinite series*:

$$S_\infty = a_1 + a_2 + a_3 + ... \tag{1.21}$$

1.2.1.2 Sigma Notation

In this section we introduce a notation that will make our lives a little easier.

A sum may be written out using the summation symbol $\sum$. This symbol is *sigma*, which is the capital letter "S" in the Greek alphabet. It indicates that you must sum the expression to the *right* of it:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + ... + a_{n-1} + a_n \tag{1.22}$$

where

- i is the index of the sum;
- m is the lower bound (or start index), shown below the summation symbol;
- n is the upper bound (or end index), shown above the summation symbol;
- a_i are the terms of a sequence.

The index i is increased from m to n in steps of 1.

If we are summing from $n = 1$ (which implies summing from the first term in a sequence), then we can use either S_n- or $\sum$-notation since they mean the same thing:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n \tag{1.23}$$

[2]This content is available online at <http://cnx.org/content/m39301/1.1/>.

For example, in the following sum,

$$\sum_{i=1}^{5} i \tag{1.24}$$

we have to add together all the terms in the sequence $a_i = i$ from $i = 1$ up until $i = 5$:

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15 \tag{1.25}$$

1.2.1.2.1 Examples

1. $$\begin{aligned} \sum_{i=1}^{6} 2^i &= 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 \\ &= 2 + 4 + 8 + 16 + 32 + 64 \\ &= 126 \end{aligned} \tag{1.26}$$
2. $$\sum_{i=3}^{10} \left(3x^i\right) = 3x^3 + 3x^4 + ... + 3x^9 + 3x^{10} \tag{1.27}$$
   for any value x.

TIP: Notice that in the second example we used three dots (...) to indicate that we had left out part of the sum. We do this to avoid writing out every term of a sum.

1.2.1.2.2 Some Basic Rules for Sigma Notation

1. Given two sequences, a_i and b_i,
   $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \tag{1.28}$$
2. For any constant c that is not dependent on the index i,
   $$\begin{aligned} \sum_{i=1}^{n} c \cdot a_i &= c \cdot a_1 + c \cdot a_2 + c \cdot a_3 + ... + c \cdot a_n \\ &= c\ (a_1 + a_2 + a_3 + ... + a_n) \\ &= c \sum_{i=1}^{n} a_i \end{aligned} \tag{1.29}$$

1.2.1.2.3 Exercises

1. What is $\sum_{k=1}^{4} 2$?
2. Determine $\sum_{i=-1}^{3} i$.
3. Expand $\sum_{k=0}^{5} i$.
4. Calculate the value of a if:
   $$\sum_{k=1}^{3} a \cdot 2^{k-1} = 28 \tag{1.30}$$

1.2.2 Finite Arithmetic Series

Remember that an arithmetic sequence is a set of numbers, such that the difference between any term and the previous term is a constant number, d, called the **constant difference**:

$$a_n = a_1 + d\,(n-1) \tag{1.31}$$

where

- n is the index of the sequence;
- a_n is the n^{th}-term of the sequence;
- a_1 is the first term;
- d is the common difference.

When we sum a finite number of terms in an arithmetic sequence, we get a *finite arithmetic series*.

The simplest arithmetic sequence is when $a_1 = 1$ and $d = 0$ in the general form (1.31); in other words all the terms in the sequence are 1:

$$\begin{array}{rcl} a_i & = & a_1 + d\,(i-1) \\ & = & 1 + 0 \cdot (i-1) \\ & = & 1 \\ \{a_i\} & = & \{1; 1; 1; 1; 1; ...\} \end{array} \tag{1.32}$$

If we wish to sum this sequence from $i = 1$ to any positive integer n, we would write

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} 1 = 1 + 1 + 1 + ... + 1 \qquad (n\text{times}) \tag{1.33}$$

Since all the terms are equal to 1, it means that if we sum to n we will be adding n-number of 1's together, which is simply equal to n:

$$\boxed{\sum_{i=1}^{n} 1 = n} \tag{1.34}$$

Another simple arithmetic sequence is when $a_1 = 1$ and $d = 1$, which is the sequence of positive integers:

$$\begin{array}{rcl} a_i & = & a_1 + d\,(i-1) \\ & = & 1 + 1 \cdot (i-1) \\ & = & i \\ \{a_i\} & = & \{1; 2; 3; 4; 5; ...\} \end{array} \tag{1.35}$$

If we wish to sum this sequence from $i = 1$ to any positive integer n, we would write

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + ... + n \tag{1.36}$$

This is an equation with a very important solution as it gives the answer to the sum of positive integers.

> NOTE: Mathematician, Karl Friedrich Gauss, discovered this proof when he was only 8 years old. His teacher had decided to give his class a problem which would distract them for the entire day by asking them to add all the numbers from 1 to 100. Young Karl realised how to do this almost instantaneously and shocked the teacher with the correct answer, 5050.

We first write S_n as a sum of terms in ascending order:

$$S_n = 1 + 2 + ... + (n-1) + n \quad (1.37)$$

We then write the same sum but with the terms in descending order:

$$S_n = n + (n-1) + ... + 2 + 1 \quad (1.38)$$

We then add corresponding pairs of terms from equations (1.37) and (1.38), and we find that the sum for each pair is the same, $(n+1)$:

$$2S_n = (n+1) + (n+1) + ... + (n+1) + (n+1) \quad (1.39)$$

We then have n-number of $(n+1)$-terms, and by simplifying we arrive at the final result:

$$\begin{array}{rcl} 2S_n & = & n\ (n+1) \\ S_n & = & \frac{n}{2}(n+1) \end{array} \quad (1.40)$$

$$S_n = \sum_{i=1}^{n} i = \frac{n}{2}(n+1) \quad (1.41)$$

Note that this is an example of a quadratic sequence.

1.2.2.1 General Formula for a Finite Arithmetic Series

If we wish to sum any arithmetic sequence, there is no need to work it out term-for-term. We will now determine the general formula to evaluate a finite arithmetic series. We start with the general formula for an arithmetic sequence and sum it from $i = 1$ to any positive integer n:

$$\begin{array}{rcl} \sum_{i=1}^{n} a_i & = & \sum_{i=1}^{n} [a_1 + d(i-1)] \\ & = & \sum_{i=1}^{n} (a_1 + di - d) \\ & = & \sum_{i=1}^{n} [(a_1 - d) + di] \\ & = & \sum_{i=1}^{n} (a_1 - d) + \sum_{i=1}^{n} (di) \\ & = & \sum_{i=1}^{n} (a_1 - d) + d\sum_{i=1}^{n} i \\ & = & (a_1 - d)n + \frac{dn}{2}(n+1) \\ & = & \frac{n}{2}(2a_1 - 2d + dn + d) \\ & = & \frac{n}{2}(2a_1 + dn - d) \\ & = & \frac{n}{2}[2a_1 + d(n-1)] \end{array} \quad (1.42)$$

So, the general formula for determining an arithmetic series is given by

$$S_n = \sum_{i=1}^{n} [a_1 + d(i-1)] = \frac{n}{2}[2a_1 + d(n-1)] \quad (1.43)$$

For example, if we wish to know the series S_{20} for the arithmetic sequence $a_i = 3 + 7(i-1)$, we could either calculate each term individually and sum them:

$$\begin{aligned} S_{20} &= \sum_{i=1}^{20} [3 + 7(i-1)] \\ &= 3 + 10 + 17 + 24 + 31 + 38 + 45 + 52 + \\ &\quad 59 + 66 + 73 + 80 + 87 + 94 + 101 + \\ &\quad 108 + 115 + 122 + 129 + 136 \\ &= 1390 \end{aligned} \tag{1.44}$$

or, more sensibly, we could use equation (1.43) noting that $a_1 = 3$, $d = 7$ and $n = 20$ so that

$$\begin{aligned} S_{20} &= \sum_{i=1}^{20} [3 + 7(i-1)] \\ &= \frac{20}{2} [2 \cdot 3 + 7(20-1)] \\ &= 1390 \end{aligned} \tag{1.45}$$

This example demonstrates how useful equation (1.43) is.

1.2.2.2 Exercises

1. The sum to n terms of an arithmetic series is $S_n = \frac{n}{2}(7n + 15)$.
   a. How many terms of the series must be added to give a sum of 425?
   b. Determine the 6^{th} term of the series.
2. The sum of an arithmetic series is 100 times its first term, while the last term is 9 times the first term. Calculate the number of terms in the series if the first term is not equal to zero.
3. The common difference of an arithmetic series is 3. Calculate the values of n for which the n^{th} term of the series is 93, and the sum of the first n terms is 975.
4. The sum of n terms of an arithmetic series is $5n^2 - 11n$ for all values of n. Determine the common difference.
5. The sum of an arithmetic series is 100 times the value of its first term, while the last term is 9 times the first term. Calculate the number of terms in the series if the first term is not equal to zero.
6. The third term of an arithmetic sequence is -7 and the 7^{th} term is 9. Determine the sum of the first 51 terms of the sequence.
7. Calculate the sum of the arithmetic series $4 + 7 + 10 + \cdots + 901$.
8. The common difference of an arithmetic series is 3. Calculate the values of n for which the n^{th} term of the series is 93 and the sum of the first n terms is 975.

1.2.3 Finite Squared Series

When we sum a finite number of terms in a quadratic sequence, we get a *finite quadratic series*. The general form of a quadratic series is quite complicated, so we will only look at the simple case when $D = 2$ and $d = (a_2 - a_1) = 3$, where D is the common second difference and d is the finite difference. This is the sequence of squares of the integers:

$$\begin{aligned} a_i &= i^2 \\ a_i &= 1^2;\ 2^2;\ 3^2;\ 4^2;\ 5^2;\ 6^2;\ ... \\ &= 1;\ 4;\ 9;\ 16;\ 25;\ 36;\ ... \end{aligned} \tag{1.46}$$

If we wish to sum this sequence and create a series, then we write

$$S_n = \sum_{i=1}^{n} i^{\,2} = 1 + 4 + 9 + ... + n^2 \tag{1.47}$$

which can be written, in general, as

$$\boxed{S_n = \sum_{i=1}^{n} i^2 = \frac{n}{6}\left(2n+1\right)\left(n+1\right)} \tag{1.48}$$

The proof for equation (1.48) can be found under the Advanced block that follows:

1.2.3.1 Derivation of the Finite Squared Series

We will now prove the formula for the finite squared series:

$$S_n = \sum_{i=1}^{n} i^{\,2} = 1 + 4 + 9 + ... + n^2 \tag{1.49}$$

We start off with the expansion of $(k+1)^3$.

$$\begin{aligned} (k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\ (k+1)^3 - k^3 &= 3k^2 + 3k + 1 \end{aligned} \tag{1.50}$$

$$\begin{aligned} k = 1 &: \quad 2^3 - 1^3 = 3(1)^2 + 3\left(1\right) + 1 \\ k = 2 &: \quad 3^3 - 2^3 = 3(2)^2 + 3\left(2\right) + 1 \\ k = 3 &: \quad 4^3 - 3^3 = 3(3)^2 + 3\left(3\right) + 1 \\ &\vdots \\ k = n &: \quad (n+1)^3 - n^3 = 3n^2 + 3n + 1 \end{aligned} \tag{1.51}$$

If we add all the terms on the right and left, we arrive at

$$\begin{aligned} (n+1)^3 - 1 &= \sum_{i=1}^{n} \left(3i^2 + 3i + 1\right) \\ n^3 + 3n^2 + 3n + 1 - 1 &= 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ n^3 + 3n^2 + 3n &= 3\sum_{i=1}^{n} i^2 + \frac{3n}{2}\left(n+1\right) + n \\ \sum_{i=1}^{n} i^2 &= \frac{1}{3}\left[n^3 + 3n^2 + 3n - \frac{3n}{2}\left(n+1\right) - n\right] \\ &= \frac{1}{3}\left(n^3 + 3n^2 + 3n - \frac{3}{2}n^2 - \frac{3}{2}n - n\right) \\ &= \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right) \\ &= \frac{n}{6}\left(2n^2 + 3n + 1\right) \end{aligned} \tag{1.52}$$

Therefore,

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n}{6}(2n+1)(n+1)} \tag{1.53}$$

1.2.4 Finite Geometric Series

When we sum a known number of terms in a geometric sequence, we get a *finite geometric series*. We can write out each term of a geometric sequence in the general form:

$$a_n = a_1 \cdot r^{n-1} \tag{1.54}$$

where

- n is the index of the sequence;
- a_n is the n^{th}-term of the sequence;
- a_1 is the first term;
- r is the common ratio (the ratio of any term to the previous term).

By simply adding together the first n terms, we are actually writing out the series

$$S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-2} + a_1 r^{n-1} \tag{1.55}$$

We may multiply the above equation by r on both sides, giving us

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n-1} + a_1 r^n \tag{1.56}$$

You may notice that all the terms on the right side of (1.55) and (1.56) are the same, except the first and last terms. If we subtract (1.55) from (1.56), we are left with just

$$\begin{aligned} rS_n - S_n &= a_1 r^n - a_1 \\ S_n (r-1) &= a_1 (r^n - 1) \end{aligned} \tag{1.57}$$

Dividing by $(r-1)$ on both sides, we arrive at the general form of a geometric series:

$$S_n = \sum_{i=1}^{n} a_1 \cdot r^{i-1} = \frac{a_1 (r^n - 1)}{r - 1} \tag{1.58}$$

The following video summarises what you have learnt so far about sequences and series:

Khan academy video on series - 1

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/VgVJrSJxkDk&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 1.4

1.2.4.1 Exercises

1. Prove that

$$a + ar + ar^2 + ... + ar^{n-1} = \frac{a\left(1 - r^n\right)}{(1 - r)} \tag{1.59}$$

2. Find the sum of the first 11 terms of the geometric series $6 + 3 + \frac{3}{2} + \frac{3}{4} + ...$
3. Show that the sum of the first n terms of the geometric series

$$54 + 18 + 6 + ... + 5\left(\frac{1}{3}\right)^{n-1} \tag{1.60}$$

is given by $81 - 3^{4-n}$.
4. The eighth term of a geometric sequence is 640. The third term is 20. Find the sum of the first 7 terms.
5. Solve for n: $\sum_{t=1}^{n} 8\left(\frac{1}{2}\right)^t = 15\frac{3}{4}$.
6. The ratio between the sum of the first three terms of a geometric series and the sum of the 4^{th}-, 5^{th}- and 6^{th}-terms of the same series is $8 : 27$. Determine the common ratio and the first 2 terms if the third term is 8.
7. Given the geometric sequence $1;\ -3;\ 9;\ \cdots$ determine:
   a. The 8^{th} term of the sequence
   b. The sum of the first 8 terms of the sequence.
8. Determine:

$$\sum_{n=1}^{4} 3 \cdot 2^{n-1} \tag{1.61}$$

1.2.5 Infinite Series

Thus far we have been working only with finite sums, meaning that whenever we determined the sum of a series, we only considered the sum of the first n terms. In this section, we consider what happens when we add infinitely many terms together. You might think that this is a silly question - surely the answer will be ∞ when one sums infinitely many numbers, no matter how small they are? The surprising answer is that while in some cases one will reach ∞ (like when you try to add all the positive integers together), there are some cases one will get a finite answer. If you don't believe this, try doing the following sum, a geometric series, on your calculator or computer:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ... \tag{1.62}$$

You might think that if you keep adding more and more terms you will eventually get larger and larger numbers, but in fact you won't even get past 1 - try it and see for yourself!

We denote the sum of an infinite number of terms of a sequence by

$$S_\infty = \sum_{i=1}^{\infty} a_i \tag{1.63}$$

When we sum the terms of a series, and the answer we get after each summation gets closer and closer to some number, we say that the series *converges*. If a series does not converge, then we say that it *diverges*.

1.2.5.1 Infinite Geometric Series

There is a simple test for knowing instantly which geometric series converges and which diverges. When r, the common ratio, is strictly between -1 and 1, i.e. $-1 < r < 1$, the infinite series will converge, otherwise it will diverge. There is also a formula for working out the value to which the series converges.

Let's start off with formula (1.58) for the finite geometric series:

$$S_n = \sum_{i=1}^{n} a_1 \cdot r^{i-1} = \frac{a_1\,(r^n - 1)}{r - 1} \tag{1.64}$$

Now we will investigate the behaviour of r^n for $-1 < r < 1$ as n becomes larger.

Take $r = \frac{1}{2}$:

$$\begin{array}{lcr} n = 1 & : & r^n = r^1 = \left(\frac{1}{2}\right)^1 = \frac{1}{2} \\ n = 2 & : & r^n = r^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} < \frac{1}{2} \\ n = 3 & : & r^n = r^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} < \frac{1}{4} \end{array} \tag{1.65}$$

Since r is in the range $-1 < r < 1$, we see that r^n gets closer to 0 as n gets larger.

Therefore,

$$\begin{array}{rcl} S_n & = & \frac{a_1\,(r^n-1)}{r-1} \\ S_\infty & = & \frac{a_1\,(0-1)}{r-1} \qquad \text{for} -1 < r < 1 \\ & = & \frac{-a_1}{r-1} \\ & = & \frac{a_1}{1-r} \end{array} \tag{1.66}$$

The sum of an infinite geometric series is given by the formula

$$\boxed{S_\infty = \sum_{i=1}^{\infty} a_1 r^{i-1} = \frac{a_1}{1-r} \qquad \text{for} \qquad -1 < r < 1} \tag{1.67}$$

where a_1 is the first term of the series and r is the common ratio.

Khan academy video on series - 2

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/U_8GRLJplZg&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 1.5

1.2.5.2 Exercises

1. What does $\left(\frac{2}{5}\right)^n$ approach as n tends towards ∞?
2. Given the geometric series:

$$2 \cdot (5)^5 + 2 \cdot (5)^4 + 2 \cdot (5)^3 + ... \tag{1.68}$$

   a. Show that the series converges
   b. Calculate the sum to infinity of the series
   c. Calculate the sum of the first 8 terms of the series, correct to two decimal places.

d. Determine $\sum_{n=9}^{\infty} 2 \cdot 5^{6-n}$ correct to two decimal places using previously calculated results.

3. Find the sum to infinity of the geometric series $3 + 1 + \frac{1}{3} + \frac{1}{9} + ...$
4. Determine for which values of x, the geometric series

$$2 + \frac{2}{3}(x+1) + \frac{2}{9}(x+1)^2 + ... \tag{1.69}$$

will converge.
5. The sum to infinity of a geometric series with positive terms is $4\frac{1}{6}$ and the sum of the first two terms is $2\frac{2}{3}$. Find a, the first term, and r, the common ratio between consecutive terms.

1.2.6 End of Chapter Exercises

1. Is $1 + 2 + 3 + 4 + ...$ an example of a *finite series* or an *infinite series*?
2. Calculate

$$\sum_{k=2}^{6} 3\left(\frac{1}{3}\right)^{k+2} \tag{1.70}$$

3. If $x + 1$; $x - 1$; $2x - 5$ are the first 3 terms of a convergent geometric series, calculate the:
   a. Value of x.
   b. Sum to infinity of the series.
4. Write the sum of the first 20 terms of the series $6 + 3 + \frac{3}{2} + \frac{3}{4} + ...$ in $\sum$,-notation.
5. Given the geometric series: $2 \cdot 5^5 + 2 \cdot 5^4 + 2 \cdot 5^3 + ...$
   a. Show that the series converges.
   b. Calculate the sum of the first 8 terms of the series, correct to TWO decimal places.
   c. Calculate the sum to infinity of the series.
   d. Use your answer to list, p. 26 above to determine

$$\sum_{n=9}^{\infty} 2 \cdot 5^{(6-n)} \tag{1.71}$$

   correct to TWO decimal places.
6. For the geometric series,

$$54 + 18 + 6 + ... + 5\left(\frac{1}{3}\right)^{n-1} \tag{1.72}$$

calculate the smallest value of n for which the sum of the first n terms is greater than 80.99.
7. Determine the value of $\sum_{k=1}^{\infty} 12\left(\frac{1}{5}\right)^{k-1}$.
8. A new soccer competition requires each of 8 teams to play every other team once.
   a. Calculate the total number of matches to be played in the competition.
   b. If each of n teams played each other once, determine a formula for the total number of matches in terms of n.
9. The midpoints of the opposite sides of square of length 4 units are joined to form 4 new smaller squares. This midpoints of the new smaller squares are then joined in the same way to make even smaller squares. This process is repeated indefinitely. Calculate the sum of the areas of all the squares so formed.
10. Thembi worked part-time to buy a Mathematics book which cost R29,50. On 1 February she saved R1,60, and saves everyday 30 cents more than she saved the previous day. (So, on the second day, she saved R1,90, and so on.) After how many days did she have enough money to buy the book?

11. Consider the geometric series:

$$5 + 2\frac{1}{2} + 1\frac{1}{4} + ... \tag{1.73}$$

   a. If A is the sum to infinity and B is the sum of the first n terms, write down the value of:
      i. A
      ii. B in terms of n.
   b. For which values of n is $(A - B) < \frac{1}{24}$?
12. A certain plant reaches a height of 118 mm after one year under ideal conditions in a greenhouse. During the next year, the height increases by 12 mm. In each successive year, the height increases by $\frac{5}{8}$ of the previous year's growth. Show that the plant will never reach a height of more than 150 mm.
13. Calculate the value of n if $\sum_{a=1}^{n} (20 - 4a) = -20$.
14. Michael saved R400 during the first month of his working life. In each subsequent month, he saved 10% more than what he had saved in the previous month.
   a. How much did he save in the 7^{th} working month?
   b. How much did he save all together in his first 12 working months?
   c. In which month of his working life did he save more than R1,500 for the first time?
15. A man was injured in an accident at work. He receives a disability grant of R4,800 in the first year. This grant increases with a fixed amount each year.
   a. What is the annual increase if, over 20 years, he would have received a total of R143,500?
   b. His initial annual expenditure is R2,600 and increases at a rate of R400 per year. After how many years does his expenses exceed his income?
16. The Cape Town High School wants to build a school hall and is busy with fundraising. Mr. Manuel, an ex-learner of the school and a successful politician, offers to donate money to the school. Having enjoyed mathematics at school, he decides to donate an amount of money on the following basis. He sets a mathematical quiz with 20 questions. For the correct answer to the first question (any learner may answer), the school will receive 1 cent, for a correct answer to the second question, the school will receive 2 cents, and so on. The donations 1, 2, 4, ... form a geometric sequence. Calculate (Give your answer to the nearest Rand)
   a. The amount of money that the school will receive for the correct answer to the 20^{th} question.
   b. The total amount of money that the school will receive if all 20 questions are answered correctly.
17. The first term of a geometric sequence is 9, and the ratio of the sum of the first eight terms to the sum of the first four terms is 97 : 81. Find the first three terms of the sequence, if it is given that all the terms are positive.
18. $(k - 4); (k + 1); m; 5k$ is a set of numbers, the first three of which form an arithmetic sequence, and the last three a geometric sequence. Find k and m if both are positive.
19. Given: The sequence $6 + p, 10 + p, 15 + p$ is geometric.
   a. Determine p.
   b. Show that the common ratio is $\frac{5}{4}$.
   c. Determine the 10^{th} term of this sequence correct to one decimal place.
20. The second and fourth terms of a convergent geometric series are 36 and 16, respectively. Find the sum to infinity of this series, if all its terms are positive.
21. Evaluate: $\sum_{k=2}^{5} \frac{k(k+1)}{2}$
22. $S_n = 4n^2 + 1$ represents the sum of the first n terms of a particular series. Find the second term.
23. Find p if: $\sum_{k=1}^{\infty} 27p^k = \sum_{t=1}^{12} (24 - 3t)$
24. Find the integer that is the closest approximation to:

$$\frac{10^{2001} + 10^{2003}}{10^{2002} + 10^{2002}} \tag{1.74}$$

25. Find the pattern and hence calculate:

$$1 - 2 + 3 - 4 + 5 - 6... + 677 - 678 + ... - 1000 \tag{1.75}$$

26. Determine if $\sum_{p=1}^{\infty} (x+2)^p$ converges. If it does, then work out what it converges to if:
    a. $x = -\frac{5}{2}$
    b. $x = -5$
27. Calculate: $\sum_{i=1}^{\infty} 5 \cdot 4^{-i}$
28. The sum of the first p terms of a sequence is $p(p+1)$. Find the 10^{th} term.
29. The powers of 2 are removed from the set of positive integers

$$1; 2; 3; 4; 5; 6; ...; 1998; 1999; 2000 \tag{1.76}$$

    Find the sum of remaining integers.
30. Observe the pattern below:

Figure 1.6

    a. If the pattern continues, find the number of letters in the column containing M's.
    b. If the total number of letters in the pattern is 361, which letter will be in the last column.
31. The following question was asked in a test: Find the value of $2^{2005} + 2^{2005}$. Here are some of the students' answers:
    a. Megan said the answer is 4^{2005}.
    b. Stefan wrote down 2^{4010}.
    c. Nina thinks it is 2^{2006}.
    d. Annatte gave the answer $2^{2005 \times 2005}$.

    Who is correct? ("None of them" is also a possibility.)
32. A shrub of height 110 cm is planted. At the end of the first year, the shrub is 120 cm tall. Thereafter, the growth of the shrub each year is half of its growth in the previous year. Show that the height of the shrub will never exceed 130 cm.

Chapter 2

Finance

2.1 Introduction, Sequences & Series, Future Value of Payments[1]

2.1.1 Introduction

In earlier grades simple interest and compound interest were studied, together with the concept of depreciation. Nominal and effective interest rates were also described. Since this chapter expands on earlier work, it would be best if you revised the work done in Grades 10 and 11.

If you master the techniques in this chapter, when you start working and earning you will be able to apply the techniques in this chapter to critically assess how to invest your money. And when you are looking at applying for a bond from a bank to buy a home, you will confidently be able to get out the calculator and work out with amazement how much you could actually save by making additional repayments. Indeed, this chapter will provide you with the fundamental concepts you will need to confidently manage your finances and with some successful investing, sit back on your yacht and enjoy the millionaire lifestyle.

2.1.2 Finding the Length of the Investment or Loan

In Grade 11, we used the Compound Interest formula $A = P(1+i)^n$ to determine the term of the investment or loan, by trial and error. Remember that P is the initial amount, A is the current amount, i is the interest rate and n is the number of time units (number of months or years). So if we invest an amount and know what the interest rate is, then we can work out how long it will take for the money to grow to the required amount.

Now that you have learnt about logarithms, you are ready to work out the proper algebraic solution. If you need to remind yourself how logarithms work, go to Chapter (on page).

The basic finance equation is:

$$A = P \cdot (1+i)^n \tag{2.1}$$

If you don't know what A, P, i and n represent, then you should definitely revise the work from Grade 10 and 11.

[1]This content is available online at <http://cnx.org/content/m39196/1.1/>.

Solving for n:

$$\begin{aligned} A &= P(1+i)^n \\ (1+i)^n &= (A/P) \\ log\left((1+i)^n\right) &= log\left(A/P\right) \\ nlog\left(1+i\right) &= log\left(A/P\right) \\ n &= log\left(A/P\right)/log\left(1+i\right) \end{aligned} \quad (2.2)$$

Remember, you do not have to memorise this formula. It is very easy to derive any time you need it. It is simply a matter of writing down what you have, deciding what you need, and solving for that variable.

Exercise 2.1.1: Term of Investment - Logarithms *(Solution on p. 41.)*

Suppose we invested R3 500 into a savings account which pays 7,5% compound interest. After an unknown period of time our account is worth R4 044,69. For how long did we invest the money? How does this compare with the trial and error answer from Chapters .

2.1.3 A Series of Payments

i

i

By this stage, you know how to do calculations such as 'If I want R1 000 in 3 years' time, how much do I need to invest now at 10% ?'

What if we extend this as follows: 'If I want to draw R1 000 next year, R1 000 the next year and R1 000 after three years ... how much do I need to initially put into a bank account earning 10% p.a. to be able to afford to be able to do this?'

The obvious way of working that out is to work out how much you need now to afford the payments individually and sum them. We'll work out how much is needed now to afford the payment of R1 000 in a year $\left(= R1\ 000 \times (1,10)^{-1} = R909,09\right)$, the amount needed now for the following year's R1 000 $\left(= R1\ 000 \times (1,10)^{-2} = R826,45\right)$ and the amount needed now for the R1 000 after 3 years $\left(= R1\ 000 \times (1,10)^{-3} = R751,31\right)$. Add these together gives you the amount needed to afford all three payments and you get R2 486,85.

So, if you put R2 486,85 into a 10% bank account now, you will be able to draw out R1 000 in a year, R1 000 a year after that, and R1 000 a year after that - and your bank account will come down to R0. You would have had exactly the right amount of money to do that (obviously!).

You can check this as follows:

Amount at Time 0 (i.e. Now)		= R2 486,85
Amount at Time 1 (i.e. a year later)	= 2 486,85(1+10%)	= R2 735,54
Amount after withdrawing R1 000	= 2 735,54 - 1 000	= R1 735,54
Amount at Time 2 (i.e. a year later)	= 1 735,54(1+10%)	= R1 909,09
Amount after withdrawing R1 000	= R1 909,09 - 1 000	= R909,09
Amount at Time 3 (i.e. a year later)	= 909,09(1+10%)	= R1 000
Amount after withdrawing R1 000	= 1 000 - 1 000	= R0

Table 2.1

Perfect! Of course, for only three years, that was not too bad. But what if I asked you how much you needed to put into a bank account now, to be able to afford R100 a month for the next 15 years. If you used the above approach you would still get the right answer, but it would take you weeks!

There is - I'm sure you guessed - an easier way! This section will focus on describing how to work with:

- **annuities** - a fixed sum payable each year or each month, either to provide a pre-determined sum at the end of a number of years or months (referred to as a future value annuity) or a fixed amount paid each year or each month to repay (amortise) a loan (referred to as a present value annuity).
- **bond repayments** - a fixed sum payable at regular intervals to pay off a loan. This is an example of a present value annuity.
- **sinking funds** - an accounting term for cash set aside for a particular purpose and invested so that the correct amount of money will be available when it is needed. This is an example of a future value annuity.

2.1.3.1 Sequences and Series

Before we progress, you need to go back and read Chapter (from page) to revise sequences and series.

In summary, if you have a series of n terms in total which looks like this:

$$a + ar + ar^2 + ... + ar^{n-1} = a\left[1 + r + r^2 + ...r^{n-1}\right] \qquad (2.3)$$

this can be simplified as:

$$\begin{array}{cc} \frac{a(r^n-1)}{r-1} & \text{usefulwhen} r > 1 \\ \frac{a(1-r^n)}{1-r} & \text{usefulwhen} 0 \leq r < 1 \end{array} \qquad (2.4)$$

2.1.3.2 Present Values of a series of Payments

So having reviewed the mathematics of Sequences and Series, you might be wondering how this is meant to have any practical purpose! Given that we are in the finance section, you would be right to guess that there must be some financial use to all this. Here is an example which happens in many people's lives - so you know you are learning something practical.

Let us say you would like to buy a property for R300 000, so you go to the bank to apply for a mortgage bond. The bank wants it to be repaid by annually payments for the next 20 years, starting at end of this year. They will charge you 15% interest per annum. At the end of the 20 years the bank would have received back the total amount you borrowed together with all the interest they have earned from lending you the money. You would obviously want to work out what the annual repayment is going to be!

Let X be the annual repayment, i is the interest rate, and M is the amount of the mortgage bond you will be taking out.

Time lines are particularly useful tools for visualizing the series of payments for calculations, and we can represent these payments on a time line as:

Image not finished

Figure 2.1: Time Line for an annuity (in arrears) of X for n periods.

The present value of all the payments (which includes interest) must equate to the (present) value of the mortgage loan amount.

Mathematically, you can write this as:

$$M = X(1+i)^{-1} + X(1+i)^{-2} + X(1+i)^{-3} + ... + X(1+i)^{-20} \tag{2.5}$$

The painful way of solving this problem would be to do the calculation for each of the terms above - which is 20 different calculations. Not only would you probably get bored along the way, but you are also likely to make a mistake.

Naturally, there is a simpler way of doing this! You can rewrite the above equation as follows:

$$\begin{array}{c} M = X\left[v^1 + v^2 + v^3 + ... + v^{20}\right] \\ \text{where} v^i = (1+i)^{-1} = 1/(1+i) \end{array} \tag{2.6}$$

Of course, you do not have to use the method of substitution to solve this. We just find this a useful method because you can get rid of the negative exponents - which can be quite confusing! As an exercise - to show you are a real financial whizz - try to solve this without substitution. It is actually quite easy.

Now, the item in square brackets is the sum of a geometric sequence, as discussion in . This can be re-written as follows, using what we know from Chapter of this text book:

$$\begin{array}{rcl} v^1 + v^2 + v^3 + ... + v^n & = & v\left(1 + v + v^2 + ... + v^{n-1}\right) \\ & = & v\left(\frac{1-v^n}{1-v}\right) \\ & = & \frac{1-v^n}{1/v-1} \\ & = & \frac{1-(1+i)^{-n}}{i} \end{array} \tag{2.7}$$

Note that we took out a common factor of v before using the formula for the geometric sequence.

So we can write:

$$M = X\left[\frac{\left(1-(1+i)^{-n}\right)}{i}\right] \tag{2.8}$$

This can be re-written:

$$X = \frac{M}{\left[\frac{\left(1-(1+i)^{-n}\right)}{i}\right]} = \frac{Mi}{1-(1+i)^{-n}} \tag{2.9}$$

So, this formula is useful if you know the amount of the mortgage bond you need and want to work out the repayment, or if you know how big a repayment you can afford and want to see what property you can buy.

For example, if I want to buy a house for R300 000 over 20 years, and the bank is going to charge me 15% per annum on the outstanding balance, then the annual repayment is:

$$\begin{array}{rcl} X & = & \frac{Mi}{1-(1+i)^{-n}} \\ & = & \frac{R300\,000\times 0,15}{1-(1+0,15)^{-20}} \\ & = & R4\,792\,844 \end{array} \tag{2.10}$$

This means, each year for the next 20 years, I need to pay the bank R47 928,44 per year before I have paid off the mortgage bond.

On the other hand, if I know I will only have R30 000 a year to repay my bond, then how big a house can I buy? That is easy ....

$$\begin{aligned} M &= X\left[\frac{\left(1-(1+i)^{-n}\right)}{i}\right] \\ &= R30\ 000\left[\frac{\left(1-(1{,}15)^{-20}\right)}{0{,}15}\right] \\ &= R187\ 779{,}90 \end{aligned} \tag{2.11}$$

So, for R30 000 a year for 20 years, I can afford to buy a house of R187 800 (rounded to the nearest hundred).

The bad news is that R187 800 does not come close to the R300 000 you wanted to pay! The good news is that you do not have to memorise this formula. In fact , when you answer questions like this in an exam, you will be expected to start from the beginning - writing out the opening equation in full, showing that it is the sum of a geometric sequence, deriving the answer, and then coming up with the correct numerical answer.

Exercise 2.1.2: Monthly mortgage repayments ***(Solution on p. 41.)***

Sam is looking to buy his first flat, and has R15 000 in cash savings which he will use as a deposit. He has viewed a flat which is on the market for R250 000, and he would like to work out how much the monthly repayments would be. He will be taking out a 30 year mortgage with monthly repayments. The annual interest rate is 11%.

Exercise 2.1.3: Monthly mortgage repayments ***(Solution on p. 42.)***

You are considering purchasing a flat for R200 000 and the bank's mortgage rate is currently 9% per annum payable monthly. You have savings of R10 000 which you intend to use for a deposit. How much would your monthly mortgage payment be if you were considering a mortgage over 20 years.

2.1.3.2.1 Show me the money$\hat{A}$[U+0085]

Now that you've done the calculations for the worked example and know what the monthly repayments are, you can work out some surprising figures. For example, R1 709,48 per month for 240 month makes for a total of R410 275,20 (=R1 709,48 × 240). That is more than double the amount that you borrowed! This seems like a lot. However, now that you've studied the effects of time (and interest) on money, you should know that this amount is somewhat meaningless. The value of money is dependant on its timing.

Nonetheless, you might not be particularly happy to sit back for 20 years making your R1 709,48 mortgage payment every month knowing that half the money you are paying are going toward interest. But there is a way to avoid those heavy interest charges. It can be done for less than R300 extra every month...

So our payment is now R2 000. The interest rate is still 9% per annum payable monthly (0,75% per month), and our principal amount borrowed is R190 000. Making this higher repayment amount every month, how long will it take to pay off the mortgage?

The present value of the stream of payments must be equal to R190 000 (the present value of the borrowed amount). So we need to solve for n in:

$$\begin{aligned} R2\ 000 \times \left[1-(1+0,75\%)^{-n}\right]/0,75\% &= R190\ 000 \\ 1-(1+0,75\%)^{-n} &= \left(\frac{190\ 000\times 0,75\%}{2\ 000}\right) \\ log(1+0,75\%)^{-n} &= log\left[\left(1-\frac{190\ 000\times 0,0075}{2\ 000}\right]\right. \\ -n\times log\left(1+0,75\%\right) &= log\left[\left(1-\frac{190\ 000\times 0,0075}{2\ 000}\right]\right. \\ -n\times 0,007472 &= -1,2465 \\ n &= 166,8\text{months} \\ &= 13,9\text{years} \end{aligned} \quad (2.12)$$

So the mortgage will be completely repaid in less than 14 years, and you would have made a total payment of 166,8× R2 000 = R333 600.

Can you see what is happened? Making regular payments of R2 000 instead of the required R1,709,48, you will have saved R76 675,20 (= R410 275,20 - R333 600) in interest, and yet you have only paid an additional amount of R290,52 for 166,8 months, or R48 458,74. You surely know by now that the difference between the additional R48 458,74 that you have paid and the R76 675,20 interest that you have saved is attributable to, yes, you have got it, compound interest!

2.1.3.3 Future Value of a series of Payments

In the same way that when we have a single payment, we can calculate a present value or a future value - we can also do that when we have a series of payments.

In the above section, we had a few payments, and we wanted to know what they are worth now - so we calculated present values. But the other possible situation is that we want to look at the future value of a series of payments.

Maybe you want to save up for a car, which will cost R45 000 - and you would like to buy it in 2 years time. You have a savings account which pays interest of 12% per annum. You need to work out how much to put into your bank account now, and then again each month for 2 years, until you are ready to buy the car.

Can you see the difference between this example and the ones at the start of the chapter where we were only making a single payment into the bank account - whereas now we are making a series of payments into the same account? This is a sinking fund.

So, using our usual notation, let us write out the answer. Make sure you agree how we come up with this. Because we are making monthly payments, everything needs to be in months. So let A be the closing balance you need to buy a car, P is how much you need to pay into the bank account each month, and $i12$ is the monthly interest rate. (Careful - because 12% is the annual interest rate, so we will need to work out later what the monthly interest rate is!)

$$A = P(1+i12)^{24} + P(1+i12)^{23} + ... + P(1+i12)^{1} \quad (2.13)$$

Here are some important points to remember when deriving this formula:

1. We are calculating future values, so in this example we use $(1+i12)^{n}$ and not $(1+i12)^{-n}$. Check back to the start of the chapter if this is not obvious to you by now.
2. If you draw a timeline you will see that the time between the first payment and when you buy the car is 24 months, which is why we use 24 in the first exponent.
3. Again, looking at the timeline, you can see that the 24th payment is being made one month before you buy the car - which is why the last exponent is a 1.
4. Always check that you have got the right number of payments in the equation. Check right now that you agree that there are 24 terms in the formula above.

So, now that we have the right starting point, let us simplify this equation:

$$\begin{aligned} A &= P\left[(1+i12)^{24}+(1+i12)^{23}+...+(1+i12)^{1}\right] \\ &= P\left[X^{24}+X^{23}+...+X^{1}\right] \text{using} X=(1+i12) \end{aligned} \tag{2.14}$$

Note that this time X has a positive exponent not a negative exponent, because we are doing future values. This is not a rule you have to memorise - you can see from the equation what the obvious choice of X should be.

Let us re-order the terms:

$$A = P\left[X^{1}+X^{2}+...+X^{24}\right] = P\cdot X\left[1+X+X^{2}+...+X^{23}\right] \tag{2.15}$$

This is just another sum of a geometric sequence, which as you know can be simplified as:

$$\begin{aligned} A &= P\cdot X\left[X^{n}-1\right]/\left((1+i12)-1\right) \\ &= P\cdot X\left[X^{n}-1\right]/i12 \end{aligned} \tag{2.16}$$

So if we want to use our numbers, we know that A = R45 000, n=24 (because we are looking at monthly payments, so there are 24 months involved) and $i = 12\%$ per annum.

BUT (and it is a big but) we need a monthly interest rate. Do not forget that the trick is to keep the time periods and the interest rates in the same units - so if we have monthly payments, make sure you use a monthly interest rate! Using the formula from Grade 11, we know that $(1+i) = (1+i12)^{12}$. So we can show that $i12 = 0,0094888 = 0,94888\%$.

Therefore,

$$\begin{aligned} 45\ 000 &= P(1,0094888)\left[(1,0094888)^{24}-1\right]/0,0094888 \\ P &= 1662,67 \end{aligned} \tag{2.17}$$

This means you need to invest R166 267 each month into that bank account to be able to pay for your car in 2 years time.

2.1.3.4 Exercises - Present and Future Values

1. You have taken out a mortgage bond for R875 000 to buy a flat. The bond is for 30 years and the interest rate is 12% per annum payable monthly.
   a. What is the monthly repayment on the bond?
   b. How much interest will be paid in total over the 30 years?
2. How much money must be invested now to obtain regular annuity payments of R 5 500 per month for five years ? The money is invested at 11,1% p.a., compounded monthly. (Answer to the nearest hundred rand)

2.2 Investments & Loans, Loan Schedules, Capital Outstanding, Formulae[2]

2.2.1 Investments and Loans

By now, you should be well equipped to perform calculations with compound interest. This section aims to allow you to use these valuable skills to critically analyse investment and loan options that you will come across in your later life. This way, you will be able to make informed decisions on options presented to you.

[2]This content is available online at <http://cnx.org/content/m39194/1.1/>.

At this stage, you should understand the mathematical theory behind compound interest. However, the numerical implications of compound interest are often subtle and far from obvious.

Recall the example 'Show me the money' in "Present Values of a series of Payments". For an extra payment of R29 052 a month, we could have paid off our loan in less than 14 years instead of 20 years. This provides a good illustration of the long term effect of compound interest that is often surprising. In the following section, we'll aim to explain the reason for the drastic reduction in the time it takes to repay the loan.

2.2.1.1 Loan Schedules

So far, we have been working out loan repayment amounts by taking all the payments and discounting them back to the present time. We are not considering the repayments individually. Think about the time you make a repayment to the bank. There are numerous questions that could be raised: how much do you still owe them? Since you are paying off the loan, surely you must owe them less money, but how much less? We know that we'll be paying interest on the money we still owe the bank. When exactly do we pay interest? How much interest are we paying?

The answer to these questions lie in something called the load schedule.

We will continue to use the earlier example. There is a loan amount of R190 000. We are paying it off over 20 years at an interest of 9% per annum payable monthly. We worked out that the repayments should be R1 709,48.

Consider the first payment of R1 709,48 one month into the loan. First, we can work out how much interest we owe the bank at this moment. We borrowed R190 000 a month ago, so we should owe:

$$\begin{aligned} I &= M \times i12 \\ &= R190\ 000 \times 0,75\% \\ &= R1\ 425 \end{aligned} \tag{2.18}$$

We are paying them R1 425 in interest. We call this the interest component of the repayment. We are only paying off R1 709,48 - R1 425 = R284.48 of what we owe! This is called the capital component. That means we still owe R190 000 - R284,48 = R189 715,52. This is called the capital outstanding. Let's see what happens at the end of the second month. The amount of interest we need to pay is the interest on the capital outstanding.

$$\begin{aligned} I &= M \times i12 \\ &= R189\ 715,52 \times 0,75\% \\ &= R1\ 422,87 \end{aligned} \tag{2.19}$$

Since we don't owe the bank as much as we did last time, we also owe a little less interest. The capital component of the repayment is now R1 709,48 - R1 422,87 = R286,61. The capital outstanding will be R189 715,52 - R286,61 = R189 428,91. This way, we can break each of our repayments down into an interest part and the part that goes towards paying off the loan.

This is a simple and repetitive process. Table 2.2 is a table showing the breakdown of the first 12 payments. This is called a loan schedule.

Time	Repayment			Interest Component			Capital Component			Capital Outstanding		
0										R	190 000	00
1	R	1 709	48	R	1 425	00	R	284	48	R	189 715	52
2	R	1 709	48	R	1 422	87	R	286	61	R	189 428	91
3	R	1 709	48	R	1 420	72	R	288	76	R	189 140	14
4	R	1 709	48	R	1 418	55	R	290	93	R	188 849	21
5	R	1 709	48	R	1 416	37	R	293	11	R	188 556	10
6	R	1 709	48	R	1 414	17	R	295	31	R	188 260	79
7	R	1 709	48	R	1 411	96	R	297	52	R	187 963	27
8	R	1 709	48	R	1 409	72	R	299	76	R	187 663	51
9	R	1 709	48	R	1 407	48	R	302	00	R	187 361	51
10	R	1 709	48	R	1 405	21	R	304	27	R	187 057	24
11	R	1 709	48	R	1 402	93	R	306	55	R	186 750	69
12	R	1 709	48	R	1 400	63	R	308	85	R	186 441	84

Table 2.2: A loan schedule with repayments of R1 709,48 per month.

Now, let's see the same thing again, but with R2 000 being repaid each year. We expect the numbers to change. However, how much will they change by? As before, we owe R1 425 in interest in interest. After one month. However, we are paying R2 000 this time. That leaves R575 that goes towards paying off the capital outstanding, reducing it to R189 425. By the end of the second month, the interest owed is R1 420,69 (That's R189 425$\times i12$). Our R2 000 pays for that interest, and reduces the capital amount owed by R2 000 - R1 420,69 = R579,31. This reduces the amount outstanding to R188 845,69.

Doing the same calculations as before yields a new loan schedule shown in Table 2.3.

Time	Repayment			Interest Component			Capital Component			Capital Outstanding		
0										R	190 000	00
1	R	2 000	00	R	1 425	00	R	575	00	R	189 425	00
2	R	2 000	00	R	1 420	69	R	579	31	R	188 845	69
3	R	2 000	00	R	1 416	34	R	583	66	R	188 262	03
4	R	2 000	00	R	1 411	97	R	588	03	R	187 674	00
5	R	2 000	00	R	1 407	55	R	592	45	R	187 081	55
6	R	2 000	00	R	1 403	11	R	596	89	R	186 484	66
7	R	2 000	00	R	1 398	63	R	601	37	R	185 883	30
8	R	2 000	00	R	1 394	12	R	605	88	R	185 277	42
9	R	2 000	00	R	1 389	58	R	610	42	R	184 667	00
10	R	2 000	00	R	1 385	00	R	615	00	R	184 052	00
11	R	2 000	00	R	1 380	39	R	619	61	R	183 432	39
12	R	2 000	00	R	1 375	74	R	624	26	R	182 808	14

Table 2.3: A loan schedule with repayments of R2 000 per month.

The important numbers to notice is the “Capital Component" column. Note that when we are paying off R2 000 a month as compared to R1 709,48 a month, this column more than double. In the beginning of paying off a loan, very little of our money is used to pay off the capital outstanding. Therefore, even a small increase in repayment amounts can significantly increase the speed at which we are paying off the capital.

What’s more, look at the amount we are still owing after one year (i.e. at time 12). When we were paying R1 709,48 a month, we still owe R186 441,84. However, if we increase the repayments to R2 000 a month, the amount outstanding decreases by over R3 000 to R182 808,14. This means we would have paid off over R7 000 in our first year instead of less than R4 000. This increased speed at which we are paying off the capital portion of the loan is what allows us to pay off the whole loan in around 14 years instead of the original 20. Note however, the effect of paying R2 000 instead of R1 709,48 is more significant in the beginning of the loan than near the end of the loan.

It is noted that in this instance, by paying slightly more than what the bank would ask you to pay, you can pay off a loan a lot quicker. The natural question to ask here is: why are banks asking us to pay the lower amount for much longer then? Are they trying to cheat us out of our money?

There is no simple answer to this. Banks provide a service to us in return for a fee, so they are out to make a profit. However, they need to be careful not to cheat their customers for fear that they’ll simply use another bank. The central issue here is one of scale. For us, the changes involved appear big. We are paying off our loan 6 years earlier by paying just a bit more a month. To a bank, however, it doesn’t matter much either way. In all likelihood, it doesn’t affect their profit margins one bit!

Remember that the bank calculates repayment amounts using the same methods as we’ve been learning. They decide on the correct repayment amounts for a given interest rate and set of terms. Smaller repayment amounts will make the bank more money, because it will take you longer to pay off the loan and more interest will acumulate. Larger repayment amounts mean that you will pay off the loan faster, so you will accumulate less interest i.e. the bank will make less money off of you. It’s a simple matter of less money now or more money later. Banks generally use a 20 year repayment period by default.

Learning about financial mathematics enables you to duplicate these calculations for yourself. This way, you can decide what’s best for you. You can decide how much you want to repay each month and you’ll know of its effects. A bank wouldn’t care much either way, so you should pick something that suits you.

Exercise 2.2.1: Monthly Payments ***(Solution on p. 42.)***

Stefan and Marna want to buy a house that costs R 1 200 000. Their parents offer to put down a 20% payment towards the cost of the house. They need to get a moratage for the balance. What are their monthly repayments if the term of the home loan is 30 years and the interest is 7,5%, compounded monthly?

2.2.1.2 Exercises - Investments and Loans

1. A property costs R1 800 000. Calculate the monthly repayments if the interest rate is 14% p.a. compounded monthly and the loan must be paid off in 20 years time.
2. A loan of R 4 200 is to be returned in two equal annual instalments. If the rate of interest of 10% per annum, compounded annually, calculate the amount of each instalment.

2.2.1.3 Calculating Capital Outstanding

As defined in "Loan Schedules" (Section 2.2.1.1: Loan Schedules), Capital outstanding is the amount we still owe the people we borrowed money from at a given moment in time. We also saw how we can calculate this using loan schedules. However, there is a significant disadvantage to this method: it is very time consuming. For example, in order to calculate how much capital is still outstanding at time 12 using the loan schedule, we’ll have to first calculate how much capital is outstanding at time 1 through to 11 as well. This is already quite a bit more work than we'd like to do. Can you imagine calculating the amount outstanding after 10 years (time 120)?

Fortunately, there is an easier method. However, it is not immediately clear why this works, so let's take some time to examine the concept.

2.2.1.3.1 Prospective method for Capital Outstanding

Let's say that after a certain number of years, just after we made a repayment, we still owe amount Y. What do we know about Y? We know that using the loan schedule, we can calculate what it equals to, but that is a lot of repetitive work. We also know that Y is the amount that we are still going to pay off. In other words, all the repayments we are still going to make in the future will exactly pay off Y. This is true because in the end, after all the repayments, we won't be owing anything.

Therefore, the present value of all outstanding future payments equal the present amount outstanding. This is the prospective method for calculating capital outstanding.

Let's return to a previous example. Recall the case where we were trying to repay a loan of R200 000 over 20 years. A R10 000 deposit was put down, so the amount being payed off was R190 000. At an interest rate of 9% compounded monthly, the monthly repayment was R1 709,48. In Table 2.2, we can see that after 12 months, the amount outstanding was R186 441,84. Let's try to work this out using the the prospective method.

After time 12, there are still $19 \times 12 = 228$ repayments left of R1 709,48 each. The present value is:

$$\begin{aligned} n &= 228 \\ i &= 0,75\% \\ Y &= R1\ 709,48 \times \frac{1-1{,}0075^{-228}}{0{,}0075} \\ &= R186\ 441,92 \end{aligned} \tag{2.20}$$

Oops! This seems to be almost right, but not quite. We should have got R186 441,84. We are 8 cents out. However, this is in fact not a mistake. Remember that when we worked out the monthly repayments, we rounded to the nearest cents and arrived at R1 709,48. This was because one cannot make a payment for a fraction of a cent. Therefore, the rounding off error was carried through. That's why the two figures don't match exactly. In financial mathematics, this is largely unavoidable.

2.2.2 Formulae Sheet

As an easy reference, here are the key formulae that we derived and used during this chapter. While memorising them is nice (there are not many), it is the application that is useful. Financial experts are not paid a salary in order to recite formulae, they are paid a salary to use the right methods to solve financial problems.

2.2.2.1 Definitions

P	Principal (the amount of money at the starting point of the calculation)
i	interest rate, normally the effective rate per annum
n	period for which the investment is made
iT	the interest rate paid T times per annum, i.e. $iT = \frac{\text{NominalInterestRate}}{T}$

Table 2.4

2.2.2.2 Equations

$$\left.\begin{array}{c}\text{PresentValue} - \text{simple}\\ \text{FutureValue} - \text{simple}\\ \text{Solvefor}i\\ \text{Solvefor}n\end{array}\right\} = P\left(1 + i \cdot n\right) (2.21)$$

$$\left.\begin{array}{c}\text{PresentValue} - \text{compound}\\ \text{FutureValue} - \text{compound}\\ \text{Solvefor}i\\ \text{Solvefor}n\end{array}\right\} = P(1+i)^n (2.22)$$

TIP: Always keep the interest and the time period in the same units of time (e.g. both in years, or both in months etc.).

2.2.3 End of Chapter Exercises

1. Thabo is about to invest his R8 500 bonus in a special banking product which will pay 1% per annum for 1 month, then 2% per annum for the next 2 months, then 3% per annum for the next 3 months, 4% per annum for the next 4 months, and 0% for the rest of the year. The are going to charge him R100 to set up the account. How much can he expect to get back at the end of the period?
2. A special bank account pays simple interest of 8% per annum. Calculate the opening balance required to generate a closing balance of R5 000 after 2 years.
3. A different bank account pays compound interest of 8% per annum. Calculate the opening balance required to generate a closing balance of R5 000 after 2 years.
4. Which of the two answers above is lower, and why?
5. 7 Months after an initial deposit, the value of a bank account which pays compound interest of 7,5% per annum is R3 650,81. What was the value of the initial deposit?
6. Thabani and Lungelo are both using UKZN Bank for their saving. Suppose Lungelo makes a deposit of X today at interest rate of i for six years. Thabani makes a deposit of $3X$ at an interest rate of 0.05. Thabani made his deposit 3 years after Lungelo made his first deposit. If after 6 years, their investments are equal, calculate the value of i and find X. If the sum of their investment is R20 000, use X you got to find out how much Thabani got in 6 years.
7. Sipho invests R500 at an interest rate of $log\,(1, 12)$ for 5 years. Themba, Sipho's sister invested R200 at interest rate i for 10 years on the same date that her brother made his first deposit. If after 5 years, Themba's accumulation equals Sipho's, find the interest rate i and find out whether Themba will be able to buy her favorite cell phone after 10 years which costs R2 000.
8. Calculate the real cost of a loan of R10 000 for 5 years at 5% capitalised monthly. Repeat this for the case where it is capitalised half yearly i.e. Every 6 months.
9. Determine how long, in years, it will take for the value of a motor vehicle to decrease to 25% of its original value if the rate of depreciation, based on the reducing-balance method, is 21% per annum.

Solutions to Exercises in Chapter 2

Solution to Exercise 2.1.1 (p. 30)

Step 1.
- P=R3 500
- i=7,5%
- A=R4 044,69

We are required to find n.

Step 2. We know that:

$$\begin{array}{rcl} A & = & P(1+i)^n \\ (1+i)^n & = & (A/P) \\ log\left((1+i)^n\right) & = & log\,(A/P) \\ nlog\,(1+i) & = & log\,(A/P) \\ n & = & log\,(A/P)\,/log\,(1+i) \end{array} \quad (2.23)$$

Step 3.

$$\begin{array}{rcl} n & = & log\,(A/P)\,/log\,(1+i) \\ & = & \frac{log\left(\frac{4\ 044{,}69}{3\ 500}\right)}{log(1+0.075)} \quad \mathrm{Rememberthat} : 7.5\% = \frac{7.5}{100} = 0.075 \\ & = & 2.0 \end{array} \quad (2.24)$$

Step 4. The R3 500 was invested for 2 years.

Solution to Exercise 2.1.2 (p. 33)

Step 1. The following is given:

- Deposit amount = R15 000
- Price of flat = R250 000
- interest rate, $i = 11\%$

We are required to find the monthly repayment for a 30-year mortgage.

Step 2. We know that:

$$X = \frac{M}{\left[\frac{\left(1-(1+i)^{-n}\right)}{i}\right]} \quad (2.25)$$

In order to use this equation, we need to calculate M, the amount of the mortgage bond, which is the purchase price of property less the deposit which Sam pays up-front.

$$\begin{array}{rcl} M & = & R250\ 000 - R15\ 000 \\ & = & R235\ 000 \end{array} \quad (2.26)$$

Now because we are considering monthly repayments, but we have been given an annual interest rate, we need to convert this to a monthly interest rate, $i12$. (If you are not clear on this, go back and revise .)

$$\begin{array}{rcl} (1+i12)^{12} & = & (1+i) \\ (1+i12)^{12} & = & 1{,}11 \\ i12 & = & 0{,}873459\% \end{array} \quad (2.27)$$

We know that the mortgage bond is for 30 years, which equates to 360 months.

Step 3. Now it is easy, we can just plug the numbers in the formula, but do not forget that you can always deduce the formula from first principles as well!

$$\begin{aligned} X &= \frac{M}{\left[\frac{(1-(1+i)^{-n})}{i}\right]} \\ &= \frac{R235\ 000}{\left[\frac{(1-(1.00876459)^{-360})}{0,008734594}\right]} \\ &= R2\ 146,39 \end{aligned} \tag{2.28}$$

Step 4. That means that to buy a flat for R250 000, after Sam pays a R15 000 deposit, he will make repayments to the bank each month for the next 30 years equal to R2 146,39.

Solution to Exercise 2.1.3 (p. 33)

Step 1. The following is given:

- Deposit amount = R10 000
- Price of flat = R200 000
- interest rate, $i = 9\%$

We are required to find the monthly repayment for a 20-year mortgage.

Step 2. We are considering monthly mortgage repayments, so it makes sense to use months as our time period. The interest rate was quoted as 9% per annum payable monthly, which means that the monthly effective rate $= \frac{9\%}{12} = 0,75\%$ per month. Once we have converted 20 years into 240 months, we are ready to do the calculations!

First we need to calculate M, the amount of the mortgage bond, which is the purchase price of property minus the deposit which Sam pays up-front.

$$\begin{aligned} M &= R200\ \ 000 - R10\ \ 000 \\ &= R190\ \ 000 \end{aligned} \tag{2.29}$$

The present value of our mortgage payments X (which includes interest), must equate to the present mortgage amount

$$\begin{aligned} M = X \times (1+0,75\%)^{-1} &\quad + \\ X \times (1+0,75\%)^{-2} &\quad + \\ X \times (1+0,75\%)^{-3} &\quad + \\ X \times (1+0,75\%)^{-4} &\quad + \quad ... \\ X \times (1+0,75\%)^{-239} + X \times (1+0,75\%)^{-240} & \end{aligned} \tag{2.30}$$

But it is clearly much easier to use our formula than work out 240 factors and add them all up!

Step 3.

$$\begin{aligned} X \times \frac{1-(1+0,75\%)^{-240}}{0,75\%} &= R190\ 000 \\ X \times 111,14495 &= R190\ 000 \\ X &= R1\ 709,48 \end{aligned} \tag{2.31}$$

Step 4. So to repay a R190 000 mortgage over 20 years, at 9% interest payable monthly, will cost you R1 709,48 per month for 240 months.

Solution to Exercise 2.2.1 (p. 38)

Step 1. $R1\ 200\ 00 - R240\ 000 = R960\ 000$

Step 2. Use the formula:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i} \tag{2.32}$$

Where
$P = 960\ 000$
$n = 30 \times 12 = 360 \text{months}$
$i = 0,075 \div 12 = 0,00625$

Step 3.

$$\begin{aligned} R960\ 000 &= \frac{x\left[1-(1+0{,}00625)^{-360}\right]}{0{,}00625} \\ &= x\,(143,0176273) \\ x &= R6\ 712,46 \end{aligned} \tag{2.33}$$

Step 4. The monthly repayments $= R6\ 712,46$

Chapter 3

Factorising cubic polynomials

3.1 Introduction, Factor Theorem, Factorising Cubic Polynomials[1]

3.1.1 Introduction

In grades 10 and 11, you learnt how to solve different types of equations. Most of the solutions, relied on being able to factorise some expression and the factorisation of quadratics was studied in detail. This chapter focusses on the factorisation of cubic polynomials, that is expressions with the highest power equal to 3.

3.1.2 The Factor Theorem

The *factor theorem* describes the relationship between the root of a polynomial and a factor of the polynomial.

Definition 3.1: Factor Theorem

For any polynomial, $f(x)$, for all values of a which satisfy $f(a) = 0$, $(x - a)$ is a factor of $f(x)$. Or, more concisely:

$$f(x) = (x - a)\, q(x) \tag{3.1}$$

is a polynomial.

In other words: If the remainder when dividing $f(x)$ by $(x - a)$ is zero, then $(x - a)$ is a factor of $f(x)$.

So if $f\left(-\frac{b}{a}\right) = 0$, then $(ax + b)$ is a factor of $f(x)$.

Exercise 3.1.1: Factor Theorem *(Solution on p. 49.)*

Use the Factor Theorem to determine whether $y - 1$ is a factor of $f(y) = 2y^4 + 3y^2 - 5y + 7$.

Exercise 3.1.2: Factor Theorem *(Solution on p. 49.)*

Using the Factor Theorem, verify that $y + 4$ is a factor of $g(y) = 5y^4 + 16y^3 - 15y^2 + 8y + 16$.

3.1.3 Factorisation of Cubic Polynomials

A cubic polynomial is a polynomial of the form

$$ax^3 + bx^2 + cx + d \tag{3.2}$$

where a is nonzero. We have seen in Grade 10 that the sum and difference of cubes is factorised as follows.:

$$(x + y)\left(x^2 - xy + y^2\right) = x^3 + y^3 \tag{3.3}$$

[1]This content is available online at <http://cnx.org/content/m39279/1.1/>.

and

$$(x-y)\left(x^2+xy+y^2\right)=x^3-y^3 \tag{3.4}$$

We also saw that the quadratic term does not have rational roots.

There are many methods of factorising a cubic polynomial. The general method is similar to that used to factorise quadratic equations. If you have a cubic polynomial of the form:

$$f(x)=ax^3+bx^2+cx+d \tag{3.5}$$

then in an ideal world you would get factors of the form:

$$(Ax+B)(Cx+D)(Ex+F). \tag{3.6}$$

But sometimes you will get factors of the form:

$$(Ax+B)\left(Cx^2+Ex+D\right) \tag{3.7}$$

We will deal with simplest case first. When $a=1$, then $A=C=E=1$, and you only have to determine B, D and F. For example, find the factors of:

$$x^3-2x^2-5x+6. \tag{3.8}$$

In this case we have

$$\begin{aligned} a &= 1\\ b &= -2\\ c &= -5\\ d &= 6 \end{aligned} \tag{3.9}$$

The factors will have the general form shown in (3.6), with $A=C=E=1$. We can then use values for a, b, c and d to determine values for B, D and F. We can re-write (3.6) with $A=C=E=1$ as:

$$(x+B)(x+D)(x+F). \tag{3.10}$$

If we multiply this out we get:

$$\begin{aligned} (x+B)(x+D)(x+F) &= (x+B)\left(x^2+Dx+Fx+DF\right)\\ &= x^3+Dx^2+Fx^2+Bx^2+DFx+BDx+BFx+BDF\\ &= x^3+(D+F+B)x^2+(DF+BD+BF)x+BDF \end{aligned} \tag{3.11}$$

We can therefore write:

$$\begin{aligned} b &= -2=D+F+B\\ c &= -5=DF+BD+BF\\ d &= 6=BDF. \end{aligned} \tag{3.12}$$

This is a set of three equations in three unknowns. However, we know that B, D and F are factors of 6 because $BDF=6$. Therefore we can use a trial and error method to find B, D and F.

This can become a very tedious method, therefore the **Factor Theorem** can be used to find the factors of cubic polynomials.

Exercise 3.1.3: Factorisation of Cubic Polynomials ***(Solution on p. 49.)***

Factorise $f(x)=x^3+x^2-9x-9$ into three linear factors.

In general, to factorise a cubic polynomial, you find one factor by trial and error. Use the factor theorem to confirm that the guess is a root. Then divide the cubic polynomial by the factor to obtain a quadratic. Once you have the quadratic, you can apply the standard methods to factorise the quadratic.

For example the factors of $x^3 - 2x^2 - 5x + 6$ can be found as follows: There are three factors which we can write as

$$(x - a)(x - b)(x - c). \quad (3.13)$$

Exercise 3.1.4: Factorisation of Cubic Polynomials *(Solution on p. 49.)*
Use the Factor Theorem to factorise

$$x^3 - 2x^2 - 5^x + 6. \quad (3.14)$$

3.2 Factor Theorem Exercises, Solving Cubic Equations[2]

3.2.1 Exercises - Using Factor Theorem

1. Find the remainder when $4x^3 - 4x^2 + x - 5$ is divided by $(x + 1)$.
2. Use the factor theorem to factorise $x^3 - 3x^2 + 4$ completely.
3. $f(x) = 2x^3 + x^2 - 5x + 2$
   a. Find $f(1)$.
   b. Factorise $f(x)$ completely
4. Use the Factor Theorem to determine all the factors of the following expression:

$$x^3 + x^2 - 17x + 15 \quad (3.15)$$

5. Complete: If $f(x)$ is a polynomial and p is a number such that $f(p) = 0$, then $(x - p)$ is .....

3.2.2 Solving Cubic Equations

Once you know how to factorise cubic polynomials, it is also easy to solve cubic equations of the kind

$$ax^3 + bx^2 + cx + d = 0 \quad (3.16)$$

Exercise 3.2.1: Solution of Cubic Equations *(Solution on p. 50.)*
Solve

$$6x^3 - 5x^2 - 17x + 6 = 0. \quad (3.17)$$

Sometimes it is not possible to factorise the trinomial ("second bracket"). This is when the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (3.18)$$

[2]This content is available online at <http://cnx.org/content/m39276/1.1/>.

can be used to solve the cubic equation fully.

For example:

Exercise 3.2.2: Solution of Cubic Equations *(Solution on p. 50.)*

Solve for x: $x^3 - 2x^2 - 6x + 4 = 0$.

3.2.2.1 Exercises - Solving of Cubic Equations

1. Solve for x: $x^3 + x^2 - 5x + 3 = 0$
2. Solve for y: $y^3 - 3y^2 - 16y - 12 = 0$
3. Solve for m: $m^3 - m^2 - 4m - 4 = 0$
4. Solve for x: $x^3 - x^2 = 3\left(3x + 2\right)$

   TIP: :

   Remove brackets and write as an equation equal to zero.
5. Solve for x if $2x^3 - 3x^2 - 8x = 3$

3.2.3 End of Chapter Exercises

1. Solve for x: $16\left(x + 1\right) = x^2\left(x + 1\right)$
2. a. Show that $x - 2$ is a factor of $3x^3 - 11x^2 + 12x - 4$

   b. Hence, by factorising completely, solve the equation

$$3x^3 - 11x^2 + 12x - 4 = 0 \tag{3.19}$$

3. $2x^3 - x^2 - 2x + 2 = Q\left(x\right).\left(2x - 1\right) + R$ for all values of x. What is the value of R?
4. a. Use the factor theorem to solve the following equation for m:

$$8m^3 + 7m^2 - 17m + 2 = 0 \tag{3.20}$$

   b. Hence, or otherwise, solve for x:

$$2^{3x+3} + 7 \cdot 2^{2x} + 2 = 17 \cdot 2^x \tag{3.21}$$

5. **A challenge**: Determine the values of p for which the function

$$f\left(x\right) = 3p^3 - \left(3p - 7\right)x^2 + 5x - 3 \tag{3.22}$$

   leaves a remainder of 9 when it is divided by $\left(x - p\right)$.

Solutions to Exercises in Chapter 3

Solution to Exercise 3.1.1 (p. 45)

Step 1. In order for $y-1$ to be a factor, $f(1)$ must be 0.
Step 2.

$$\begin{aligned} f(y) &= 2y^4+3y^2-5y+7 \\ \therefore f(1) &= 2(1)^4+3(1)^2-5(1)+7 \\ &= 2+3-5+7 \\ &= 7 \end{aligned} \tag{3.23}$$

Step 3. Since $f(1)\neq 0$, $y-1$ is not a factor of $f(y)=2y^4+3y^2-5y+7$.

Solution to Exercise 3.1.2 (p. 45)

Step 1. In order for $y+4$ to be a factor, $g(-4)$ must be 0.
Step 2.

$$\begin{aligned} g(y) &= 5y^4+16y^3-15y^2+8y+16 \\ \therefore g(-4) &= 5(-4)^4+16(-4)^3-15(-4)^2+8(-4)+16 \\ &= 5(256)+16(-64)-15(16)+8(-4)+16 \\ &= 1280-1024-240-32+16 \\ &= 0 \end{aligned} \tag{3.24}$$

Step 3. Since $g(-4)=0$, $y+4$ is a factor of $g(y)=5y^4+16y^3-15y^2+8y+16$.

Solution to Exercise 3.1.3 (p. 46)

Step 1. Try

$$f(1)=(1)^3+(1)^2-9(1)-9=1+1-9-9=-16 \tag{3.25}$$

Therefore $(x-1)$ is not a factor
Try

$$f(-1)=(-1)^3+(-1)^2\text{[U+0096]}9(-1)\text{[U+0096]}9=\text{[U+0096]}1+1+9\text{[U+0096]}9=0 \tag{3.26}$$

Thus $(x+1)$ is a factor, because $f(-1)=0$.
Now divide $f(x)$ by $(x+1)$ using division by inspection:
Write $x^3+x^2-9x-9=(x+1)(\qquad)$
The first term in the second bracket must be x^2 to give x^3 if one works backwards.
The last term in the second bracket must be -9 because $+1\times -9=-9$.
So we have $x^3+x^2-9x-9=(x+1)\left(x^2+?x-9\right)$.
Now, we must find the coefficient of the middle term (x).
$(+1)\left(x^2\right)$ gives the x^2 in the original polynomial. So, the coefficient of the x-term must be 0.
So $f(x)=(x+1)\left(x^2-9\right)$.
Step 2. x^2-9 can be further factorised to $(x-3)(x+3)$,
and we are now left with $f(x)=(x+1)(x-3)(x+3)$

Solution to Exercise 3.1.4 (p. 47)

Step 1. Try

$$f(1)=(1)^3-2(1)^2-5(1)+6=1-2-5+6=0 \tag{3.27}$$

Therefore $(x-1)$ is a factor.

Step 2. $x^3 - 2x^2 - 5x + 6 = (x-1)(\qquad)$
The first term in the second bracket must be x^2 to give x^3 if one works backwards.
The last term in the second bracket must be -6 because $-1 \times -6 = +6$.
So we have $x^3 - 2x^2 - 5x + 6 = (x-1)\left(x^2 + ?x - 6\right)$.
Now, we must find the coefficient of the middle term (x).
$(-1)\left(x^2\right)$ gives $-x^2$. So, the coefficient of the x-term must be -1.
So $f(x) = (x-1)\left(x^2 - x - 6\right)$.
Step 3. $x^2 - x - 6$ can be further factorised to $(x-3)(x+2)$,
and we are now left with $x^3 - 2x^2 - 5x + 6 = (x-1)(x-3)(x+2)$

Solution to Exercise 3.2.1 (p. 47)

Step 1. Try

$$f(1) = 6(1)^3 - 5(1)^2 - 17(1) + 6 = 6 - 5 - 17 + 6 = -10 \tag{3.28}$$

Therefore $(x-1)$ is NOT a factor.
Try

$$f(2) = 6(2)^3 - 5(2)^2 - 17(2) + 6 = 48 - 20 - 34 + 6 = 0 \tag{3.29}$$

Therefore $(x-2)$ IS a factor.
Step 2. $6x^3 - 5x^2 - 17x + 6 = (x-2)(\qquad)$
The first term in the second bracket must be $6x^2$ to give $6x^3$ if one works backwards.
The last term in the second bracket must be -3 because $-2 \times -3 = +6$.
So we have $6x^3 - 5x^2 - 17x + 6 = (x-2)\left(6x^2 + ?x - 3\right)$.
Now, we must find the coefficient of the middle term (x).
$(-2)\left(6x^2\right)$ gives $-12x^2$. So, the coefficient of the x-term must be 7.
So, $6x^3 - 5x^2 - 17x + 6 = (x-2)\left(6x^2 + 7x - 3\right)$.
Step 3. $6x^2 + 7x - 3$ can be further factorised to $(2x+3)(3x-1)$,
and we are now left with $6x^3 - 5x^2 - 17x + 6 = (x-2)(2x+3)(3x-1)$
Step 4.

$$\begin{aligned} 6x^3 - 5x^2 - 17x + 6 &= 0 \\ (x-2)(2x+3)(3x-1) &= 0 \\ x &= 2; \tfrac{1}{3}; -\tfrac{3}{2} \end{aligned} \tag{3.30}$$

Solution to Exercise 3.2.2 (p. 48)

Step 1. Try

$$f(1) = (1)^3 - 2(1)^2 - 6(1) + 4 = 1 - 2 - 6 + 4 = -1 \tag{3.31}$$

Therefore $(x-1)$ is NOT a factor.
Try

$$f(2) = (2)^3 - 2(2)^2 - 6(2) + 4 = 8 - 8 - 12 + 4 = -8 \tag{3.32}$$

Therefore $(x-2)$ is NOT a factor.

$$f(-2) = (-2)^3 - 2(-2)^2 - 6(-2) + 4 = -8 - 8 + 12 + 4 = 0 \tag{3.33}$$

Therefore $(x+2)$ IS a factor.

Step 2. $x^3 - 2x^2 - 6x + 4 = (x+2)\left(\quad \right)$
The first term in the second bracket must be x^2 to give x^3.
The last term in the second bracket must be 2 because $2 \times 2 = +4$.
So we have $x^3 - 2x^2 - 6x + 4 = (x+2)\left(x^2 + ?x + 2\right)$.
Now, we must find the coefficient of the middle term (x).
$(2)\left(x^2\right)$ gives $2x^2$. So, the coefficient of the x-term must be -4. $(2x^2 - 4x^2 = -2x^2)$
So $x^3 - 2x^2 - 6x + 4 = (x+2)\left(x^2 - 4x + 2\right)$.
$x^2 - 4x + 2$ cannot be factorised any futher and we are now left with
$(x+2)\left(x^2 - 4x + 2\right) = 0$

Step 3.

$$\begin{array}{rcl} (x+2)\left(x^2-4x+2\right) & = & 0 \\ (x+2)=0 & or & \left(x^2-4x+2\right)=0 \end{array} \tag{3.34}$$

Step 4. Always write down the formula first and then substitute the values of a, b and c.

$$\begin{array}{rcl} x & = & \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ & = & \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(2)}}{2(1)} \\ & = & \frac{4\pm\sqrt{8}}{2} \\ & = & 2\pm\sqrt{2} \end{array} \tag{3.35}$$

Step 5. $x = -2$ *or* $x = 2 \pm \sqrt{2}$

Chapter 4

Functions and graphs

4.1 Introduction, Definition, Notation[1]

4.1.1 Introduction

In grades 10 and 11 you have learnt about linear functions and quadratic functions as well as the hyperbolic functions and exponential functions and many more. In grade 12 you are expected to demonstrate the ability to work with various types of functions and relations including the inverses of some functions and generate graphs of the inverse relations of functions, in particular the inverses of:

$$y = ax + q \tag{4.1}$$

$$y = ax^2 \tag{4.2}$$

$$y = ax; a > 0 \tag{4.3}$$

.

4.1.2 Definition of a Function

A *function* is a relation for which there is only one value of y corresponding to any value of x. We sometimes write $y = f(x)$, which is notation meaning 'y is a function of x'. This definition makes complete sense when compared to our real world examples — each person has only one height, so height is a function of people; on each day, in a specific town, there is only one average temperature.

However, some very common mathematical constructions are not functions. For example, consider the relation $x^2 + y^2 = 4$. This relation describes a circle of radius 2 centred at the origin, as in Figure 4.1. If we let $x = 0$, we see that $y^2 = 4$ and thus either $y = 2$ or $y = -2$. Since there are two y values which are possible for the same x value, the relation $x^2 + y^2 = 4$ is **not** the graph a function.

There is a simple test to check if a relation is a function, by looking at its graph. This test is called the *vertical line test*. If it is possible to draw any vertical line (a line of constant x) which crosses the graph of the relation more than once, then the relation is **not** a function. If more than one intersection point exists, then the intersections correspond to multiple values of y for a single value of x.

We can see this with our previous example of the circle by looking at its graph again in Figure 4.1.

[1]This content is available online at <http://cnx.org/content/m39286/1.1/>.

Image not finished

Figure 4.1: Graph of $x^2 + y^2 = 4$

We see that we can draw a vertical line, for example the dotted line in the drawing, which cuts the circle more than once. Therefore this is **not** a function.

4.1.2.1 Exercises

1. State whether each of the following equations are functions or not:
   a. $x + y = 4$
   b. $y = \frac{x}{4}$
   c. $y = 2^x$
   d. $x^2 + y^2 = 4$
2. The table gives the average per capita income, d, in a region of the country as a function of u, the percentage unemployed. Write down the equation to show that income is a function of the persent unemployed.

u	1	2	3	4
d	22500	22000	21500	21000

Table 4.1

4.1.3 Notation used for Functions

In grade 10 you were introduced to the notation used to "name" a function. In a function $y = f(x)$, y is called the *dependent variable*, because the value of y depends on what you choose as x. We say x is the *independent variable*, since we can choose x to be any number. Similarly if $g(t) = 2t + 1$, then t is the independent variable and g is the function name. If $f(x) = 3x - 5$ and you are ask to determine $f(3)$, then you have to work out the value for $f(x)$ when $x = 3$. For example,

$$\begin{aligned} f(x) &= 3x - 5 \\ f(3) &= 3(3) - 5 \\ &= 4 \end{aligned} \qquad (4.4)$$

4.2 Graphs of Inverse Functions[2]

4.2.1 Graphs of Inverse Functions

In earlier grades, you studied various types of functions and understood the effect of various parameters in the general equation. In this section, we will consider *inverse functions*.

[2] This content is available online at <http://cnx.org/content/m39282/1.1/>.

An inverse function is a function which "does the reverse" of a given function. More formally, if f is a function with domain X, then f^{-1} is its inverse function if and only if for every $x \in X$ we have:

$$f^{-1}(f(x)) == x \tag{4.5}$$

A simple way to think about this is that a function, say $y = f(x)$, gives you a y-value if you substitute an x-value into $f(x)$. The inverse function tells you tells you which x-value was used to get a particular y-value when you substitue the y-value into $f^{-1}(x)$. There are some things which can complicate this for example, think about a *sin* function, there are many x-values that give you a peak as the function oscillates. This means that the inverse of a *sin* function would be tricky to define because if you substitute the peak y-value into it you won't know which of the x-values was used to get the peak.

$$\begin{aligned}
y &= f(x) \quad \text{we have a function} \\
y_1 &= f(x_1) \\
&\text{we substitute a specific x} - \text{value into the function to get a specific y} - \text{value} \\
&\text{consider the inverse function} \\
x &= f^{-1}(y) \\
x &= f^{-1}(y) \\
&\text{substituting the specific y} - \text{value into the inverse should return the specific x} - \text{value} \\
&= f^{-1}(y_1) \\
&= x_1
\end{aligned} \tag{4.6}$$

This works both ways, if we don't have any complications like in the case of the *sin* function, so we can write:

$$f^{-1}(f(x)) = f\left(f^{-1}(x)\right) = x \tag{4.7}$$

For example, if the function $x \rightarrow 3x + 2$ is given, then its inverse function is $x \rightarrow \frac{(x-2)}{3}$. This is usually written as:

$$\begin{aligned}
f &: \quad x \rightarrow 3x + 2 \\
f^{-1} &: \quad x \rightarrow \frac{(x-2)}{3}
\end{aligned} \tag{4.8}$$

The superscript "-1" is not an exponent.

If a function f has an inverse then f is said to be invertible.

If f is a real-valued function, then for f to have a valid inverse, it must pass the **horizontal line test**, that is a horizontal line $y = k$ placed anywhere on the graph of f must pass through f exactly once for all real k.

It is possible to work around this condition, by defining a "multi-valued" function as an inverse.

If one represents the function f graphically in a xy-coordinate system, the inverse function of the equation of a straight line, f^{-1}, is the reflection of the graph of f across the line $y = x$.

Algebraically, one computes the inverse function of f by solving the equation

$$y = f(x) \tag{4.9}$$

for x, and then exchanging y and x to get

$$y = f^{-1}(x) \tag{4.10}$$

Khan academy video on inverse functions - 1

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/W84lObmOp8M&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 4.2

4.2.1.1 Inverse Function of $y = ax + q$

The inverse function of $y = ax + q$ is determined by solving for x as:

$$\begin{aligned} y &= ax + q \\ ax &= y - q \\ x &= \frac{y-q}{a} \\ &= \frac{1}{a}y - \frac{q}{a} \end{aligned} \tag{4.11}$$

Therefore the inverse of $y = ax + q$ is $y = \frac{1}{a}x - \frac{q}{a}$.

The inverse function of a straight line is also a straight line, except for the case where the straight line is a perfectly horizontal line, in which case the inverse is undefined.

For example, the straight line equation given by $y = 2x - 3$ has as inverse the function, $y = \frac{1}{2}x + \frac{3}{2}$. The graphs of these functions are shown in Figure 4.3. It can be seen that the two graphs are reflections of each other across the line $y = x$.

Image not finished

Figure 4.3: The graphs of the function $f(x) = 2x - 3$ and its inverse $f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}$. The line $y = x$ is shown as a dashed line.

4.2.1.1.1 Domain and Range

We have seen that the domain of a function of the form $y = ax + q$ is $\{x : x \in \mathbb{R}\}$ and the range is $\{y : y \in \mathbb{R}\}$. Since the inverse function of a straight line is also a straight line, the inverse function will have the same domain and range as the original function.

4.2.1.1.2 Intercepts

The general form of the inverse function of the form $y = ax + q$ is $y = \frac{1}{a}x - \frac{q}{a}$.

By setting $x = 0$ we have that the y-intercept is $y_{int} = -\frac{q}{a}$. Similarly, by setting $y = 0$ we have that the x-intercept is $x_{int} = q$.

It is interesting to note that if $f(x) = ax + q$, then $f^{-1}(x) = \frac{1}{a}x - \frac{q}{a}$ and the y-intercept of $f(x)$ is the x-intercept of $f^{-1}(x)$ and the x-intercept of $f(x)$ is the y-intercept of $f^{-1}(x)$.

4.2.1.2 Exercises

1. Given $f(x) = 2x - 3$, find $f^{-1}(x)$
2. Consider the function $f(x) = 3x - 7$.
    a. Is the relation a function?
    b. If it is a function, identify the domain and range.
3. Sketch the graph of the function $f(x) = 3x - 1$ and its inverse on the same set of axes.
4. The inverse of a function is $f^{-1}(x) = 2x - 4$, what is the function $f(x)$?

4.2.1.3 Inverse Function of $y = ax^2$

The inverse relation, possibly a function, of $y = ax^2$ is determined by solving for x as:

$$\begin{aligned} y &= ax^2 \\ x^2 &= \frac{y}{a} \\ x &= \pm\sqrt{\frac{y}{a}} \end{aligned} \tag{4.12}$$

Image not finished

Figure 4.4: The function $f(x) = x^2$ and its inverse $f^{-1}(x) = \pm\sqrt{x}$. The line $y = x$ is shown as a dashed line.

We see that the inverse "function" of $y = ax^2$ is not a function because it fails the vertical line test. If we draw a vertical line through the graph of $f^{-1}(x) = \pm\sqrt{x}$, the line intersects the graph more than once. There has to be a restriction on the domain of a parabola for the inverse to also be a function. Consider the function $f(x) = -x^2 + 9$. The inverse of f can be found by witing $f(y) = x$. Then

$$\begin{aligned} x &= -y^2 + 9 \\ y^2 &= 9 - x \\ y &= \pm\sqrt{9 - x} \end{aligned} \tag{4.13}$$

If we restrict the domain of $f(x)$ to be $x \geq 0$, then $\sqrt{9 - x}$ is a function. If the restriction on the domain of f is $x \leq 0$ then $-\sqrt{9 - x}$ would be a function, inverse to f.

Khan academy video on inverse functions - 2

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/aeyFb2eVH1c&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 4.5

Khan academy video on inverse functions - 3

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/Bq9cq9FZuNM&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 4.6

4.2.1.4 Exercises

1. The graph of f^{-1} is shown. Find the equation of f, given that the graph of f is a parabola. (Do not simplify your answer)

Figure 4.7

2. $f(x) = 2x^2$.
   a. Draw the graph of f and state its domain and range.
   b. Find f^{-1} and, if it exists, state the domain and range.
   c. What must the domain of f be, so that f^{-1} is a function ?
3. Sketch the graph of $x = -\sqrt{10 - y^2}$. Label a point on the graph other than the intercepts with the axes.
4. a. Sketch the graph of $y = x^2$ labelling a point other than the origin on your graph.
   b. Find the equation of the inverse of the above graph in the form $y = ...$
   c. Now sketch the graph of $y = \sqrt{x}$.
   d. The tangent to the graph of $y = \sqrt{x}$ at the point A(9;3) intersects the x-axis at B. Find the equation of this tangent and hence or otherwise prove that the y-axis bisects the straight line AB.
5. Given: $g(x) = -1 + \sqrt{x}$, find the inverse of $g(x)$ in the form $g^{-1}(x) = ...$

4.2.1.5 Inverse Function of $y = a^x$

The inverse function of $y = ax^2$ is determined by solving for x as follows:

$$\begin{aligned} y &= a^x \\ log(y) &= log(a^x) \\ &= xlog(a) \\ \therefore \quad x &= \frac{log(y)}{log(a)} \end{aligned} \qquad (4.14)$$

The inverse of $y = 10^x$ is $x = 10^y$, which we write as $y = log(x)$. Therefore, if $f(x) = 10^x$, then $f^{-1} = log(x)$.

Image not finished

Figure 4.8: The function $f(x) = 10^x$ and its inverse $f^{-1}(x) = log(x)$. The line $y = x$ is shown as a dashed line.

The exponential function and the logarithmic function are inverses of each other; the graph of the one is the graph of the other, reflected in the line $y = x$. The domain of the function is equal to the range of the inverse. The range of the function is equal to the domain of the inverse.

4.2.1.6 Exercises

1. Given that $f(x) = \left(\frac{1}{5}\right)^x$, sketch the graphs of f and f^{-1} on the same system of axes indicating a point on each graph (other than the intercepts) and showing clearly which is f and which is f^{-1}.
2. Given that $f(x) = 4^{-x}$,
   a. Sketch the graphs of f and f^{-1} on the same system of axes indicating a point on each graph (other than the intercepts) and showing clearly which is f and which is f^{-1}.
   b. Write f^{-1} in the form $y = ...$
3. Given $g(x) = -1 + \sqrt{x}$, find the inverse of $g(x)$ in the form $g^{-1}(x) = ...$
4. Answer the following questions:
   a. Sketch the graph of $y = x^2$, labeling a point other than the origin on your graph.
   b. Find the equation of the inverse of the above graph in the form $y = ...$
   c. Now, sketch $y = \sqrt{x}$.
   d. The tangent to the graph of $y = \sqrt{x}$ at the point $A(9;3)$ intersects the x-axis at B. Find the equation of this tangent, and hence, or otherwise, prove that the y-axis bisects the straight line AB.

4.2.2 End of Chapter Exercises

1. Sketch the graph of $x = -\sqrt{10 - y^2}$. Is this graph a function ? Verify your answer.
2. $f(x) = \frac{1}{x-5}$,
   a. determine the y-intercept of $f(x)$
   b. determine x if $f(x) = -1$.
3. Below, you are given 3 graphs and 5 equations.

Image not finished

Figure 4.9

   a. $y = log_3 x$
   b. $y = -log_3 x$
   c. $y = log_3(-x)$

d. $y = 3^{-x}$
e. $y = 3^{x}$

Write the equation that best describes each graph.

4. Given $g(x) = -1 + \sqrt{x}$, find the inverse of $g(x)$ in the form $g^{-1}(x) = ...$
5. Consider the equation $h(x) = 3^{x}$
   a. Write down the inverse in the form $h^{-1}(x) = ...$
   b. Sketch the graphs of $h(x)$ and $h^{-1}(x)$ on the same set of axes, labelling the intercepts with the axes.
   c. For which values of x is $h^{-1}(x)$ undefined ?
6. a. Sketch the graph of $y = 2x^2 + 1$, labelling a point other than the origin on your graph.
   b. Find the equation of the inverse of the above graph in the form $y = ...$
   c. Now, sketch $y = \sqrt{x}$.
   d. The tangent to the graph of $y = \sqrt{x}$ at the point $A(9;3)$ intersects the x-axis at B. Find the equation of this tangent, and hence, or otherwise, prove that the y-axis bisects the straight line AB.

Chapter 5

Differential calculus

5.1 Introduction, Limits, Average Gradient[1]

5.1.1 Why do I have to learn this stuff?

Calculus is one of the central branches of mathematics and was developed from algebra and geometry. Calculus is built on the concept of limits, which will be discussed in this chapter. Calculus consists of two complementary ideas: differential calculus and integral calculus. We will only be dealing with differential calculus in this text. Differential calculus is concerned with the instantaneous rate of change of quantities with respect to other quantities, or more precisely, the local behaviour of functions. This can be illustrated by the slope of a function's graph. Examples of typical differential calculus problems include: finding the acceleration and velocity of a free-falling body at a particular moment and finding the optimal number of units a company should produce to maximize its profit.

Calculus is fundamentally different from the mathematics that you have studied previously. Calculus is more dynamic and less static. It is concerned with change and motion. It deals with quantities that approach other quantities. For that reason it may be useful to have an overview of the subject before beginning its intensive study.

Calculus is a tool to understand many phenomena, both natural and man-made, like how the wind blows, how water flows, how light travels, how sound travels, how the planets move and even economics.

In this section we give a glimpse of some of the main ideas of calculus by showing how limits arise when we attempt to solve a variety of problems.

5.1.1.1 Integral Calculus

Integral calculus is concerned with the accumulation of quantities, such as areas under a curve, linear distance traveled, or volume displaced. Differential and integral calculus act inversely to each other. Examples of typical integral calculus problems include finding areas and volumes, finding the amount of water pumped by a pump with a set power input but varying conditions of pumping losses and pressure and finding the amount of rain that fell in a certain area if the rain fell at a specific rate.

> NOTE: Both Isaac Newton (4 January 1643 – 31 March 1727) and Gottfried Leibnitz (1 July 1646 – 14 November 1716 (Hanover, Germany)) are credited with the 'invention' of calculus. Newton was the first to apply calculus to general physics, while Leibnitz developed most of the notation that is still in use today.
>
> When Newton and Leibniz first published their results, there was some controversy over whether Leibniz's work was independent of Newton's. While Newton derived his results years before Leibniz,

[1] This content is available online at <http://cnx.org/content/m39270/1.1/>.

it was only some time after Leibniz published in 1684 that Newton published. Later, Newton would claim that Leibniz got the idea from Newton's notes on the subject; however examination of the papers of Leibniz and Newton show they arrived at their results independently, with Leibniz starting first with integration and Newton with differentiation. This controversy between Leibniz and Newton divided English-speaking mathematicians from those in Europe for many years, which slowed the development of mathematical analysis. Today, both Newton and Leibniz are given credit for independently developing calculus. It is Leibniz, however, who is credited with giving the new discipline the name it is known by today: "calculus". Newton's name for it was "the science of fluxions".

5.1.2 Limits

5.1.2.1 A Tale of Achilles and the Tortoise

NOTE: Zeno (circa 490 BC - circa 430 BC) was a pre-Socratic Greek philosopher of southern Italy who is famous for his paradoxes.

One of Zeno's paradoxes can be summarised by:

Achilles and a tortoise agree to a race, but the tortoise is unhappy because Achilles is very fast. So, the tortoise asks Achilles for a head-start. Achilles agrees to give the tortoise a 1 000 m head start. Does Achilles overtake the tortoise?

We know how to solve this problem. We start by writing:

$$\begin{aligned} x_A &= v_A t \\ x_t &= 1000m + v_t t \end{aligned} \tag{5.1}$$

where

x_A	distance covered by Achilles
v_A	Achilles' speed
t	time taken by Achilles to overtake tortoise
x_t	distance covered by the tortoise
v_t	the tortoise's speed

Table 5.1

If we assume that Achilles runs at 2 m·s^{-1}and the tortoise runs at 0,25 m·s^{-1}then Achilles will overtake the tortoise when both of them have covered the same distance. This means that Achilles overtakes the tortoise at a time calculated as:

$$\begin{aligned} x_A &= x_t \\ v_A t &= 1000 + v_t t \\ \left(2m \cdot s^{-1}\right) t &= 1000m + \left(0,25m \cdot s^{-1}\right) t \\ \left(2m \cdot s^{-1} - 0,25m \cdot s^{-1}\right) t &= 1000m \\ t &= \frac{1000m}{1\frac{3}{4} m \cdot s^{-1}} \\ &= \frac{1000m}{\frac{7}{4} m \cdot s^{-1}} \\ &= \frac{(4)(1000)}{7} s \\ &= \frac{4000}{7} s \\ &= 571\frac{3}{7} s \end{aligned} \tag{5.2}$$

However, Zeno (the Greek philosopher who thought up this problem) looked at it as follows: Achilles takes $t = \frac{1000}{2} = 500s$ to travel the 1 000 m head start that the tortoise had. However, in this 500 s, the tortoise has travelled a further $x = (500)(0,25) = 125m$. Achilles then takes another $t = \frac{125}{2} = 62,5s$ to travel the 125 m. In this 62,5 s, the tortoise travels a further $x = (62,5)(0,25) = 15,625m$. Zeno saw that Achilles would always get closer but wouldn't actually overtake the tortoise.

5.1.2.2 Sequences, Series and Functions

So what does Zeno, Achilles and the tortoise have to do with calculus?

Well, in Grades 10 and 11 you studied sequences. For the sequence $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...$ which is defined by the expression $a_n = 1 - \frac{1}{n}$ the terms get closer to 1 as n gets larger. Similarly, for the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$ which is defined by the expression $a_n = \frac{1}{n}$ the terms get closer to 0 as n gets larger. We have also seen that the infinite geometric series has a finite total. The infinite geometric series is

$$S_\infty = \sum_{i=1}^{\infty} a_1.r^{i-1} = \frac{a_1}{1-r} \qquad \text{for} \qquad r \neq -1 \tag{5.3}$$

where a_1 is the first term of the series and r is the common ratio.

We see that there are some functions where the value of the function gets close to or **approaches** a certain value.

Similarly, for the function: $y = \frac{x^2+4x-12}{x+6}$ The numerator of the function can be factorised as: $y = \frac{(x+6)(x-2)}{x+6}$. Then we can cancel the $x - 6$ from numerator and denominator and we are left with: $y = x - 2$. However, we are only able to cancel the $x + 6$ term if $x \neq -6$. If $x = -6$, then the denominator becomes 0 and the function is not defined. This means that the domain of the function does not include $x = -6$. But we can examine what happens to the values for y as x gets close to -6. These values are listed in Table 5.2 which shows that as x gets closer to -6, y gets close to 8.

x	$y = \frac{(x+6)(x-2)}{x+6}$
-9	-11
-8	-10
-7	-9
-6.5	-8.5
-6.4	-8.4
-6.3	-8.3
-6.2	-8.2
-6.1	-8.1
-6.09	-8.09
-6.08	-8.08
-6.01	-8.01
-5.9	-7.9
-5.8	-7.8
-5.7	-7.7
-5.6	-7.6
-5.5	-7.5
-5	-7
-4	-6
-3	-5

Table 5.2: Values for the function $y = \frac{(x+6)(x-2)}{x+6}$ as x gets close to -6.

The graph of this function is shown in Figure 5.1. The graph is a straight line with slope 1 and intercept -2, but with a missing section at $x = -6$.

Image not finished

Figure 5.1: Graph of $y = \frac{(x+6)(x-2)}{x+6}$.

We say that a function is continuous if there are no values of the independent variable for which the function is undefined.

5.1.2.3 Limits

We can now introduce a new notation. For the function $y = \frac{(x+6)(x-2)}{x+6}$, we can write: $lim_{x \to -6} \frac{(x+6)(x-2)}{x+6} = -8$. This is read: *the limit of* $\frac{(x+6)(x-2)}{x+6}$ *as* x *tends to -6 is 8.*

5.1.2.3.1 Investigation : Limits

If $f(x) = x + 1$, determine:

f(-0.1)	
f(-0.05)	
f(-0.04)	
f(-0.03)	
f(-0.02)	
f(-0.01)	
f(0.00)	
f(0.01)	
f(0.02)	
f(0.03)	
f(0.04)	
f(0.05)	
f(0.1)	

Table 5.3

What do you notice about the value of $f(x)$ as x gets close to 0.

Exercise 5.1.1: Limits Notation *(Solution on p. 77.)*

Summarise the following situation by using limit notation: As x gets close to 1, the value of the function $y = x + 2$ gets close to 3.

We can also have the situation where a function has a different value depending on whether x approaches from the left or the right. An example of this is shown in Figure 5.2.

Image not finished

Figure 5.2: Graph of $y = \frac{1}{x}$.

As $x \to 0$ from the left, $y = \frac{1}{x}$ approaches $-\infty$. As $x \to 0$ from the right, $y = \frac{1}{x}$ approaches $+\infty$. This is written in limits notation as: $lim_{x \to 0^-} \frac{1}{x} = -\infty$ for x approaching zero from the left and $lim_{x \to 0^+} \frac{1}{x} = \infty$ for x approaching zero from the right. You can calculate the limit of many different functions using a set method.

Method:

Limits: If you are required to calculate a limit like $lim_{x \to a}$ then:

1. Simplify the expression completely.
2. If it is possible, cancel all common terms.
3. Let x approach a.

Exercise 5.1.2: Limits *(Solution on p. 77.)*

Determine $lim_{x\to 1} 10$

Exercise 5.1.3: Limits *(Solution on p. 77.)*

Determine $lim_{x\to 2} x$

Exercise 5.1.4: Limits *(Solution on p. 77.)*

Determine $lim_{x\to 10} \frac{x^2-100}{x-10}$

5.1.2.4 Average Gradient and Gradient at a Point

In Grade 10 you learnt about average gradients on a curve. The average gradient between any two points on a curve is given by the gradient of the straight line that passes through both points. In Grade 11 you were introduced to the idea of a gradient at a single point on a curve. We saw that this was the gradient of the tangent to the curve at the given point, but we did not learn how to determine the gradient of the tangent.

Now let us consider the problem of trying to find the gradient of a tangent t to a curve with equation $y = f(x)$ at a given point P.

Image not finished

Figure 5.3

We know how to calculate the average gradient between two points on a curve, but we need two points. The problem now is that we only have one point, namely P. To get around the problem we first consider a secant to the curve that passes through point P and another point on the curve Q. We can now find the average gradient of the curve between points P and Q.

Image not finished

Figure 5.4

If the x-coordinate of P is a, then the y-coordinate is $f(a)$. Similarly, if the x-coordinate of Q is $a-h$, then the y-coordinate is $f(a-h)$. If we choose a as x_2 and $a-h$ as x_1, then: $y_1 = f(a-h) y_2 = f(a)$. We can now calculate the average gradient as:

$$\begin{aligned} \frac{y_2-y_1}{x_2-x_1} &= \frac{f(a)-f(a-h)}{a-(a-h)} \\ &= \frac{f(a)-f(a-h)}{h} \end{aligned} \quad (5.4)$$

Now imagine that Q moves along the curve toward P. The secant line approaches the tangent line as its limiting position. This means that the average gradient of the secant *approaches* the gradient of the tangent to the curve at P. In we see that as point Q approaches point P, h gets closer to 0. When $h = 0$, points P and Q are equal. We can now use our knowledge of limits to write this as:

$$\text{gradient at P} = \lim_{h\to 0} \frac{f(a)-f(a-h)}{h}. \quad (5.5)$$

and we say that the gradient at point P is the limit of the average gradient as Q approaches P along the curve.

Khan academy video on calculus - 1

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/ANyVpMS3HL4&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 5.5

Exercise 5.1.5: Limits ***(Solution on p. 77.)***
For the function $f(x) = 2x^2 - 5x$, determine the gradient of the tangent to the curve at the point $x = 2$.

Exercise 5.1.6: Limits ***(Solution on p. 77.)***
For the function $f(x) = 5x^2 - 4x + 1$, determine the gradient of the tangent to curve at the point $x = a$.

5.1.2.4.1 Limits

Determine the following

1. $lim_{x \to 3} \frac{x^2-9}{x+3}$
2. $lim_{x \to 3} \frac{x+3}{x^2+3x}$
3. $lim_{x \to 2} \frac{3x^2-4x}{3-x}$
4. $lim_{x \to 4} \frac{x^2-x-12}{x-4}$
5. $lim_{x \to 2} 3x + \frac{1}{3x}$

5.2 Differentiation (First Principles, Rules) and Sketching Graphs[2]

5.2.1 Differentiation from First Principles

The tangent problem has given rise to the branch of calculus called **differential calculus** and the equation: $lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ defines the **derivative of the function** $f(x)$**.** Using to calculate the derivative is called **finding the derivative from first principles**.

Definition 5.1: Derivative
The derivative of a function $f(x)$ is written as $f'(x)$ and is defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (5.6)$$

There are a few different notations used to refer to derivatives. If we use the traditional notation $y = f(x)$ to indicate that the dependent variable is y and the independent variable is x, then some common alternative notations for the derivative are as follows: $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$

[2] This content is available online at <http://cnx.org/content/m39313/1.1/>.

The symbols D and $\frac{d}{dx}$ are called **differential operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative. It is very important that you learn to identify these different ways of denoting the derivative, and that you are consistent in your usage of them when answering questions.

TIP: Though we choose to use a fractional form of representation, $\frac{dy}{dx}$ is a limit and **is not** a fraction, i.e. $\frac{dy}{dx}$ does not mean $dy \div dx$. $\frac{dy}{dx}$ means y differentiated with respect to x. Thus, $\frac{dp}{dx}$ means p differentiated with respect to x. The '$\frac{d}{dx}$' is the "operator", operating on some function of x.

Video on calculus - 2

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/GpMgMcA2SBY?version=3>

Figure 5.6

Exercise 5.2.1: Derivatives - First Principles *(Solution on p. 78.)*

Calculate the derivative of $g(x) = x - 1$ from first principles.

5.2.1.1 Derivatives

1. Given $g(x) = -x^2$
   a. determine $\frac{g(x+h)-g(x)}{h}$
   b. hence, determine $lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$
   c. explain the meaning of your answer in (b).
2. Find the derivative of $f(x) = -2x^2 + 3x$ using first principles.
3. Determine the derivative of $f(x) = \frac{1}{x-2}$ using first principles.
4. Determine $f'(3)$ from first principles if $f(x) = -5x^2$.
5. If $h(x) = 4x^2 - 4x$, determine $h'(x)$ using first principles.

5.2.2 Rules of Differentiation

Calculating the derivative of a function from first principles is very long, and it is easy to make mistakes. Fortunately, there are rules which make calculating the derivative simple.

5.2.2.1 Investigation : Rules of Differentiation

From first principles, determine the derivatives of the following:

1. $f(x) = b$
2. $f(x) = x$
3. $f(x) = x^2$
4. $f(x) = x^3$
5. $f(x) = 1/x$

You should have found the following:

$f(x)$	$f'(x)$
b	0
x	1
x^2	$2x$
x^3	$3x^2$
$1/x = x^{-1}$	$-x^{-2}$

Table 5.4

If we examine these results we see that there is a pattern, which can be summarised by: $\frac{d}{dx}\ (x^n) = nx^{n-1}$

There are two other rules which make differentiation simpler. For any two functions $f(x)$ and $g(x)$: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$ This means that we differentiate each term separately.

The final rule applies to a function $f(x)$ that is multiplied by a constant k. $\frac{d}{dx}[k.f(x)] = kf'(x)$

Video on calculus - 3

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/SjecBgK-SNY?version=3>

Figure 5.7

Exercise 5.2.2: Rules of Differentiation ***(Solution on p. 78.)***

Determine the derivative of $x-1$ using the rules of differentiation.

5.2.2.2 Summary of Differentiation Rules

Given two functions, $f(x)$ and $g(X)$ we know that:

$\frac{d}{dx}b = 0$
$\frac{d}{dx}(x^n) = nx^{n-1}$
$\frac{d}{dx}(kf) = k\frac{df}{dx}$
$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

Table 5.5

5.2.2.2.1 Rules of Differentiation

1. Find $f'(x)$ if $f(x) = \frac{x^2-5x+6}{x-2}$.
2. Find $f'(y)$ if $f(y) = \sqrt{y}$.
3. Find $f'(z)$ if $f(z) = (z-1)(z+1)$.
4. Determine $\frac{dy}{dx}$ if $y = \frac{x^3+2\sqrt{x}-3}{x}$.
5. Determine the derivative of $y = \sqrt{x^3} + \frac{1}{3x^3}$.

5.2.3 Applying Differentiation to Draw Graphs

Thus far we have learnt about how to differentiate various functions, but I am sure that you are beginning to ask, *What is the point of learning about derivatives?* Well, we know one important fact about a derivative: it is a gradient. So, any problems involving the calculations of gradients or rates of change can use derivatives. One simple application is to draw graphs of functions by firstly determine the gradients of straight lines and secondly to determine the turning points of the graph.

5.2.3.1 Finding Equations of Tangents to Curves

In "Average Gradient and Gradient at a Point" we saw that finding the gradient of a tangent to a curve is the same as finding the gradient (or slope) of the same curve at the point of the tangent. We also saw that the gradient of a function at a point is just its derivative.

Since we have the gradient of the tangent and the point on the curve through which the tangent passes, we can find the equation of the tangent.

Exercise 5.2.3: Finding the Equation of a Tangent to a Curve *(Solution on p. 78.)*

Find the equation of the tangent to the curve $y = x^2$ at the point (1,1) and draw both functions.

5.2.3.2 Curve Sketching

Differentiation can be used to sketch the graphs of functions, by helping determine the turning points. We know that if a graph is increasing on an interval and reaches a turning point, then the graph will start decreasing after the turning point. The turning point is also known as a stationary point because the gradient at a turning point is 0. We can then use this information to calculate turning points, by calculating the points at which the derivative of a function is 0.

TIP: If $x = a$ is a turning point of $f(x)$, then: $f'(a) = 0$ This means that the derivative is 0 at a turning point.

Take the graph of $y = x^2$ as an example. We know that the graph of this function has a turning point at (0,0), but we can use the derivative of the function: $y' = 2x$ and set it equal to 0 to find the x-value for which the graph has a turning point.

$$\begin{aligned} 2x &= 0 \\ x &= 0 \end{aligned} \tag{5.7}$$

We then substitute this into the equation of the graph (i.e. $y = x^2$) to determine the y-coordinate of the turning point: $f(0) = (0)^2 = 0$ This corresponds to the point that we have previously calculated.

Exercise 5.2.4: Calculation of Turning Points *(Solution on p. 79.)*

Calculate the turning points of the graph of the function $f(x) = 2x^3 - 9x^2 + 12x - 15$.

We are now ready to sketch graphs of functions.

Method:

Sketching Graphs: Suppose we are given that $f(x) = ax^3 + bx^2 + cx + d$, then there are **five** steps to be followed to sketch the graph of the function:

1. If $a > 0$, then the graph is increasing from left to right, and may have a maximum and a minimum. As x increases, so does $f(x)$. If $a < 0$, then the graph decreasing is from left to right, and has first a minimum and then a maximum. $f(x)$ decreases as x increases.
2. Determine the value of the y-intercept by substituting $x = 0$ into $f(x)$
3. Determine the x-intercepts by factorising $ax^3 + bx^2 + cx + d = 0$ and solving for x. First try to eliminate constant common factors, and to group like terms together so that the expression is expressed as economically as possible. Use the factor theorem if necessary.
4. Find the turning points of the function by working out the derivative $\frac{df}{dx}$ and setting it to zero, and solving for x.
5. Determine the y-coordinates of the turning points by substituting the x values obtained in the previous step, into the expression for $f(x)$.
6. Use the information you're given to plot the points and get a rough idea of the gradients between points. Then fill in the missing parts of the function in a smooth, continuous curve.

Exercise 5.2.5: Sketching Graphs ***(Solution on p. 79.)***
Draw the graph of $g(x) = x^2 - x + 2$

Exercise 5.2.6: Sketching Graphs ***(Solution on p. 80.)***
Sketch the graph of $g(x) = -x^3 + 6x^2 - 9x + 4$.

5.2.3.2.1 Sketching Graphs

1. Given $f(x) = x^3 + x^2 - 5x + 3$:
   a. Show that $(x - 1)$ is a factor of $f(x)$ and hence fatorise $f(x)$ fully.
   b. Find the coordinates of the intercepts with the axes and the turning points and sketch the graph
2. Sketch the graph of $f(x) = x^3 - 4x^2 - 11x + 30$ showing all the relative turning points and intercepts with the axes.
3. a. Sketch the graph of $f(x) = x^3 - 9x^2 + 24x - 20$, showing all intercepts with the axes and turning points.
   b. Find the equation of the tangent to $f(x)$ at $x = 4$.

5.2.3.3 Local minimum, Local maximum and Point of Inflextion

If the derivative ($\frac{dy}{dx}$) is zero at a point, the gradient of the tangent at that point is zero. It means that a turning point occurs as seen in the previous example.

Figure 5.8

From the drawing the point (1;0) represents a **local minimum** and the point (3;4) the **local maximum**.
A graph has a horizontal **point of inflexion** where the derivative is zero but the sign of the sign of the gradient does not change. That means the graph always increases or always decreases.

Figure 5.9

From this drawing, the point (3;1) is a horizontal point of inflexion, because the sign of the derivative does not change from positive to negative.

5.3 Solving Problems[3]

5.3.1 Using Differential Calculus to Solve Problems

We have seen that differential calculus can be used to determine the stationary points of functions, in order to sketch their graphs. However, determining stationary points also lends itself to the solution of problems that require some variable to be *optimised*.

For example, if fuel used by a car is defined by:

$$f(v) = \frac{3}{80}v^2 - 6v + 245 \quad (5.8)$$

where v is the travelling speed, what is the most economical speed (that means the speed that uses the least fuel)?

If we draw the graph of this function we find that the graph has a minimum. The speed at the minimum would then give the most economical speed.

Figure 5.10

We have seen that the coordinates of the turning point can be calculated by differentiating the function and finding the x-coordinate (speed in the case of the example) for which the derivative is 0.

Differentiating (5.8), we get: $f'(v) = \frac{3}{40}v - 6$ If we set $f'(v) = 0$ we can calculate the speed that corresponds to the turning point.

$$\begin{array}{rcl} f'(v) & = & \frac{3}{40}v - 6 \\ 0 & = & \frac{3}{40}v - 6 \\ v & = & \frac{6 \times 40}{3} \\ & = & 80 \end{array} \quad (5.9)$$

This means that the most economical speed is 80 $\mathrm{km \cdot hr^{-1}}$.

[3]This content is available online at <http://cnx.org/content/m39273/1.1/>.

Video on calculus - 4

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/1TK69SVQNPk?version=3>

Figure 5.11

Exercise 5.3.1: Optimisation Problems *(Solution on p. 81.)*

The sum of two positive numbers is 10. One of the numbers is multiplied by the square of the other. If each number is greater than 0, find the numbers that make this product a maximum.

Exercise 5.3.2: Optimisation Problems *(Solution on p. 82.)*

Michael wants to start a vegetable garden, which he decides to fence off in the shape of a rectangle from the rest of the garden. Michael only has 160 m of fencing, so he decides to use a wall as one border of the vegetable garden. Calculate the width and length of the garden that corresponds to largest possible area that Michael can fence off.

Figure 5.12

5.3.1.1 Solving Optimisation Problems using Differential Calculus

1. The sum of two positive numbers is 20. One of the numbers is multiplied by the square of the other. Find the numbers that make this product a maximum.
2. A wooden block is made as shown in the diagram. The ends are right-angled triangles having sides $3x$, $4x$ and $5x$. The length of the block is y. The total surface area of the block is 3600cm^2.

Figure 5.13

   a. Show that $y = \frac{300 - x^2}{x}$.
   b. Find the value of x for which the block will have a maximum volume. (Volume = area of base × height.)

3. The diagram shows the plan for a verandah which is to be built on the corner of a cottage. A railing $ABCDE$ is to be constructed around the four edges of the verandah.

Figure 5.14

If $AB = DE = x$ and $BC = CD = y$, and the length of the railing must be 30 metres, find the values of x and y for which the verandah will have a maximum area.

5.3.1.2 Rate of Change problems

Two concepts were discussed in this chapter: **Average** rate of change $= \frac{f(b)-f(a)}{b-a}$ and **Instantaneous** rate of change $= lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. When we mention *rate of change*, the latter is implied. Instantaneous rate of change is the **derivative**. When *Average rate of change* is required, it will be specifically refer to as **average** rate of change.

Velocity is one of the most common forms of rate of change. Again, **average** velocity = **average** rate of change and **instantaneous** velocity = **instantaneous** rate of change = **derivative**. Velocity refers to the increase of distance(s) for a corresponding increade in time (t). The notation commonly used for this is: $v(t) = \frac{ds}{dt} = s'(t)$

where $s'(t)$ is the position function. Acceleration is the change in velocity for a corersponding increase in time. Therefore, acceleration is the derivative of velocity $a(t) = v'(t)$ This implies that acceleration is the second derivative of the distance(s).

Exercise 5.3.3: Rate of Change ***(Solution on p. 82.)***

The height (in metres) of a golf ball that is hit into the air after t seconds, is given by $h(t) = 20t - 5t^2$. Determine

1. the average velocity of the ball during the first two seconds
2. the velocity of the ball after 1,5 seconds
3. when the velocity is zero
4. the velocity at which the ball hits the ground
5. the acceleration of the ball

5.3.2 End of Chapter Exercises

1. Determine $f'(x)$ from **first principles** if:
   a. $f(x) = x^2 - 6x$
   b. $f(x) = 2x - x^2$
2. Given: $f(x) = -x^2 + 3x$, find $f'(x)$ using *first principles*.
3. Determine $\frac{dx}{dy}$ if:
   a. $y = (2x)^2 - \frac{1}{3x}$
   b. $y = \frac{2\sqrt{x}-5}{\sqrt{x}}$
4. Given: $f(x) = x^3 - 3x^2 + 4$
   a. Calculate $f(-1)$, and hence solve the equation$f(x) = 0$
   b. Determine $f'(x)$

c. Sketch the graph of f neatly and clearly, showing the co-ordinates of the turning points as well as the intercepts on both axes.
d. Determine the co-ordinates of the points on the graph of f where the gradient is 9.

5. Given: $f(x) = 2x^3 - 5x^2 - 4x + 3$. The x-intercepts of f are: (-1;0) ($\frac{1}{2}$;0) and (3;0).

a. Determine the co-ordinates of the turning points of f.
b. Draw a neat sketch graph of f. Clearly indicate the co-ordinates of the intercepts with the axes, as well as the co-ordinates of the turning points.
c. For which values of k will the equation $f(x) = k$, have exactly two real roots?
d. Determine the equation of the tangent to the graph of $f(x) = 2x^3 - 5x^2 - 4x + 3$ at the point where $x = 1$.

6. Answer the following questions:

a. Sketch the graph of $f(x) = x^3 - 9x^2 + 24x - 20$, showing all intercepts with the axes and turning points.
b. Find the equation of the tangent to $f(x)$ at $x = 4$.

7. Calculate: $lim_{x \to 1} \frac{1-x^3}{1-x}$
8. Given: $f(x) = 2x^2 - x$

a. Use the definition of the derivative to calculate $f^{'}(x)$.
b. Hence, calculate the co-ordinates of the point at which the gradient of the tangent to the graph of f is 7.

9. If $xy - 5 = \sqrt{x^3}$, determine $\frac{dx}{dy}$
10. Given: $g(x) = \left(x^{-2} + x^2\right)^2$. Calculate $g^{'}(2)$.
11. Given: $f(x) = 2x - 3$

a. Find: $f^{-1}(x)$
b. Solve: $f^{-1}(x) = 3f^{'}(x)$

12. Find $f^{'}(x)$ for each of the following:

a. $f(x) = \frac{\sqrt[5]{x^3}}{3} + 10$
b. $f(x) = \frac{\left(2x^2-5\right)(3x+2)}{x^2}$

13. Determine the minimum value of the sum of a *positive* number and its reciprocal.
14. If the displacement s (in metres) of a particle at time t (in seconds) is governed by the equation $s = \frac{1}{2}t^3 - 2t$, find its acceleration after 2 seconds. (Acceleration is the rate of change of velocity, and velocity is the rate of change of displacement.)
15. After doing some research, a transport company has determined that the rate at which petrol is consumed by one of its large carriers, travelling at an average speed of x km per hour, is given by: $P(x) = \frac{55}{2x} + \frac{x}{200}$ litresperkilometre

a. Assume that the petrol costs R4,00 per litre and the driver earns R18,00 per hour (travelling time). Now deduce that the total cost, C, in Rands, for a 2 000 km trip is given by: $C(x) = \frac{256000}{x} + 40x$
b. Hence determine the average speed to be maintained to effect a minimum cost for a 2 000 km trip.

16. During an experiment the temperature T (in degrees Celsius), varies with time t (in hours), according to the formula: $T(t) = 30 + 4t - \frac{1}{2}t^2 \quad t \in [1; 10]$

a. Determine an expression for the rate of change of temperature with time.
b. During which time interval was the temperature dropping?

17. The depth, d, of water in a kettle t minutes after it starts to boil, is given by $d = 86 - \frac{1}{8}t - \frac{1}{4}t^3$, where d is measured in millimetres.

a. How many millimetres of water are there in the kettle just before it starts to boil?

b. As the water boils, the level in the kettle drops. Find the *rate* at which the water level is decreasing when $t = 2$ minutes.
c. How many minutes after the kettle starts boiling will the water level be dropping at a rate of $12\frac{1}{8}$ mm/minute?

Solutions to Exercises in Chapter 5

Solution to Exercise 5.1.1 (p. 65)

Step 1. This is written as: $lim_{x\to 1}x+2=3$ in limit notation.

Solution to Exercise 5.1.2 (p. 66)

Step 1. There is nothing to simplify.
Step 2. There are no terms to cancel.
Step 3. $lim_{x\to 1}10=10$

Solution to Exercise 5.1.3 (p. 66)

Step 1. There is nothing to simplify.
Step 2. There are no terms to cancel.
Step 3. $lim_{x\to 2}x=2$

Solution to Exercise 5.1.4 (p. 66)

Step 1. The numerator can be factorised. $\frac{x^2-100}{x-10}=\frac{(x+10)(x-10)}{x-10}$
Step 2. $x-10$ can be cancelled from the numerator and denominator. $\frac{(x+10)(x-10)}{x-10}=x+10$
Step 3. $lim_{x\to 10}\frac{x^2-100}{x-10}=20$

Solution to Exercise 5.1.5 (p. 67)

Step 1. We know that the gradient at a point x is given by: $lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ In our case $x=2$. It is simpler to substitute $x=2$ at the end of the calculation.
Step 2.

$$\begin{array}{rcl} f(x+h) & = & 2(x+h)^2-5(x+h) \\ & = & 2\left(x^2+2xh+h^2\right)-5x-5h \\ & = & 2x^2+4xh+2h^2-5x-5h \end{array} \quad (5.10)$$

Step 3.

$$\begin{array}{rcl} \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} & = & \frac{2x^2+4xh+2h^2-5x-5h-\left(2x^2-5x\right)}{h}; \quad h\neq 0 \\ & = & \lim\limits_{h\to 0}\frac{2x^2+4xh+2h^2-5x-5h-2x^2+5x}{h} \\ & = & \lim\limits_{h\to 0}\frac{4xh+2h^2-5h}{h} \\ & = & \lim\limits_{h\to 0}\frac{h(4x+2h-5)}{h} \\ & = & \lim\limits_{h\to 0}4x+2h-5 \\ & = & 4x-5 \end{array} \quad (5.11)$$

Step 4. $4x-5=4(2)-5=3$
Step 5. The gradient of the tangent to the curve $f(x)=2x^2-5x$ at $x=2$ is 3.

Solution to Exercise 5.1.6 (p. 67)

Step 1. We know that the gradient at a point x is given by: $lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ In our case $x=a$. It is simpler to substitute $x=a$ at the end of the calculation.

Step 2.

$$\begin{aligned} f(x+h) &= 5(x+h)^2 - 4(x+h) + 1 \\ &= 5(x^2 + 2xh + h^2) - 4x - 4h + 1 \\ &= 5x^2 + 10xh + 5h^2 - 4x - 4h + 1 \end{aligned} \quad (5.12)$$

Step 3.

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \frac{5x^2+10xh+5h^2-4x-4h+1-(5x^2-4x+1)}{h} \\ &= \lim_{h\to 0} \frac{5x^2+10xh+5h^2-4x-4h+1-5x^2+4x-1}{h} \\ &= \lim_{h\to 0} \frac{10xh+5h^2-4h}{h} \\ &= \lim_{h\to 0} \frac{h(10x+5h-4)}{h} \\ &= \lim_{h\to 0} 10x+5h-4 \\ &= 10x-4 \end{aligned} \quad (5.13)$$

Step 4. $10x - 4 = 10a - 5$
Step 5. The gradient of the tangent to the curve $f(x) = 5x^2 - 4x + 1$ at $x = 1$ is $10a - 5$.

Solution to Exercise 5.2.1 (p. 68)

Step 1. We know that the gradient at a point x is given by: $g'(x) = lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$
Step 2. $g(x+h) = x + h - 1$
Step 3.

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{x+h-1-(x-1)}{h} \\ &= \lim_{h\to 0} \frac{x+h-1-x+1}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned} \quad (5.14)$$

Step 4. The derivative $g'(x)$ of $g(x) = x - 1$ is 1.

Solution to Exercise 5.2.2 (p. 69)

Step 1. We will apply two rules of differentiation: $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$
Step 2. In our case $f(x) = x$ and $g(x) = 1$. $f'(x) = 1$ and $g'(x) = 0$
Step 3. The derivative of $x - 1$ is 1 which is the same result as was obtained earlier, from first principles.

Solution to Exercise 5.2.3 (p. 70)

Step 1. We are required to determine the equation of the tangent to the curve defined by $y = x^2$ at the point (1,1). The tangent is a straight line and we can find the equation by using derivatives to find the gradient of the straight line. Then we will have the gradient and one point on the line, so we can find the equation using: $y - y_1 = m(x - x_1)$ from grade 11 Coordinate Geometry.
Step 2. Using our rules of differentiation we get: $y' = 2x$
Step 3. In order to determine the gradient at the point (1,1), we substitute the x-value into the equation for the derivative. So, y' at $x = 1$ is: $m = 2(1) = 2$

Step 4.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= (2)(x - 1) \\ y &= 2x - 2 + 1 \\ y &= 2x - 1 \end{aligned} \quad (5.15)$$

Step 5. The equation of the tangent to the curve defined by $y = x^2$ at the point (1,1) is $y = 2x - 1$.

Step 6.

Figure 5.15

Solution to Exercise 5.2.4 (p. 70)

Step 1. Using the rules of differentiation we get: $f'(x) = 6x^2 - 18x + 12$
Step 2.

$$\begin{aligned} 6x^2 - 18x + 12 &= 0 \\ x^2 - 3x + 2 &= 0 \\ (x - 2)(x - 1) &= 0 \end{aligned} \quad (5.16)$$

Therefore, the turning points are at $x = 2$ and $x = 1$.
Step 3.

$$\begin{aligned} f(2) &= 2(2)^3 - 9(2)^2 + 12(2) - 15 \\ &= 16 - 36 + 24 - 15 \\ &= -11 \end{aligned} \quad (5.17)$$

$$\begin{aligned} f(1) &= 2(1)^3 - 9(1)^2 + 12(1) - 15 \\ &= 2 - 9 + 12 - 15 \\ &= -10 \end{aligned} \quad (5.18)$$

Step 4. The turning points of the graph of $f(x) = 2x^3 - 9x^2 + 12x - 15$ are (2,-11) and (1,-10).

Solution to Exercise 5.2.5 (p. 71)

Step 1. The y-intercept is obtained by setting $x = 0$. $g(0) = (0)^2 - 0 + 2 = 2$ The turning point is at (0,2).
Step 2. The x-intercepts are found by setting $g(x) = 0$.

$$\begin{aligned} g(x) &= x^2 - x + 2 \\ 0 &= x^2 - x + 2 \end{aligned} \quad (5.19)$$

Using the quadratic formula and looking at $b^2 - 4ac$ we can see that this would be negative and so this function does not have real roots. Therefore, the graph of $g(x)$ does not have any x-intercepts.

Step 3. Work out the derivative $\frac{dg}{dx}$ and set it to zero to for the x coordinate of the turning point. $\frac{dg}{dx} = 2x - 1$

$$\begin{aligned} \frac{dg}{dx} &= 0 \\ 2x - 1 &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned} \tag{5.20}$$

Step 4. y coordinate of turning point is given by calculating $g\left(\frac{1}{2}\right)$.

$$\begin{aligned} g\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 2 \\ &= \frac{1}{4} - \frac{1}{2} + 2 \\ &= \frac{7}{4} \end{aligned} \tag{5.21}$$

The turning point is at $\left(\frac{1}{2}, \frac{7}{4}\right)$

Step 5. **Image not finished**

Figure 5.16

Solution to Exercise 5.2.6 (p. 71)

Step 1. Find the turning points by setting $g'(x) = 0$.
If we use the rules of differentiation we get $g'(x) = -3x^2 + 12x - 9$

$$\begin{aligned} g'(x) &= 0 \\ -3x^2 + 12x - 9 &= 0 \\ x^2 - 4x + 3 &= 0 \\ (x - 3)(x - 1) &= 0 \end{aligned} \tag{5.22}$$

The x-coordinates of the turning points are: $x = 1$ and $x = 3$.
The y-coordinates of the turning points are calculated as:

$$\begin{aligned} g(x) &= -x^3 + 6x^2 - 9x + 4 \\ g(1) &= -(1)^3 + 6(1)^2 - 9(1) + 4 \\ &= -1 + 6 - 9 + 4 \\ &= 0 \end{aligned} \tag{5.23}$$

$$\begin{aligned} g(x) &= -x^3 + 6x^2 - 9x + 4 \\ g(3) &= -(3)^3 + 6(3)^2 - 9(3) + 4 \\ &= -27 + 54 - 27 + 4 \\ &= 4 \end{aligned} \tag{5.24}$$

Therefore the turning points are: $(1, 0)$ and $(3, 4)$.

Step 2. We find the y-intercepts by finding the value for $g(0)$.

$$\begin{array}{rcl} g(x) & = & -x^3+6x^2-9x+4 \\ y_{int}=g(0) & = & -(0)^3+6(0)^2-9(0)+4 \\ & = & 4 \end{array} \quad (5.25)$$

Step 3. We find the x-intercepts by finding the points for which the function $g(x)=0$.
$g(x)=-x^3+6x^2-9x+4$
Use the factor theorem to confirm that $(x-1)$ is a factor. If $g(1)=0$, then $(x-1)$ is a factor.

$$\begin{array}{rcl} g(x) & = & -x^3+6x^2-9x+4 \\ g(1) & = & -(1)^3+6(1)^2-9(1)+4 \\ & = & -1+6-9+4 \\ & = & 0 \end{array} \quad (5.26)$$

Therefore, $(x-1)$ is a factor.
If we divide $g(x)$ by $(x-1)$ we are left with: $-x^2+5x-4$ This has factors $-(x-4)(x-1)$
Therefore: $g(x)=-(x-1)(x-1)(x-4)$
The x-intercepts are: $x_{int}=1,4$

Step 4. 

Figure 5.17

Solution to Exercise 5.3.1 (p. 73)

Step 1. Let the two numbers be a and b. Then we have:
$a+b=10$
We are required to minimise the product of a and b. Call the product P. Then:
$P=a\cdot b$
We can solve for b from to get:
$b=10-a$
Substitute this into to write P in terms of a only.
$P=a(10-a)=10a-a^2$
Step 2. The derivative of is: $P'(a)=10-2a$
Step 3. Set $P'(a)=0$ to find the value of a which makes P a maximum.

$$\begin{array}{rcl} P'(a) & = & 10-2a \\ 0 & = & 10-2a \\ 2a & = & 10 \\ a & = & \frac{10}{2} \\ a & = & 5 \end{array} \quad (5.27)$$

Substitute into to solve for the width.

$$\begin{aligned} b &= 10 - a \\ &= 10 - 5 \\ &= 5 \end{aligned} \quad (5.28)$$

Step 4. The product is maximised if a and b are both equal to 5.

Solution to Exercise 5.3.2 (p. 73)

Step 1. The important pieces of information given are related to the area and modified perimeter of the garden. We know that the area of the garden is: $A = w \cdot l$ We are also told that the fence covers only 3 sides and the three sides should add up to 160 m. This can be written as: $160 = w + l + l$
However, we can use to write w in terms of l: $w = 160 - 2l$ Substitute into to get: $A = (160 - 2l)\, l = 160l - 2l^2$
Step 2. Since we are interested in maximising the area, we differentiate to get: $A'(l) = 160 - 4l$
Step 3. To find the stationary point, we set $A'(l) = 0$ and solve for the value of l that maximises the area.

$$\begin{aligned} A'(l) &= 160 - 4l \\ 0 &= 160 - 4l \\ \therefore 4l &= 160 \\ l &= \tfrac{160}{4} \\ l &= 40m \end{aligned} \quad (5.29)$$

Substitute into to solve for the width.

$$\begin{aligned} w &= 160 - 2l \\ &= 160 - 2\,(40) \\ &= 160 - 80 \\ &= 80m \end{aligned} \quad (5.30)$$

Step 4. A width of 80 m and a length of 40 m will yield the maximal area fenced off.

Solution to Exercise 5.3.3 (p. 74)

Step 1.

$$\begin{aligned} v_{ave} &= \tfrac{h(2)-h(0)}{2-0} \\ &= \tfrac{\left[20(2)-5(2)^2\right]-\left[20(0)-5(0)^2\right]}{2} \\ &= \tfrac{40-20}{2} \\ &= 10\ ms^{-1} \end{aligned} \quad (5.31)$$

Step 2.

$$\begin{aligned} v(t) &= \tfrac{dh}{dt} \\ &= 20 - 10t \end{aligned} \quad (5.32)$$

Velocity after 1,5 seconds:

$$\begin{aligned} v(1,5) &= 20 - 10\,(1,5) \\ &= 5\ ms^{-1} \end{aligned} \quad (5.33)$$

Step 3.

$$\begin{aligned} v(t) &= 0 \\ 20 - 10t &= 0 \\ 10t &= 20 \\ t &= 2 \end{aligned} \quad (5.34)$$

Therefore the velocity is zero after 2 seconds

Step 4. The ball hits the ground when $h(t) = 0$

$$\begin{aligned} 20t - 5t^2 &= 0 \\ 5t(4 - t) &= 0 \\ t = 0 \quad &or \quad t = 4 \end{aligned} \quad (5.35)$$

The ball hits the ground after 4 seconds. The velocity after 4 seconds will be:

$$\begin{aligned} v(4) &= h'(4) \\ &= 20 - 10(4) \\ &= -20\ ms^{-1} \end{aligned} \quad (5.36)$$

The ball hits the gound at a speed of $20ms^{-1}$. Notice that the sign of the velocity is negative which means that the ball is moving downward (the reverse of upward, which is when the velocity is positive).

Step 5.

$$\begin{aligned} a &= v'(t) \\ &= -10\ ms^{-1} \end{aligned} \quad (5.37)$$

Just because gravity is constant does not mean we should think of acceleration as a constant. We should still consider it a function.

Chapter 6

Linear programming

6.1 Introduction, Terminology, The Feasible Region[1]

6.1.1 Introduction

In Grade 11 you were introduced to linear programming and solved problems by looking at points on the edges of the feasible region. In Grade 12 you will look at how to solve linear programming problems in a more general manner.

6.1.2 Terminology

Here is a recap of some of the important concepts in linear programming.

6.1.2.1 Feasible Region and Points

Constraints mean that we cannot just take any x and y when looking for the x and y that optimise our objective function. If we think of the variables x and y as a point (x,y) in the xy-plane then we call the set of all points in the xy-plane that satisfy our constraints the **feasible region**. Any point in the feasible region is called a **feasible point**.

For example, the constraints

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \end{aligned} \quad (6.1)$$

mean that every (x,y) we can consider must lie in the first quadrant of the xy plane. The constraint

$$x \geq y \quad (6.2)$$

means that every (x,y) must lie on or below the line $y=x$ and the constraint

$$x \leq 20 \quad (6.3)$$

means that x must lie on or to the left of the line $x=20$.

We can use these constraints to draw the feasible region as shown by the shaded region in Figure 6.1.

TIP: The constraints are used to create bounds of the solution.

[1]This content is available online at <http://cnx.org/content/m39293/1.1/>.

Image not finished

Figure 6.1: The feasible region corresponding to the constraints $x \geq 0$, $y \geq 0$, $x \geq y$ and $x \leq 20$.

TIP:

$ax + by = c$	If $b \neq 0$, feasible points must lie *on* the line $y = -\frac{a}{b}x + \frac{c}{b}$
	If $b = 0$, feasible points must lie *on* the line $x = c/a$
$ax + by \leq c$	If $b \neq 0$, feasible points must lie *on* or *below* the line $y = -\frac{a}{b}x + \frac{c}{b}$.
	If $b = 0$, feasible points must lie *on* or *to the left of* the line $x = c/a$.

Table 6.1

When a constraint is linear, it means that it requires that any feasible point (x, y) lies on one side of or on a line. Interpreting constraints as graphs in the xy plane is very important since it allows us to construct the feasible region such as in Figure 6.1.

6.1.3 Linear Programming and the Feasible Region

If the objective function and all of the constraints are linear then we call the problem of optimising the objective function subject to these constraints a **linear program**. All optimisation problems we will look at will be linear programs.

The major consequence of the constraints being linear is that *the feasible region is always a polygon.* This is evident since the constraints that define the feasible region all contribute a line segment to its boundary (see Figure 6.1). It is also always true that the feasible region is a convex polygon.

The objective function being linear means that *the feasible point(s) that gives the solution of a linear program always lies on one of the vertices of the feasible region.* This is very important since, as we will soon see, it gives us a way of solving linear programs.

We will now see why the solutions of a linear program always lie on the vertices of the feasible region. Firstly, note that if we think of $f(x, y)$ as lying on the z axis, then the function $f(x, y) = ax + by$ (where a and b are real numbers) is the definition of a plane. If we solve for y in the equation defining the objective function then

$$\begin{aligned} & f(x, y) = ax + by \\ \therefore \quad & y = \frac{-a}{b}x + \frac{f(x,y)}{b} \end{aligned} \quad (6.4)$$

What this means is that if we find all the points where $f(x, y) = c$ for any real number c (i.e. $f(x, y)$ is constant with a value of c), then we have the equation of a line. This line we call a **level line** of the objective function.

Consider again the feasible region described in Figure 6.1. Let's say that we have the objective function $f(x, y) = x - 2y$ with this feasible region. If we consider Equation (6.4) corresponding to

$$f(x, y) = -20 \quad (6.5)$$

then we get the level line

$$y = \frac{1}{2}x + 10 \tag{6.6}$$

which has been drawn in . Level lines corresponding to

$$\begin{array}{rcc} f(x,y) = -10 & \text{or} & y = \frac{x}{2} + 5 \\ f(x,y) = 0 & \text{or} & y = \frac{x}{2} \\ f(x,y) = 10 & \text{or} & y = \frac{x}{2} - 5 \\ f(x,y) = 20 & \text{or} & y = \frac{x}{2} - 10 \end{array} \tag{6.7}$$

have also been drawn in. It is very important to realise that these are not the only level lines; in fact, there are infinitely many of them and they are *all parallel to each other*. Remember that if we look at any one level line $f(x,y)$ has the *same* value for every point (x,y) that lies on that line. Also, $f(x,y)$ will always have different values on different level lines.

Image not finished

Figure 6.2: The feasible region corresponding to the constraints $x \geq 0$, $y \geq 0$, $x \geq y$ and $x \leq 20$ with objective function $f(x,y) = x - 2y$. The dashed lines represent various level lines of $f(x,y)$.

If a ruler is placed on the level line corresponding to $f(x,y) = -20$ in Figure 6.2 and moved down the page parallel to this line then it is clear that the ruler will be moving over level lines which correspond to *larger* values of $f(x,y)$. So if we wanted to maximise $f(x,y)$ then we simply move the ruler down the page until we reach the "lowest" point in the feasible region. This point will then be the feasible point that maximises $f(x,y)$. Similarly, if we wanted to minimise $f(x,y)$ then the "highest" feasible point will give the minimum value of $f(x,y)$.

Since our feasible region is a polygon, these points *will always lie on vertices in the feasible region*. The fact that the value of our objective function along the line of the ruler increases as we move it down and decreases as we move it up depends on this particular example. Some other examples might have that the function increases as we move the ruler up and decreases as we move it down.

It is a general property, though, of linear objective functions that they will consistently increase or decrease as we move the ruler up or down. Knowing which direction to move the ruler in order to maximise/minimise $f(x,y) = ax+by$ is as simple as looking at the sign of b (i.e. "is b negative, positive or zero?"). If b is *positive*, then $f(x,y)$*increases* as we move the ruler up and $f(x,y)$*decreases* as we move the ruler down. The opposite happens for the case when b is negative: $f(x,y)$*decreases* as we move the ruler up and $f(x,y)$*increases* as we move the ruler down. If $b = 0$ then we need to look at the sign of a.

If a is positive then $f(x,y)$ increases as we move the ruler to the right and decreases if we move the ruler to the left. Once again, the opposite happens for a negative. If we look again at the objective function mentioned earlier,

$$f(x,y) = x - 2y \tag{6.8}$$

with $a = 1$ and $b = -2$, then we should find that $f(x,y)$ increases as we move the ruler down the page since $b = -2 < 0$. This is exactly what we found happening in Figure 6.2.

The main points about linear programming we have encountered so far are

- The feasible region is always a polygon.
- Solutions occur at vertices of the feasible region.
- Moving a ruler parallel to the level lines of the objective function up/down to the top/bottom of the feasible region shows us which of the vertices is the solution.
- The direction in which to move the ruler is determined by the sign of b and also possibly by the sign of a.

These points are sufficient to determine a method for solving any linear program.

6.2 Method, Exercises[2]

6.2.1 Method: Linear Programming

If we wish to maximise the objective function $f(x,y)$ then:

1. Find the gradient of the level lines of $f(x,y)$ (this is always going to be $-\frac{a}{b}$ as we saw in Equation)
2. Place your ruler on the xy plane, making a line with gradient $-\frac{a}{b}$ (i.e. b units on the x-axis and $-a$ units on the y-axis)
3. The solution of the linear program is given by appropriately moving the ruler. Firstly we need to check whether b is negative, positive or zero.
   a. If $b > 0$, move the ruler up the page, keeping the ruler parallel to the level lines all the time, until it touches the "highest" point in the feasible region. This point is then the solution.
   b. If $b < 0$, move the ruler in the opposite direction to get the solution at the "lowest" point in the feasible region.
   c. If $b = 0$, check the sign of a
      1. If $a < 0$ move the ruler to the "leftmost" feasible point. This point is then the solution.
      2. If $a > 0$ move the ruler to the "rightmost" feasible point. This point is then the solution.

Exercise 6.2.1: Prizes! *(Solution on p. 91.)*

As part of their opening specials, a furniture store has promised to give away at least 40 prizes with a total value of at least R2 000. The prizes are kettles and toasters.

1. If the company decides that there will be at least 10 of each prize, write down two more inequalities from these constraints.
2. If the cost of manufacturing a kettle is R60 and a toaster is R50, write down an objective function C which can be used to determine the cost to the company of both kettles and toasters.
3. Sketch the graph of the feasibility region that can be used to determine all the possible combinations of kettles and toasters that honour the promises of the company.
4. How many of each prize will represent the cheapest option for the company?
5. How much will this combination of kettles and toasters cost?

Exercise 6.2.2: Search Line Method *(Solution on p. 92.)*

As a production planner at a factory manufacturing lawn cutters your job will be to advise the management on how many of each model should be produced per week in order to maximise the profit on the local production. The factory is producing two types of lawn cutters: Quadrant and Pentagon. Two of the production processes that the lawn cutters must go through are: bodywork and engine work.

- The factory cannot operate for less than 360 hours on engine work for the lawn cutters.

[2]This content is available online at <http://cnx.org/content/m39297/1.1/>.

- The factory has a maximum capacity of 480 hours for bodywork for the lawn cutters.
- Half an hour of engine work and half an hour of bodywork is required to produce one Quadrant.

- The ratio of Pentagon lawn cutters to Quadrant lawn cutters produced per week must be at least 3:2.
- A minimum of 200 Quadrant lawn cutters must be produced per week.

Let the number of Quadrant lawn cutters manufactured in a week be x.
Let the number of Pentagon lawn cutters manufactured in a week be y.
Two of the constraints are:

$$x \geq 200 \tag{6.9}$$

$$3x + 2y \geq 2\ 160 \tag{6.10}$$

1. Write down the remaining constraints in terms of x and y to represent the abovementioned information.
2. Use graph paper to represent the constraints graphically.
3. Clearly indicate the feasible region by shading it.
4. If the profit on one Quadrant lawn cutter is R1 200 and the profit on one Pentagon lawn cutter is R400, write down an equation that will represent the profit on the lawn cutters.
5. Using a search line and your graph, determine the number of Quadrant and Pentagon lawn cutters that will yield a maximum profit.
6. Determine the maximum profit per week.

6.2.2 End of Chapter Exercises

1. Polkadots is a small company that makes two types of cards, type X and type Y. With the available labour and material, the company can make not more than 150 cards of type X and not more than 120 cards of type Y per week. Altogether they cannot make more than 200 cards per week. There is an order for at least 40 type X cards and 10 type Y cards per week. Polkadots makes a profit of R5 for each type X card sold and R10 for each type Y card. Let the number of type X cards be x and the number of type Y cards be y, manufactured per week.
   a. One of the constraint inequalities which represents the restrictions above is $x \leq 150$. Write the other constraint inequalities.
   b. Represent the constraints graphically and shade the feasible region.
   c. Write the equation that represents the profit P (the objective function), in terms of x and y.
   d. On your graph, draw a straight line which will help you to determine how many of each type must be made weekly to produce the maximum P
   e. Calculate the maximum weekly profit.
2. A brickworks produces “face bricks" and “clinkers". Both types of bricks are produced and sold in batches of a thousand. Face bricks are sold at R150 per thousand, and clinkers at R100 per thousand, where an income of at least R9,000 per month is required to cover costs. The brickworks is able to produce at most 40,000 face bricks and 90,000 clinkers per month, and has transport facilities to deliver at most 100,000 bricks per month. The number of clinkers produced must be at least the same number of face bricks produced. Let the number of face bricks *in thousands* be x, and the number of clinkers *in thousands* be y.
   a. List all the constraints.

b. Graph the feasible region.
c. If the sale of face bricks yields a profit of R25 per thousand and clinkers R45 per thousand, use your graph to determine the maximum profit.
d. If the profit margins on face bricks and clinkers are interchanged, use your graph to determine the maximum profit.

3. A small cell phone company makes two types of cell phones: *Easyhear* and *Longtalk*. Production figures are checked weekly. At most, 42 *Easyhear* and 60 *Longtalk* phones can be manufactured each week. At least 30 cell phones must be produced each week to cover costs. In order not to flood the market, the number of *Easyhear* phones cannot be more than twice the number of *Longtalk* phones. It takes $\frac{2}{3}$ hour to assemble an *Easyhear* phone and $\frac{1}{2}$ hour to put together a *Longtalk* phone. The trade unions only allow for a 50-hour week. Let x be the number of *Easyhear* phones and y be the number of *Longtalk* phones manufactured each week.

   a. Two of the constraints are:

   $$0 \leq x \leq 42 \qquad \text{and} \qquad 0 \leq y \leq 60 \tag{6.11}$$

   Write down the other three constraints.
   b. Draw a graph to represent the feasible region
   c. If the profit on an *Easyhear* phone is R225 and the profit on a *Longtalk* is R75, determine the maximum profit per week.

4. *Hair for Africa* is a firm that specialises in making two kinds of up-market shampoo, *Glowhair* and *Longcurls*. They must produce at least two cases of *Glowhair* and one case of *Longcurls* per day to stay in the market. Due to a limited supply of chemicals, they cannot produce more than 8 cases of *Glowhair* and 6 cases of *Longcurls* per day. It takes half-an-hour to produce one case of *Glowhair* and one hour to produce a case of *Longcurls*, and due to restrictions by the unions, the plant may operate for at most 7 hours per day. The workforce at *Hair for Africa*, which is still in training, can only produce a maximum of 10 cases of shampoo per day. Let x be the number of cases of *Glowhair* and y the number of cases of *Longcurls* produced per day.

   a. Write down the inequalities that represent all the constraints.
   b. Sketch the feasible region.
   c. If the profit on a case of *Glowhair* is R400 and the profit on a case of *Longcurls* is R300, determine the maximum profit that *Hair for Africa* can make per day.

5. A transport contracter has 6 5-ton trucks and 8 3-ton trucks. He must deliver at least 120 tons of sand per day to a construction site, but he may not deliver more than 180 tons per day. The 5-ton trucks can each make three trips per day at a cost of R30 per trip, and the 3-ton trucks can each make four trips per day at a cost of R120 per trip. How must the contracter utilise his trucks so that he has minimum expense?

Solutions to Exercises in Chapter 6

Solution to Exercise 6.2.1 (p. 88)

Step 1. Let the number of kettles be x_k and the number of toasters be y_t and write down two constraints apart from $x_k \geq 0$ and $y_t \geq 0$ that must be adhered to.

Step 2. Since there will be at least 10 of each prize we can write:

$$x_k \geq 10 \tag{6.12}$$

and

$$y_t \geq 10 \tag{6.13}$$

Also the store has promised to give away at least 40 prizes in total. Therefore:

$$x_k + y_t \geq 40 \tag{6.14}$$

Step 3. The cost of manufacturing a kettle is R60 and a toaster is R50. Therefore the cost the total cost C is:

$$C = 60x_k + 50y_t \tag{6.15}$$

Step 4.

Figure 6.3

Step 5. From the graph, the coordinates of vertex A are (30,10) and the coordinates of vertex B are (10,30).

Step 6. The seach line is the gradient of the objective function. That is, if the equation $C = 60x + 50y$ is now written in the standard form $y = ...$, then the gradient is:

$$m = -\frac{6}{5}, \tag{6.16}$$

which is shown with the broken line on the graph.

Figure 6.4

Step 7. At vertex A, the cost is:

$$\begin{aligned} C &= 60x_k + 50y_t \\ &= 60(30) + 50(10) \\ &= 1800 + 500 \\ &= 2300 \end{aligned} \tag{6.17}$$

At vertex B, the cost is:

$$\begin{aligned} C &= 60x_k + 50y_t \\ &= 60\,(10) + 50\,(30) \\ &= 600 + 1500 \\ &= 2100 \end{aligned} \quad (6.18)$$

Step 8. The cheapest combination of prizes is 10 kettles and 30 toasters, costing the company R2 100.

Solution to Exercise 6.2.2 (p. 88)

Step 1.

$$\frac{1}{2}x + \frac{1}{5}y \leq 480 \quad (6.19)$$

$$\frac{y}{x} \geq \frac{3}{2} \quad (6.20)$$

Step 2. **Image not finished**

Figure 6.5

Step 3.

$$P = 1\ 200x + 400y \quad (6.21)$$

Step 4. By moving the search line upwards, we see that the point of maximum profit is at (600,900). Therefore

$$P = 1\ 200\,(600) + 400\,(900) \quad (6.22)$$

$$P = R1\ 080\ 000 \quad (6.23)$$

Chapter 7

Geometry

7.1 Introduction, Circle Geometry[1]

7.1.1 Introduction

7.1.1.1 Discussion : Discuss these Research Topics

Research one of the following geometrical ideas and describe it to your group:

1. taxicab geometry,
2. spherical geometry,
3. fractals,
4. the Koch snowflake.

7.1.2 Circle Geometry

7.1.2.1 Terminology

The following is a recap of terms that are regularly used when referring to circles.

arc: An arc is a part of the circumference of a circle.
chord: A chord is defined as a straight line joining the ends of an arc.
radius: The radius, r, is the distance from the centre of the circle to any point on the circumference.
diameter: The diameter is a special chord that passes through the centre of the circle. The diameter is the straight line from a point on the circumference to another point on the circumference, that passes through the centre of the circle.
segment: A segment is the part of the circle that is cut off by a chord. A chord divides a circle into two segments.
tangent: A tangent is a line that makes contact with a circle at one point on the circumference. (AB is a tangent to the circle at point P).

[1]This content is available online at <http://cnx.org/content/m39327/1.1/>.

Image not finished

Figure 7.1: Parts of a Circle

7.1.2.2 Axioms

An axiom is an established or accepted principle. For this section, the following are accepted as axioms.

1. The Theorem of Pythagoras, which states that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. In [U+25B5]ABC, this means that $(AB)^2 + (BC)^2 = (AC)^2$

   Image not finished

   Figure 7.2: A right-angled triangle

2. A tangent is perpendicular to the radius, drawn at the point of contact with the circle.

7.1.2.3 Theorems of the Geometry of Circles

A theorem is a general proposition that is not self-evident but is proved by reasoning (these proofs need not be learned for examination purposes).

Theorem 1 The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord.
Proof:

Image not finished

Figure 7.3

Consider a circle, with centre O. Draw a chord AB and draw a perpendicular line from the centre of the circle to intersect the chord at point P. **The aim is to prove that** $AP = BP$

1. [U+25B5]OAP and [U+25B5]OBP are right-angled triangles.
2. $OA = OB$ as both of these are radii and OP is common to both triangles.

Apply the Theorem of Pythagoras to each triangle, to get:

$$\begin{aligned} OA^2 &= OP^2 + AP^2 \\ OB^2 &= OP^2 + BP^2 \end{aligned} \quad (7.1)$$

However, $OA = OB$. So,

$$\begin{aligned} OP^2 + AP^2 &= OP^2 + BP^2 \\ \therefore AP^2 &= BP^2 \\ \text{and AP} &= BP \end{aligned} \quad (7.2)$$

This means that OP bisects AB.

Theorem 2 The line drawn from the centre of a circle, that bisects a chord, is perpendicular to the chord.

Proof:

Image not finished

Figure 7.4

Consider a circle, with centre O. Draw a chord AB and draw a line from the centre of the circle to bisect the chord at point P. **The aim is to prove that** $OP \perp AB$ In [U+25B5]OAP and [U+25B5]OBP,

1. $AP = PB$ (given)
2. $OA = OB$ (radii)
3. OP is common to both triangles.

$\therefore$ [U+25B5]$OAP \equiv$ [U+25B5]OBP (SSS).

$$\begin{aligned} O\hat{P}A &= O\hat{P}B \\ O\hat{P}A + O\hat{P}B &= 180^\circ \quad \text{(APB is a str. line)} \\ \therefore O\hat{P}A &= O\hat{P}B = 90^\circ \\ \therefore OP &\perp AB \end{aligned} \quad (7.3)$$

Theorem 3 The perpendicular bisector of a chord passes through the centre of the circle.

Proof:

Image not finished

Figure 7.5

Consider a circle. Draw a chord AB. Draw a line PQ perpendicular to AB such that PQ bisects AB at point P. Draw lines AQ and BQ. **The aim is to prove that Q is the centre of the circle, by showing that $AQ = BQ$.** In [U+25B5]OAP and [U+25B5]OBP,

1. $AP = PB$ (given)
2. $\angle QPA = \angle QPB$ ($QP \perp AB$)
3. QP is common to both triangles.

$\therefore$ [U+25B5]$QAP \equiv$ [U+25B5]QBP (SAS). From this, $QA = QB$. Since the centre of a circle is the only point inside a circle that has points on the circumference at an equal distance from it, Q must be the centre of the circle.

7.1.2.3.1 Circles I

1. Find the value of x:

Image not finished

Figure 7.6

Image not finished

Figure 7.7

Theorem 4 The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circumference of the circle.

Proof:

Image not finished

Figure 7.8

Consider a circle, with centre O and with A and B on the circumference. Draw a chord AB. Draw radii OA and OB. Select any point P on the circumference of the circle. Draw lines PA and PB. Draw PO and extend to R. **The aim is to prove that** $\hat{AOB}= 2\cdot \hat{APB}$**.**$\hat{AOR}=\hat{PAO} + \hat{APO}$ (exterior angle = sum of interior opp. angles) But, $\hat{PAO}=\hat{APO}$ ([U+25B5]AOP is an isosceles [U+25B5]) $\therefore \hat{AOR}= 2\ \hat{APO}$ Similarly, $\hat{BOR}= 2\ \hat{BPO}$. So,

$$\begin{aligned} \hat{AOB} &= \hat{AOR} + \hat{BOR} \\ &= 2\ \hat{APO} + 2\ \hat{BPO} \\ &= 2\left(\hat{APO} + \hat{BPO}\right) \\ &= 2\left(\hat{APB}\right) \end{aligned} \tag{7.4}$$

7.1.2.3.2 Circles II

1. Find the angles (a to f) indicated in each diagram:

Figure 7.9

Theorem 5 The angles subtended by a chord at the circumference of a circle on the same side of the chord are equal.

Proof:

Figure 7.10

Consider a circle, with centre O. Draw a chord AB. Select any points P and Q on the circumference of the circle, such that both P and Q are on the same side of the chord. Draw lines PA, PB, QA and QB.

The aim is to prove that $A\hat{Q}B = A\hat{P}B$**.**

$$\begin{array}{rcl} A\hat{O}B & = & 2\,A\hat{Q}B \quad \angle \text{ at centre } = \text{twice} \angle \text{at circumference} \\ \text{and } A\hat{O}B & = & 2\,A\hat{P}B \quad \angle \text{ at centre } = \text{twice} \angle \text{ at circumference} \\ \therefore 2\,A\hat{Q}B & = & 2\,A\hat{P}B \\ \therefore A\hat{Q}B & = & A\hat{P}B \end{array} \tag{7.5}$$

Theorem 6 (Converse of Theorem) If a line segment subtends equal angles at two other points on the same side of the line, then these four points lie on a circle.

Proof:

Figure 7.11

Consider a line segment AB, that subtends equal angles at points P and Q on the same side of AB. **The aim is to prove that points A, B, P and Q lie on the circumference of a circle.** By contradiction. Assume that point P does not lie on a circle drawn through points A, B and Q. Let the circle cut AP (or

AP extended) at point R.

$$\begin{array}{rcl} \hat{AQB} & = & \hat{ARB}\ \angle\text{'s on same side of chord} \\ \text{but } \hat{\text{AQB}} & = & \hat{APB}\ \text{(given)} \\ \therefore \hat{ARB} & = & \hat{APB} \\ \text{but this cannot be true since } \hat{\text{ARB}} & = & \hat{APB} + \hat{RBP}\ \ (\text{ext. } \angle \text{ of [U+25B5]}) \end{array} \tag{7.6}$$

$\therefore$ the assumption that the circle does not pass through P, must be false, and A, B, P and Q lie on the circumference of a circle.

7.1.2.3.3 Circles III

1. Find the values of the unknown letters.

Figure 7.12

7.1.2.3.4 Cyclic Quadrilaterals

Cyclic quadrilaterals are quadrilaterals with all four vertices lying on the circumference of a circle. The vertices of a cyclic quadrilateral are said to be *concyclic*.

Theorem 7 The opposite angles of a cyclic quadrilateral are supplementary.

Proof:

Figure 7.13

Consider a circle, with centre O. Draw a cyclic quadrilateral $ABPQ$. Draw AO and PO. **The aim is to prove that $\hat{ABP} + \hat{AQP} = 180°$ and $\hat{QAB} + \hat{QPB} = 180°$.**

$$\begin{array}{rcl} \hat{O_1} & = & 2\ \hat{ABP}\ \ \angle\text{'s at centre} \\ \hat{O_2} & = & 2\ \hat{AQP}\ \ \angle\text{'s at centre} \\ \text{But, } \hat{O_1} + \hat{O_2} & = & 360° \\ \therefore 2\ \hat{ABP} + 2\ \hat{AQP} & = & 360° \\ \therefore \hat{ABP} + \hat{AQP} & = & 180° \\ \text{Similarly, } \hat{\text{QAB}} + \hat{\text{QPB}} & = & 180° \end{array} \tag{7.7}$$

Theorem 8 (Converse of Theorem) If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Proof:

Figure 7.14

Consider a quadrilateral $ABPQ$, such that $A\hat{B}P + A\hat{Q}P = 180°$ and $Q\hat{A}B + Q\hat{P}B = 180°$. **The aim is to prove that points A, B, P and Q lie on the circumference of a circle.** By contradiction. Assume that point P does not lie on a circle drawn through points A, B and Q. Let the circle cut AP (or AP extended) at point R. Draw BR.

$$\begin{aligned} Q\hat{A}B + Q\hat{R}B &= 180° \text{ opp. } \angle\text{'s } of\ cyclic\ quad \\ \text{but } \mathrm{Q\hat{A}B + Q\hat{P}B} &= 180° \text{ (given)} \\ \therefore Q\hat{R}B &= Q\hat{P}B \\ \text{but this cannot be true since } \mathrm{Q\hat{R}B} &= Q\hat{P}B + R\hat{B}P \text{ (ext.}\angle\text{of [U+25B5])} \end{aligned} \quad (7.8)$$

$\therefore$ the assumption that the circle does not pass through P, must be false, and A, B, P and Q lie on the circumference of a circle and $ABPQ$ is a cyclic quadrilateral.

7.1.2.3.4.1 Circles IV

1. Find the values of the unknown letters.

Figure 7.15

Theorem 9 Two tangents drawn to a circle from the same point outside the circle are equal in length.

Proof:

Figure 7.16

Consider a circle, with centre O. Choose a point P outside the circle. Draw two tangents to the circle from point P, that meet the circle at A and B. Draw lines OA, OB and OP. **The aim is to prove that $AP = BP$.** In [U+25B5]OAP and [U+25B5]OBP,

1. $OA = OB$ (radii)
2. $\angle OAP = \angle OPB = 90°$ ($OA \perp AP$ and $OB \perp BP$)
3. OP is common to both triangles.

[U+25B5]$OAP \equiv$ [U+25B5]OBP (right angle, hypotenuse, side) $\therefore AP = BP$

7.1.2.3.4.2 Circles V

1. Find the value of the unknown lengths.

Figure 7.17

Theorem 10 The angle between a tangent and a chord, drawn at the point of contact of the chord, is equal to the angle which the chord subtends in the alternate segment.
Proof:

Figure 7.18

Consider a circle, with centre O. Draw a chord AB and a tangent SR to the circle at point B. Chord AB subtends angles at points P and Q on the minor and major arcs, respectively. Draw a diameter BT and join A to T. **The aim is to prove that** $A\hat{P}B = A\hat{B}R$ **and** $A\hat{Q}B = A\hat{B}S$**.** First prove that $A\hat{Q}B = A\hat{B}S$ as this result is needed to prove that $\hat{APB}=\hat{ABR}$.

$$\begin{aligned} \hat{ABS}+\hat{ABT} &= 90^\circ \text{ (TB} \perp \text{SR)} \\ \hat{BAT} &= 90^\circ\ (\angle\text{'s at centre}) \\ \therefore \hat{ABT}+\hat{ATB} &= 90^\circ \text{ (sum of angles in [U+25B5] BAT)} \\ \therefore \hat{ABS} &= \hat{ABT} \\ \text{However, } \hat{\text{AQB}} &= \hat{ATB} \text{ (angles subtended by same chord AB)} \\ \therefore \hat{AQB} &= \hat{ABS} \end{aligned} \tag{7.9}$$

$$\begin{aligned} \hat{SBQ}+\hat{QBR} &= 180^\circ \text{ (SBT is a str. line)} \\ \hat{APB}+\hat{AQB} &= 180^\circ \text{ (ABPQ is a cyclic quad)} \\ \therefore \hat{SBQ}+\hat{QBR} &= \hat{APB}+\hat{AQB} \\ \hat{\text{AQB}} &= \hat{ABS} \\ \therefore \hat{APB} &= \hat{ABR} \end{aligned}$$

7.1.2.3.4.3 Circles VI

1. Find the values of the unknown letters.

Image not finished

Figure 7.19

Theorem 11 (Converse of) If the angle formed between a line, that is drawn through the end point of a chord, and the chord, is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

Proof:

Image not finished

Figure 7.20

Consider a circle, with centre O and chord AB. Let line SR pass through point B. Chord AB subtends an angle at point Q such that $\hat{ABS}=\hat{AQB}$. **The aim is to prove that SBR is a tangent to the circle.**

By contradiction. Assume that SBR is not a tangent to the circle and draw XBY such that XBY is a tangent to the circle.

$$\begin{aligned} \hat{ABX} &= \hat{AQB} \quad (\text{tan} - \text{chord theorem}) \\ \text{However,} \quad \hat{\text{ABS}} &= \hat{AQB} \quad (\text{given}) \\ \therefore \hat{ABX} &= \hat{ABS} \\ \text{But,} \quad \hat{\text{ABX}} &= \hat{ABS} + \hat{XBS} \\ \text{can only be true if,} \quad \hat{\text{XBS}} &= 0 \end{aligned} \tag{7.10}$$

If $\hat{XBS}$ is zero, then both XBY and SBR coincide and SBR is a tangent to the circle.

7.1.2.3.4.4 Applying Theorem

1. *Show that Theorem* also applies to the following two cases:

Figure 7.21

Exercise 7.1.1: Circle Geometry I ***(Solution on p. 111.)***

Figure 7.22

BD is a tangent to the circle with centre O. $BO \perp AD$. Prove that:

1. $CFOE$ is a cyclic quadrilateral
2. $FB = BC$
3. [U+25B5]$COE///$[U+25B5]CBF
4. $CD^2 = ED.AD$
5. $\frac{OE}{BC} = \frac{CD}{CO}$

Exercise 7.1.2: Circle Geometry II ***(Solution on p. 112.)***

Figure 7.23

FD is drawn parallel to the tangent CB Prove that:

1. $FADE$ is cyclic
2. [U+25B5]$AFE|||$[U+25B5]CBD
3. $\frac{FC.AG}{GH} = \frac{DC.FE}{BD}$

7.2 Coordinate Geometry, Equation of Tangent, Transformations[2]

7.2.1 Co-ordinate Geometry

7.2.1.1 Equation of a Circle

We know that every point on the circumference of a circle is the same distance away from the centre of the circle. Consider a point (x_1, y_1) on the circumference of a circle of radius r with centre at (x_0, y_0).

Image not finished

Figure 7.24

In Figure 7.24, [U+25B5]OPQ is a right-angled triangle. Therefore, from the Theorem of Pythagoras, we know that:

$$OP^2 = PQ^2 + OQ^2 \tag{7.11}$$

But,

$$\begin{array}{rcl} PQ & = & y_1 - y_0 \\ OQ & = & x_1 - x_0 \\ OP & = & r \\ \therefore \quad r^2 & = & (y_1 - y_0)^2 + (x_1 - x_0)^2 \end{array} \tag{7.12}$$

But, this same relation holds for any point P on the circumference. In fact, the relation holds for all points P on the circumference. Therefore, we can write:

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{7.13}$$

for a circle with centre at (x_0, y_0) and radius r.

For example, the equation of a circle with centre $(0, 0)$ and radius 4 is:

$$\begin{array}{rcl} (y - y_0)^2 + (x - x_0)^2 & = & r^2 \\ (y - 0)^2 + (x - 0)^2 & = & 4^2 \\ y^2 + x^2 & = & 16 \end{array} \tag{7.14}$$

[2]This content is available online at <http://cnx.org/content/m39288/1.1/>.

Khan academy video on circles - 1

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/6r1GQCxyMKI&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 7.25

Exercise 7.2.1: Equation of a Circle I *(Solution on p. 113.)*

Find the equation of a circle (centre O) with a diameter between two points, P at $(-5,5)$ and Q at $(5,-5)$.

Exercise 7.2.2: Equation of a Circle II *(Solution on p. 114.)*

Find the center and radius of the circle
$x^2 - 14x + y^2 + 4y = -28$.

7.2.1.2 Equation of a Tangent to a Circle at a Point on the Circle

We are given that a tangent to a circle is drawn through a point P with co-ordinates (x_1, y_1). In this section, we find out how to determine the equation of that tangent.

Image not finished

Figure 7.26

We start by making a list of what we know:

1. We know that the equation of the circle with centre (x_0, y_0) and radius r is $(x-x_0)^2 + (y-y_0)^2 = r^2$.
2. We know that a tangent is perpendicular to the radius, drawn at the point of contact with the circle.

As we have seen in earlier grades, there are two steps to determining the equation of a straight line:

Step 1. Calculate the gradient of the line, m.
Step 2. Calculate the y-intercept of the line, c.

The same method is used to determine the equation of the tangent. First we need to find the gradient of the tangent. We do this by finding the gradient of the line that passes through the centre of the circle and point P (line f in), because this line is a radius line and the tangent is perpendicular to it.

$$m_f = \frac{y_1 - y_0}{x_1 - x_0} \quad (7.15)$$

The tangent (line g) is perpendicular to this line. Therefore,

$$m_f \times m_g = -1 \quad (7.16)$$

So,

$$m_g = -\frac{1}{m_f} \tag{7.17}$$

Now, we know that the tangent passes through (x_1, y_1) so the equation is given by:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= -\frac{1}{m_f}(x - x_1) \\ y - y_1 &= -\frac{1}{\frac{y_1 - y_0}{x_1 - x_0}}(x - x_1) \\ y - y_1 &= -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1) \end{aligned} \tag{7.18}$$

For example, find the equation of the tangent to the circle at point $(1, 1)$. The centre of the circle is at $(0, 0)$. The equation of the circle is $x^2 + y^2 = 2$.

Use

$$y - y_1 = -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1) \tag{7.19}$$

with $(x_0, y_0) = (0, 0)$ and $(x_1, y_1) = (1, 1)$.

$$\begin{aligned} y - y_1 &= -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1) \\ y - 1 &= -\frac{1-0}{1-0}(x - 1) \\ y - 1 &= -\frac{1}{1}(x - 1) \\ y &= -(x - 1) + 1 \\ y &= -x + 1 + 1 \\ y &= -x + 2 \end{aligned} \tag{7.20}$$

7.2.1.2.1 Co-ordinate Geometry

1. Find the equation of the cicle:
   a. with centre $(0; 5)$ and radius 5
   b. with centre $(2; 0)$ and radius 4
   c. with centre $(5; 7)$ and radius 18
   d. with centre $(-2; 0)$ and radius 6
   e. with centre $(-5; -3)$ and radius $\sqrt{3}$
2. a. Find the equation of the circle with centre $(2; 1)$ which passes through $(4; 1)$.
   b. Where does it cut the line $y = x + 1$?
   c. Draw a sketch to illustrate your answers.
3. a. Find the equation of the circle with center $(-3; -2)$ which passes through $(1; -4)$.
   b. Find the equation of the circle with center $(3; 1)$ which passes through $(2; 5)$.
   c. Find the point where these two circles cut each other.
4. Find the center and radius of the following circles:
   a. $(x - 9)^2 + (y - 6)^2 = 36$
   b. $(x - 2)^2 + (y - 9)^2 = 1$
   c. $(x + 5)^2 + (y + 7)^2 = 12$
   d. $(x + 4)^2 + (y + 4)^2 = 23$
   e. $3(x - 2)^2 + 3(y + 3)^2 = 12$

f. $x^2 - 3x + 9 = y^2 + 5y + 25 = 17$

5. Find the $x-$ and $y-$ intercepts of the following graphs and draw a sketch to illustrate your answer:
   a. $(x+7)^2 + (y-2)^2 = 8$
   b. $x^2 + (y-6)^2 = 100$
   c. $(x+4)^2 + y^2 = 16$
   d. $(x-5)^2 + (y+1)^2 = 25$
6. Find the center and radius of the following circles:
   a. $x^2 + 6x + y^2 - 12y = -20$
   b. $x^2 + 4x + y^2 - 8y = 0$
   c. $x^2 + y^2 + 8y = 7$
   d. $x^2 - 6x + y^2 = 16$
   e. $x^2 - 5x + y^2 + 3y = -\frac{3}{4}$
   f. $x^2 - 6nx + y^2 + 10ny = 9n^2$
7. Find the equations to the tangent to the circle:
   a. $x^2 + y^2 = 17$ at the point $(1;4)$
   b. $x^2 + y^2 = 25$ at the point $(3;4)$
   c. $(x+1)^2 + (y-2)^2 = 25$ at the point $(3;5)$
   d. $(x-2)^2 + (y-1)^2 = 13$ at the point $(5;3)$

7.2.2 Transformations

7.2.2.1 Rotation of a Point about an angle θ

First we will find a formula for the co-ordinates of P after a rotation of θ.

We need to know something about polar co-ordinates and compound angles before we start.

7.2.2.1.1 Polar co-ordinates

Figure 7.27

Notice that	:	$sin\alpha = \frac{y}{r} \therefore y = rsin\alpha$
and		$cos\alpha = \frac{x}{r} \therefore x = rcos\alpha$

Table 7.1

so P can be expressed in two ways:

1. $P(x;y)$ rectangular co-ordinates
2. $P(rcos\alpha; rsin\alpha)$ polar co-ordinates.

7.2.2.1.2 Compound angles

(See derivation of formulae in Ch. 12)

$$\begin{aligned} sin\left(\alpha+\beta\right) &= sin\alpha cos\beta + sin\beta cos\alpha \\ cos\left(\alpha+\beta\right) &= cos\alpha cos\beta - sin\alpha sin\beta \end{aligned} \quad (7.21)$$

7.2.2.1.3 Now consider P' after a rotation of θ

$$\begin{gathered} P\left(x;y\right) = P\left(rcos\alpha;rsin\alpha\right) \\ P'\left(rcos\left(\alpha+\theta\right);rsin\left(\alpha+\theta\right)\right) \end{gathered} \quad (7.22)$$

Expand the co-ordinates of P'

$$\begin{aligned} x-\text{co}-\text{ordinate of} P' &= rcos\left(\alpha+\theta\right) \\ &= r\left[cos\alpha cos\theta - sin\alpha sin\theta\right] \\ &= rcos\alpha cos\theta - rsin\alpha sin\theta \\ &= xcos\theta - ysin\theta \end{aligned} \quad (7.23)$$

$$\begin{aligned} y-\text{co}-\text{ordinate of P'} &= rsin\left(\alpha+\theta\right) \\ &= r\left[sin\alpha cos\theta + sin\theta cos\alpha\right] \\ &= rsin\alpha cos\theta + rcos\alpha sin\theta \\ &= ycos\theta + xsin\theta \end{aligned} \quad (7.24)$$

Figure 7.28

which gives the formula $P' = \left[\left(xcos\theta - ysin\theta;ycos\theta + xsin\theta\right)\right]$.
So to find the co-ordinates of $P\left(1;\sqrt{3}\right)$ after a rotation of 45°, we arrive at:

$$\begin{aligned} P' &= \left[\left(xcos\theta - ysin\theta\right);\left(ycos\theta + xsin\theta\right)\right] \\ &= \left[\left(1cos45° - \sqrt{3}sin45°\right);\left(\sqrt{3}cos45° + 1sin45°\right)\right] \\ &= \left[\left(\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}}\right);\left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right] \\ &= \left(\frac{1-\sqrt{3}}{\sqrt{2}};\frac{\sqrt{3}+1}{\sqrt{2}}\right) \end{aligned} \quad (7.25)$$

7.2.2.1.3.1 Rotations

Any line OP is drawn (not necessarily in the first quadrant), making an angle of θ degrees with the x-axis. Using the co-ordinates of P and the angle α, calculate the co-ordinates of P', if the line OP is rotated about the origin through α degrees.

	P	α
1.	(2, 6)	60°
2.	(4, 2)	30°
3.	(5, -1)	45°
4.	(-3, 2)	120°
5.	(-4, -1)	225°
6.	(2, 5)	-150°

Table 7.2

Image not finished

Figure 7.29

7.2.2.2 Characteristics of Transformations

Rigid transformations like translations, reflections, rotations and glide reflections preserve shape and size, and that enlargement preserves shape but not size.

7.2.2.2.1 Geometric Transformations:

Draw a large 15×15 grid and plot [U+25B5]ABC with $A\,(2;6)$, $B\,(5;6)$ and $C\,(5;1)$. Fill in the lines $y=x$ and $y=-x$. Complete the table below , by drawing the images of [U+25B5]ABC under the given transformations. The first one has been done for you.

Image not finished

Figure 7.30

	Description			
Transformation	(translation, reflection,	Co-ordinates	Lengths	Angles
	rotation, enlargement)			
		$A^{'}(2;-6)$	$A^{'}B^{'}=3$	$\hat{B}^{'}=90°$
$(x;y)\rightarrow(x;-y)$	reflection about the x-axis	$B^{'}(5;-6)$	$B^{'}C^{'}=4$	$tan\ \hat{A}=4/3$
		$C^{'}(5;-2)$	$A^{'}C^{'}=5$	$\therefore\hat{A}=53°,\hat{C}=37°$
$(x;y)\rightarrow(x+1;y-2)$				
$(x;y)\rightarrow(-x;y)$				
$(x;y)\rightarrow(-y;x)$				
$(x;y)\rightarrow(-x;-y)$				
$(x;y)\rightarrow(2x;2y)$				
$(x;y)\rightarrow(y;x)$				
$(x;y)\rightarrow(y;x+1)$				

Table 7.3

A transformation that leaves lengths and angles unchanged is called a rigid transformation. Which of the above transformations are rigid?

7.2.3 Exercises

1. ΔABC undergoes several transformations forming $\Delta A^{'}B^{'}C^{'}$. Describe the relationship between the angles and sides of ΔABC and $\Delta A^{'}B^{'}C^{'}$ (e.g., they are twice as large, the same, etc.)

Transformation	Sides	Angles	Area
Reflect			
Reduce by a scale factor of 3			
Rotate by 90°			
Translate 4 units right			
Enlarge by a scale factor of 2			

Table 7.4

2. ΔDEF has $\hat{E}= 30°$, $DE = 4\,\mathrm{cm}$, $EF = 5\,\mathrm{cm}$. ΔDEF is enlarged by a scale factor of 6 to form $\Delta D'E'F'$.
   a. Solve ΔDEF
   b. Hence, solve $\Delta D'E'F'$
3. ΔXYZ has an area of $6\,\mathrm{cm}^2$. Find the area of $\Delta X'Y'Z'$ if the points have been transformed as follows:
   a. $(x, y) \to (x + 2; y + 3)$
   b. $(x, y) \to (y; x)$
   c. $(x, y) \to (4x; y)$
   d. $(x, y) \to (3x; y + 2)$
   e. $(x, y) \to (-x; -y)$
   f. $(x, y) \to (x; -y + 3)$
   g. $(x, y) \to (4x; 4y)$
   h. $(x, y) \to (-3x; 4y)$

Solutions to Exercises in Chapter 7

Solution to Exercise 7.1.1 (p. 102)

Step 1.

$$\begin{aligned} \hat{FOE} &= 90° \text{ (BO} \perp \text{OD)} \\ \hat{FCE} &= 90° \text{ (}\angle \text{ subtended by diameter AE)} \\ &\therefore CFOE \text{ is a cyclic quad (opposite } \angle\text{'s } \textit{supplementary}) \end{aligned} \quad (7.26)$$

Step 2. Let $\hat{OEC}= x$.

$$\begin{aligned} &\therefore \hat{FCB}= x \text{ (}\angle \text{ between tangentBD and chord CE)} \\ &\therefore \hat{BFC}= x \text{ (exterior } \angle \text{ to cyclic quad CFOE)} \\ &\therefore BF = BC \text{ (sides opposite equal } \angle\text{'s } \textit{in isosceles} \text{ [U+25B5] BFC)} \end{aligned} \quad (7.27)$$

Step 3.

$$\begin{aligned} \hat{CBF} &= 180° - 2x \text{ (sum of } \angle\text{'s } \textit{in} \text{ [U+25B5] BFC)} \\ OC &= OE \text{ (radii of circle } O) \\ &\therefore \hat{ECO}= x \text{ (isosceles [U+25B5] COE)} \\ &\therefore \hat{COE}= 180° - 2x \text{ (sum of } \angle\text{'s } \textit{in} \text{ [U+25B5] COE)} \end{aligned} \quad (7.28)$$

- $\hat{COE}=\hat{CBF}$
- $\hat{ECO}=\hat{FCB}$
- $\hat{OEC}=\hat{CFB}$

$$\therefore \text{[U+25B5]} COE ||| \text{[U+25B5]} CBF \text{ (3 } \angle\text{'s } \textit{equal}) \quad (7.29)$$

Step 4Step a. In [U+25B5] EDC

$$\begin{aligned} \hat{CED} &= 180° - x \text{ (}\angle\text{'s } \textit{on a str. line}\text{AD)} \\ \hat{ECD} &= 90° - x \text{ (complementary} \angle\text{'s)} \end{aligned} \quad (7.30)$$

Step b. In [U+25B5] ADC

$$\begin{aligned} \hat{ACE} &= 180° - x \left(\text{sum of } \angle\text{'s } \hat{\text{ACE}} \text{ and } \hat{\text{ECO}} \right) \\ \hat{CAD} &= 90° - x \text{ (sum of } \angle\text{'s } \textit{in} \text{ [U+25B5] CAE)} \end{aligned} \quad (7.31)$$

Step c. Lastly, $\hat{ADC}=\hat{EDC}$ since they are the same $\angle$.

Step d.

$$\therefore \quad [U+25B5]ADC|||[U+25B5]CDE \ (3\angle\text{'s } equal)$$
$$\therefore \quad \frac{ED}{CD} = \frac{CD}{AD} \qquad (7.32)$$
$$\therefore \quad CD^2 = ED.AD$$

Step 3Step a.

$$OE = CD \ ([U+25B5]\text{OEC is isosceles}) \qquad (7.33)$$

Step b. In [U+25B5]BCO

$$\hat{OCB} = 90° \ (\text{radius OC on tangent BD})$$
$$\hat{CBO} = 180° - 2x \ (\text{sum of } \angle\text{'s } in \ [U+25B5]\text{BFC}) \qquad (7.34)$$

Step c. In [U+25B5]OCD

$$\hat{OCD} = 90° \ (\text{radius OC on tangent BD})$$
$$\hat{COD} = 180° - 2x \ (\text{sum of } \angle\text{'s } in \ [U+25B5]\text{OCE}) \qquad (7.35)$$

Step d. Lastly, OC is a common side to both [U+25B5]'s.
Step e.

$$\therefore \quad [U+25B5]BOC|||[U+25B5]ODC \ (\text{common side and 2 equal angles})$$
$$\therefore \quad \frac{CO}{BC} = \frac{CD}{CO} \qquad (7.36)$$
$$\therefore \quad \frac{OE}{BC} = \frac{CD}{CO} \ (\text{OE} = \text{CD isosceles } [U+25B5]\text{OEC})$$

Solution to Exercise 7.1.2 (p. 102)

Step 1. Let $\angle BCD = x$

$$\therefore \quad \angle CAH = x \ (\angle \text{ between tangent BC and chord CE})$$
$$\therefore \quad \angle FDC = x \ (\text{alternate}\angle, \text{FD} \parallel \text{CB}) \qquad (7.37)$$
$$\therefore \quad \text{FADE is a cyclic quad (chord FE subtends equal } \angle\text{'}s)$$

Step 2Step a. Let $\angle FEA = y$

$$\therefore \quad \angle FDA = y \ (\angle\text{'}s \ subtended \ by \ same \ chord\text{AF in cyclic quad FADE})$$
$$\therefore \quad \angle CBD = y \ (\text{corresponding } \angle\text{'}s, \text{FD} \parallel \text{CB}) \qquad (7.38)$$
$$\therefore \quad \angle FEA = \angle CBD$$

Step b.

$$\angle BCD = \angle FAE \ (\text{above}) \qquad (7.39)$$

Step c.

$$\angle AFE = 180° - x - y \ (\angle\text{'}s \ in \ [U+25B5]\text{AFE})$$
$$\angle CBD = 180° - x - y \ (\angle\text{'}s \ in \ [U+25B5]\text{CBD}) \qquad (7.40)$$
$$\therefore \quad [U+25B5]AFE|||\ [U+25B5]CBD \ (3\angle\text{'}s \ equal)$$

Step 3Step a.

$$\begin{aligned} \frac{DC}{BD} &= \frac{FA}{FE} \\ \therefore \quad \frac{DC.FE}{BD} &= FA \end{aligned} \tag{7.41}$$

Step b.

$$\begin{aligned} \frac{AG}{GH} &= \frac{FA}{FC} \text{ (FG} \parallel \text{CH splits up lines AH and AC proportionally)} \\ \therefore \quad & FA = \frac{FC.AG}{GH} \end{aligned} \tag{7.42}$$

Step c.

$$\therefore \quad \frac{FC.AG}{GH} = \frac{DC.FE}{BD} \tag{7.43}$$

Solution to Exercise 7.2.1 (p. 104)

Step 1. Draw a picture of the situation to help you figure out what needs to be done.

Image not finished

Figure 7.31

Step 2. We know that the centre of a circle lies on the midpoint of a diameter. Therefore the co-ordinates of the centre of the circle is found by finding the midpoint of the line between P and Q. Let the co-ordinates of the centre of the circle be (x_0, y_0), let the co-ordinates of P be (x_1, y_1) and let the co-ordinates of Q be (x_2, y_2). Then, the co-ordinates of the midpoint are:

$$\begin{aligned} x_0 &= \frac{x_1+x_2}{2} \\ &= \frac{-5+5}{2} \\ &= 0 \\ y_0 &= \frac{y_1+y_2}{2} \\ &= \frac{5+(-5)}{2} \\ &= 0 \end{aligned} \tag{7.44}$$

The centre point of line PQ and therefore the centre of the circle is at $(0,0)$.

Step 3. If P and Q are two points on a diameter, then the radius is half the distance between them. The distance between the two points is:

$$\begin{aligned} r = \frac{1}{2}PQ &= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \frac{1}{2}\sqrt{(5-(-5))^2+(-5-5)^2} \\ &= \frac{1}{2}\sqrt{(10)^2+(-10)^2} \\ &= \frac{1}{2}\sqrt{100+100} \\ &= \sqrt{\frac{200}{4}} \\ &= \sqrt{50} \end{aligned} \tag{7.45}$$

Step 4.

$$x^2+y^2=50 \tag{7.46}$$

Solution to Exercise 7.2.2 (p. 104)

Step 1. We need to rewrite the equation in the form $(x-x_0)+(y-y_0)=r^2$
To do this we need to complete the square
i.e. add and subtract $\left(\frac{1}{2}\right.$ cooefficient of x^2 and $\left(\frac{1}{2}\right.$ cooefficient of y^2

Step 2.

$$\begin{aligned} x^2-14x+y^2+4y &= -28 \\ \therefore\ x^2-14x+(7)^2-(7)^2+y^2+4y+(2)^2-(2)^2 &= -28 \end{aligned} \tag{7.47}$$

Step 3.

$$\therefore\ (x-7)^2-(7)^2+(y+2)^2-(2)^2=-28 \tag{7.48}$$

Step 4.

$$\begin{aligned} \therefore\ (x-7)^2-49+(y+2)^2-4 &= -28 \\ \therefore\ (x-7)^2+(y+2)^2 &= -28+49+4 \\ \therefore\ (x-7)^2+(y+2)^2 &= 25 \end{aligned} \tag{7.49}$$

Step 5.

$$\therefore\ \text{center is } (7;-2) \text{ and the radius is 5 units} \tag{7.50}$$

Chapter 8

Trigonometry

8.1 Compound Identities, Problem Solving Strategies[1]

8.1.1 Compound Angle Identities

8.1.1.1 Derivation of $sin\left(\alpha+\beta\right)$

We have, for any angles α and β, that

$$sin\left(\alpha+\beta\right)=sin\alpha cos\beta+sin\beta cos\alpha \tag{8.1}$$

How do we derive this identity? It is tricky, so follow closely.

Suppose we have the unit circle shown below. The two points $L\left(a,b\right)$ and $K\left(x,y\right)$ are on the circle.

Image not finished

Figure 8.1

We can get the coordinates of L and K in terms of the angles α and β. For the triangle LOK, we have that

$$\begin{array}{lll} sin\beta=\frac{b}{1} & \Rightarrow & b=sin\beta \\ cos\beta=\frac{a}{1} & \Rightarrow & a=cos\beta \end{array} \tag{8.2}$$

Thus the coordinates of L are $(cos\beta;sin\beta)$. In the same way as above, we can see that the coordinates of K are $(cos\alpha;sin\alpha)$. The identity for $cos\left(\alpha-\beta\right)$ is now determined by calculating KL^2 in two ways. Using

[1] This content is available online at <http://cnx.org/content/m39319/1.1/>.

the distance formula (i.e. $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ or $d^2=(x_2-x_1)^2+(y_2-y_1)^2$), we can find KL^2:

$$\begin{aligned} KL^2 &= (cos\alpha - cos\beta)^2 + (sin\alpha - sin\beta)^2 \\ &= cos^2\alpha - 2cos\alpha cos\beta + cos^2\beta + sin^2\alpha - 2sin\alpha sin\beta + sin^2\beta \\ &= \left(cos^2\alpha + sin^2\alpha\right) + \left(cos^2\beta + sin^2\beta\right) - 2cos\alpha cos\beta - 2sin\alpha sin\beta \\ &= 1 + 1 - 2\left(cos\alpha cos\beta + sin\alpha sin\beta\right) \\ &= 2 - 2\left(cos\alpha cos\beta + sin\alpha sin\beta\right) \end{aligned} \tag{8.3}$$

The second way we can determine KL^2 is by using the cosine rule for [U+25B5]KOL:

$$\begin{aligned} KL^2 &= KO^2 + LO^2 - 2\cdot KO\cdot LO\cdot cos\left(\alpha-\beta\right) \\ &= 1^2 + 1^2 - 2\left(1\right)\left(1\right)cos\left(\alpha-\beta\right) \\ &= 2 - 2\cdot cos\left(\alpha-\beta\right) \end{aligned} \tag{8.4}$$

Equating our two values for KL^2, we have

$$\begin{aligned} 2 - 2\cdot cos\left(\alpha-\beta\right) &= 2 - 2\left(cos\alpha cos\beta + sin\alpha\cdot sin\beta\right) \\ \Rightarrow \quad cos\left(\alpha-\beta\right) &= cos\alpha\cdot cos\beta + sin\alpha\cdot sin\beta \end{aligned} \tag{8.5}$$

Now let $\alpha \to 90^\circ - \alpha$. Then

$$\begin{aligned} cos\left(90^\circ-\alpha-\beta\right) &= cos\left(90^\circ-\alpha\right)cos\beta + sin\left(90^\circ-\alpha\right)sin\beta \\ &= sin\alpha\cdot cos\beta + cos\alpha\cdot sin\beta \end{aligned} \tag{8.6}$$

But $cos\left(90^\circ-(\alpha+\beta)\right) = sin\left(\alpha+\beta\right)$. Thus

$$sin\left(\alpha+\beta\right) = sin\alpha\cdot cos\beta + cos\alpha\cdot sin\beta \tag{8.7}$$

8.1.1.2 Derivation of $sin\left(\alpha-\beta\right)$

We can use

$$sin\left(\alpha+\beta\right) = sin\alpha cos\beta + cos\alpha sin\beta \tag{8.8}$$

to show that

$$sin\left(\alpha-\beta\right) = sin\alpha cos\beta - cos\alpha sin\beta \tag{8.9}$$

We know that

$$sin\left(-\theta\right) = -sin\left(\theta\right) \tag{8.10}$$

and

$$cos\left(-\theta\right) = cos\theta \tag{8.11}$$

Therefore,

$$\begin{aligned} sin\left(\alpha-\beta\right) &= sin\left(\alpha+\left(-\beta\right)\right) \\ &= sin\alpha cos\left(-\beta\right) + cos\alpha sin\left(-\beta\right) \\ &= sin\alpha cos\beta - cos\alpha sin\beta \end{aligned} \tag{8.12}$$

8.1.1.3 Derivation of $cos\left(\alpha+\beta\right)$

We can use

$$sin\left(\alpha-\beta\right)=sin\alpha cos\beta-sin\beta cos\alpha \tag{8.13}$$

to show that

$$cos\left(\alpha+\beta\right)=cos\alpha cos\beta-sin\alpha sin\beta \tag{8.14}$$

We know that

$$sin\left(\theta\right)=cos\left(90-\theta\right). \tag{8.15}$$

Therefore,

$$\begin{array}{rcl}cos\left(\alpha+\beta\right) & = & sin\left(90-\left(\alpha+\beta\right)\right)\\ & = & sin\left(\left(90-\alpha\right)-\beta\right)\\ & = & sin\left(90-\alpha\right)cos\beta-sin\beta cos\left(90-\alpha\right)\\ & = & cos\alpha cos\beta-sin\beta sin\alpha\end{array} \tag{8.16}$$

8.1.1.4 Derivation of $cos\left(\alpha-\beta\right)$

We found this identity in our derivation of the $sin\left(\alpha+\beta\right)$ identity. We can also use the fact that

$$sin\left(\alpha+\beta\right)=sin\alpha cos\beta+cos\alpha sin\beta \tag{8.17}$$

to derive that

$$cos\left(\alpha-\beta\right)=cos\alpha cos\beta+sin\alpha sin\beta \tag{8.18}$$

As

$$cos\left(\theta\right)=sin\left(90-\theta\right), \tag{8.19}$$

we have that

$$\begin{array}{rcl}cos\left(\alpha-\beta\right) & = & sin\left(90-\left(\alpha-\beta\right)\right)\\ & = & sin\left(\left(90-\alpha\right)+\beta\right)\\ & = & sin\left(90-\alpha\right)cos\beta+cos\left(90-\alpha\right)sin\beta\\ & = & cos\alpha cos\beta+sin\alpha sin\beta\end{array} \tag{8.20}$$

8.1.1.5 Derivation of $sin2\alpha$

We know that

$$sin\left(\alpha+\beta\right)=sin\alpha cos\beta+cos\alpha sin\beta \tag{8.21}$$

When $\alpha = \beta$, we have that

$$\begin{aligned} sin(2\alpha) = sin(\alpha+\alpha) &= sin\alpha cos\alpha + cos\alpha sin\alpha \\ &= 2sin\alpha cos\alpha \\ &= sin(2\alpha) \end{aligned} \quad (8.22)$$

8.1.1.6 Derivation of $cos2\alpha$

We know that

$$cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta \quad (8.23)$$

When $\alpha = \beta$, we have that

$$\begin{aligned} cos(2\alpha) = cos(\alpha+\alpha) &= cos\alpha cos\alpha - sin\alpha sin\alpha \\ &= cos^2\alpha - sin^2\alpha \\ &= cos(2\alpha) \end{aligned} \quad (8.24)$$

However, we can also write

$$cos2\alpha = 2cos^2\alpha - 1 \quad (8.25)$$

and

$$cos2\alpha = 1 - 2sin^2\alpha \quad (8.26)$$

by using

$$sin^2\alpha + cos^2\alpha = 1. \quad (8.27)$$

8.1.1.6.1 The $cos2\alpha$ Identity

Use

$$sin^2\alpha + cos^2\alpha = 1 \quad (8.28)$$

to show that:

$$cos2\alpha = \begin{cases} 2cos^2\alpha - 1 \\ 1 - 2sin^2\alpha \end{cases} \quad (8.29)$$

8.1.1.7 Problem-solving Strategy for Identities

The most important thing to remember when asked to prove identities is:

TIP: Trigonometric Identities

When proving trigonometric identities, never assume that the left hand side is equal to the right hand side. You need to **show** that both sides are equal.

A suggestion for proving identities: It is usually much easier simplifying the more complex side of an identity to get the simpler side than the other way round.

Exercise 8.1.1: Trigonometric Identities 1 *(Solution on p. 125.)*

Prove that $sin75^\circ = \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$ without using a calculator.

Exercise 8.1.2: Trigonometric Identities 2 *(Solution on p. 125.)*

Deduce a formula for $tan\left(\alpha+\beta\right)$ in terms of $tan\alpha$ and $tan\beta$.

Hint: Use the formulae for $sin\left(\alpha+\beta\right)$ *and* $cos\left(\alpha+\beta\right)$

Exercise 8.1.3: Trigonometric Identities 3 *(Solution on p. 125.)*

Prove that

$$\frac{sin\theta + sin2\theta}{1 + cos\theta + cos2\theta} = tan\theta \tag{8.30}$$

In fact, this identity is not valid for all values of θ. Which values are those?

Exercise 8.1.4: Trigonometric Equations *(Solution on p. 126.)*

Solve the following equation for y without using a calculator:

$$\frac{1 - siny - cos2y}{sin2y - cosy} = -1 \tag{8.31}$$

8.2 Applications of Trig Functions (2D & 3D), Other Geometries[2]

8.2.1 Applications of Trigonometric Functions

8.2.1.1 Problems in Two Dimensions

Exercise 8.2.1: Problem in Two Dimensions *(Solution on p. 126.)*

For the figure below, we are given that $BC = BD = x$.

Show that $BC^2 = 2x^2\left(1 + sin\theta\right)$.

Image not finished

Figure 8.2

[2] This content is available online at <http://cnx.org/content/m39310/1.1/>.

8.2.1.1.1

1. For the diagram on the right,
   a. Find $A\hat{O}C$ in terms of θ.
   b. Find an expression for:
      1. $cos\theta$
      2. $sin\theta$
      3. $sin2\theta$
   c. Using the above, show that $sin2\theta = 2sin\theta cos\theta$.
   d. Now do the same for $cos2\theta$ and $tan\theta$.

Figure 8.3

2. DC is a diameter of circle O with radius r. $CA = r$, $AB = DE$ and $D\hat{O}E = \theta$. Show that $cos\theta = \frac{1}{4}$.

Figure 8.4

3. The figure below shows a cyclic quadrilateral with $\frac{BC}{CD} = \frac{AD}{AB}$.
   a. Show that the area of the cyclic quadrilateral is $DC \cdot DA \cdot sin\ \hat{D}$.
   b. Find expressions for $cos\ \hat{D}$ and $cos\ \hat{B}$ in terms of the quadrilateral sides.
   c. Show that $2CA^2 = CD^2 + DA^2 + AB^2 + BC^2$.
   d. Suppose that $BC = 10$, $CD = 15$, $AD = 4$ and $AB = 6$. Find CA^2.
   e. Find the angle $\hat{D}$ using your expression for $cos\ \hat{D}$. Hence find the area of $ABCD$.

Figure 8.5

8.2.1.2 Problems in 3 dimensions

Exercise 8.2.2: Height of tower *(Solution on p. 126.)*

D is the top of a tower of height h. Its base is at C. The triangle ABC lies on the ground (a horizontal plane). If we have that $BC = b$, $D\hat{B}A = \alpha$, $D\hat{B}C = x$ and $D\hat{C}B = \theta$, show that

$$h = \frac{bsin\alpha sinx}{sin\left(x+\theta\right)} \tag{8.32}$$

Image not finished

Figure 8.6

8.2.1.2.1

1. The line BC represents a tall tower, with C at its foot. Its angle of elevation from D is θ. We are also given that $BA = AD = x$.

Image not finished

Figure 8.7

   a. Find the height of the tower BC in terms of x, $tan\theta$ and $cos2\alpha$.
   b. Find BC if we are given that $x = 140m$, $\alpha = 21°$ and $\theta = 9°$.

8.2.2 Other Geometries

8.2.2.1 Taxicab Geometry

Taxicab geometry, considered by Hermann Minkowski in the 19th century, is a form of geometry in which the usual metric of Euclidean geometry is replaced by a new metric in which the distance between two points is the sum of the (absolute) differences of their coordinates.

8.2.2.2 Manhattan distance

The metric in taxi-cab geometry, is known as the *Manhattan distance*, between two points in an Euclidean space with fixed Cartesian coordinate system as the sum of the lengths of the projections of the line segment between the points onto the coordinate axes.

For example, the Manhattan distance between the point P_1 with coordinates (x_1, y_1) and the point P_2 at (x_2, y_2) is

$$|x_1 - x_2| + |y_1 - y_2| \tag{8.33}$$

Image not finished

Figure 8.8: Manhattan Distance (dotted and solid) compared to Euclidean Distance (dashed). In each case the Manhattan distance is 12 units, while the Euclidean distance is $\sqrt{36}$

The Manhattan distance changes if the coordinate system is rotated, but does not depend on the translation of the coordinate system or its reflection with respect to a coordinate axis.

Manhattan distance is also known as city block distance or taxi-cab distance. It is given these names because it is the shortest distance a car would drive in a city laid out in square blocks.

Taxicab geometry satisfies all of Euclid's axioms except for the side-angle-side axiom, as one can generate two triangles with two sides and the angle between them the same and have them not be congruent. In particular, the parallel postulate holds.

A circle in taxicab geometry consists of those points that are a fixed Manhattan distance from the center. These circles are squares whose sides make a $45°$ angle with the coordinate axes.

The great-circle distance is the shortest distance between any two points on the surface of a sphere measured along a path on the surface of the sphere (as opposed to going through the sphere's interior). Because spherical geometry is rather different from ordinary Euclidean geometry, the equations for distance take on a different form. The distance between two points in Euclidean space is the length of a straight line from one point to the other. On the sphere, however, there are no straight lines. In non-Euclidean geometry, straight lines are replaced with geodesics. Geodesics on the sphere are the great circles (circles on the sphere whose centers are coincident with the center of the sphere). The shape of the Earth more closely resembles a flattened spheroid with extreme values for the radius of curvature, or arcradius, of 6335.437 km at the equator (vertically) and 6399.592 km at the poles, and having an average great-circle radius of 6372.795 km.

8.2.3 Summary of the Trigonomertic Rules and Identities

Pythagorean Identity	Cofunction Identities	Ratio Identities
$cos^2\theta + sin^2\theta = 1$	$sin\,(90^\circ - \theta) = cos\theta$	$tan\theta = \frac{sin\theta}{cos\theta}$
	$cos\,(90^\circ - \theta) = sin\theta$	
Odd/Even Identities	Periodicity Identities	Double angle Identities
$sin\,(-\theta) = -sin\theta$	$sin\,(\theta \pm 360^\circ) = sin\theta$	$sin\,(2\theta) = 2sin\theta cos\theta$
$cos\,(-\theta) = cos\theta$	$cos\,(\theta \pm 360^\circ) = cos\theta$	$cos\,(2\theta) = cos^2\theta - sin^2\theta$
$tan\,(-\theta) = -tan\theta$	$tan\,(\theta \pm 180^\circ) = tan\theta$	$cos\,(2\theta) = 2cos^2\theta - 1$
		$tan\,(2\theta) = \frac{2tan\theta}{1-tan^2\theta}$
Addition/Subtraction Identities	Area Rule	Cosine rule
$sin\,(\theta + \phi) = sin\theta cos\phi + cos\theta sin\phi$	Area $= \frac{1}{2}bcsinA$	$a^2 = b^2 + c^2 - 2bccosA$
$sin\,(\theta - \phi) = sin\theta cos\phi - cos\theta sin\phi$	Area $= \frac{1}{2}absinC$	$b^2 = a^2 + c^2 - 2accosB$
$cos\,(\theta + \phi) = cos\theta cos\phi - sin\theta sin\phi$	$Area = \frac{1}{2}acsinB$	$c^2 = a^2 + b^2 - 2abcosC$
$cos\,(\theta - \phi) = cos\theta cos\phi + sin\theta sin\phi$		
$tan\,(\theta + \phi) = \frac{tan\phi + tan\theta}{1 - tan\theta tan\phi}$		
$tan\,(\theta - \phi) = \frac{tan\phi - tan\theta}{1 + tan\theta tan\phi}$		
Sine Rule		
$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$		

Table 8.1

8.2.4 End of Chapter Exercises

Do the following without using a calculator.

1. Suppose $cos\theta = 0.7$. Find $cos2\theta$ and $cos4\theta$.
2. If $sin\theta = \frac{4}{7}$, again find $cos2\theta$ and $cos4\theta$.

3. Work out the following:
   a. $cos15°$
   b. $cos75°$
   c. $tan105°$
   d. $cos15°$
   e. $cos3°cos42° - sin3°sin42°$
   f. $1 - 2sin^2 (22.5°)$
4. Solve the following equations:
   a. $cos3\theta \cdot cos\theta - sin3\theta \cdot sin\theta = -\frac{1}{2}$
   b. $3sin\theta = 2cos^2\theta$
5. Prove the following identities
   a. $sin^3\theta = \frac{3sin\theta - sin3\theta}{4}$
   b. $cos^2\alpha \left(1 - tan^2\alpha\right) = cos2\alpha$
   c. $4sin\theta \cdot cos\theta \cdot cos2\theta = sin4\theta$
   d. $4cos^3x - 3cosx = cos3x$
   e. $tany = \frac{sin2y}{cos2y+1}$
6. (Challenge question!) If $a + b + c = 180°$, prove that

$$sin^3a + sin^3b + sin^3c = 3cos\left(a/2\right)cos\left(b/2\right)cos\left(c/2\right) + cos\left(3a/2\right)cos\left(3b/2\right)cos\left(3c/2\right) \quad (8.34)$$

Solutions to Exercises in Chapter 8

Solution to Exercise 8.1.1 (p. 119)

Step 1. We only know the exact values of the trig functions for a few special angles (30°, 45°, 60°, etc.). We can see that $75° = 30° + 45°$. Thus we can use our double-angle identity for $sin\,(\alpha + \beta)$ to express $sin75°$ in terms of known trig function values.

Step 2.

$$\begin{array}{rcl} sin75° & = & sin\,(45° + 30°) \\ & = & sin\,(45°)\,cos\,(30°) + cos\,(45°)\,sin\,(30°) \\ & = & \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \\ & = & \frac{\sqrt{3}+1}{2\sqrt{2}} \\ & = & \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ & = & \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4} \end{array} \quad (8.35)$$

Solution to Exercise 8.1.2 (p. 119)

Step 1. We can express $tan\,(\alpha + \beta)$ in terms of cosines and sines, and then use the double-angle formulas for these. We then manipulate the resulting expression in order to get it in terms of $tan\alpha$ and $tan\beta$.

Step 2.

$$\begin{array}{rcl} tan\,(\alpha + \beta) & = & \frac{sin(\alpha+\beta)}{cos(\alpha+\beta)} \\ & = & \frac{sin\alpha \cdot cos\beta + cos\alpha \cdot sin\beta}{cos\alpha \cdot cos\beta - sin\alpha \cdot sin\beta} \\ & = & \frac{\frac{sin\alpha \cdot cos\beta}{cos\alpha \cdot cos\beta} + \frac{cos\alpha \cdot sin\beta}{cos\alpha \cdot cos\beta}}{\frac{cos\alpha \cdot cos\beta}{cos\alpha \cdot cos\beta} - \frac{sin\alpha \cdot sin\beta}{cos\alpha \cdot cos\beta}} \\ & = & \frac{tan\alpha + tan\beta}{1 - tan\alpha \cdot tan\beta} \end{array} \quad (8.36)$$

Solution to Exercise 8.1.3 (p. 119)

Step 1. The right-hand side (RHS) of the identity cannot be simplified. Thus we should try simplify the left-hand side (LHS). We can also notice that the trig function on the RHS does not have a 2θ dependance. Thus we will need to use the double-angle formulas to simplify the $sin2\theta$ and $cos2\theta$ on the LHS. We know that $tan\theta$ is undefined for some angles θ. Thus the identity is also undefined for these θ, and hence is not valid for these angles. Also, for some θ, we might have division by zero in the LHS, which is not allowed. Thus the identity won't hold for these angles also.

Step 2.

$$\begin{array}{rcl} LHS & = & \frac{sin\theta + 2\,sin\theta\,cos\theta}{1 + cos\theta + (2cos^2\theta - 1)} \\ & = & \frac{sin\theta(1+2cos\theta)}{cos\theta(1+2cos\theta)} \\ & = & \frac{sin\theta}{cos\theta} \\ & = & tan\theta \\ & = & RHS \end{array} \quad (8.37)$$

We know that $tan\theta$ is undefined when $\theta = 90° + 180°n$, where n is an integer. The LHS is undefined when $1 + cos\theta + cos2\theta = 0$. Thus we need to solve this equation.

$$\begin{array}{rrcl} & 1 + cos\theta + cos2\theta & = & 0 \\ \Rightarrow & cos\theta\,(1 + 2cos\theta) & = & 0 \end{array} \quad (8.38)$$

The above has solutions when $cos\theta = 0$, which occurs when $\theta = 90° + 180°n$, where n is an integer. These are the same values when $tan\theta$ is undefined. It also has solutions when $1 + 2cos\theta = 0$. This is true when $cos\theta = -\frac{1}{2}$, and thus $\theta = ... - 240°, -120°, 120°, 240°, ...$ To summarise, the identity is not valid when $\theta = ... - 270°, -240°, -120°, -90°, 90°, 120°, 240°, 270°, ...$

Solution to Exercise 8.1.4 (p. 119)

Step 1. Before we are able to solve the equation, we first need to simplify the left-hand side. We do this by using the double-angle formulas.
Step 2.

$$\begin{array}{rrcl} & \frac{1-siny-\left(1-2sin^2y\right)}{2sinycosy-cosy} & = & -1 \\ \Rightarrow & \frac{2sin^2y-siny}{cosy(2siny-1)} & = & -1 \\ \Rightarrow & \frac{siny(2siny-1)}{cosy(2siny-1)} & = & -1 \\ \Rightarrow & tany & = & -1 \\ \Rightarrow & y = 135° + 180°n;\ n \in\ \mathbb{Z} & & \end{array} \quad (8.39)$$

Solution to Exercise 8.2.1 (p. 119)

Step 1. We want CB, and we have CD and BD. If we could get the angle $B\hat{D}C$, then we could use the cosine rule to determine BC. This is possible, as [U+25B5]ABD is a right-angled triangle. We know this from circle geometry, that any triangle circumscribed by a circle with one side going through the origin, is right-angled. As we have two angles of [U+25B5]ABD, we know $A\hat{D}B$ and hence $B\hat{D}C$. Using the cosine rule, we can get BC^2.
Step 2.

$$A\hat{D}B = 180° - \theta - 90° = 90° - \theta \quad (8.40)$$

Thus

$$\begin{array}{rcl} B\hat{D}C & = & 180° - A\hat{D}B \\ & = & 180° - (90° - \theta) \\ & = & 90° + \theta \end{array} \quad (8.41)$$

Now the cosine rule gives

$$\begin{array}{rcl} BC^2 & = & CD^2 + BD^2 - 2 \cdot CD \cdot BD \cdot cos\left(B\hat{D}C\right) \\ & = & x^2 + x^2 - 2 \cdot x^2 \cdot cos\left(90° + \theta\right) \\ & = & 2x^2 + 2x^2\left[sin\left(90°\right)cos\left(\theta\right) + sin\left(\theta\right)cos\left(90°\right)\right] \\ & = & 2x^2 + 2x^2\left[1 \cdot cos\left(\theta\right) + sin\left(\theta\right) \cdot 0\right] \\ & = & 2x^2\left(1 - sin\theta\right) \end{array} \quad (8.42)$$

Solution to Exercise 8.2.2 (p. 120)

Step 1. We have that the triangle ABD is right-angled. Thus we can relate the height h with the angle α and either the length BA or BD (using sines or cosines). But we have two angles and a length for [U+25B5]BCD, and thus can work out all the remaining lengths and angles of this triangle. We can thus work out BD.
Step 2. We have that

$$\begin{array}{rrcl} & \frac{h}{BD} & = & sin\alpha \\ \Rightarrow & h & = & BDsin\alpha \end{array} \quad (8.43)$$

Now we need BD in terms of the given angles and length b. Considering the triangle BCD, we see that we can use the sine rule.

$$\begin{array}{rcl} \frac{sin\theta}{BD} & = & \frac{sin\left(\hat{BDC}\right)}{b} \\ \Rightarrow \quad BD & = & \frac{bsin\theta}{sin\left(\hat{bDC}\right)} \end{array} \tag{8.44}$$

But $D\hat{B}C = 180^\circ - \alpha - \theta$, and

$$\begin{array}{rcl} sin\left(180^\circ - \alpha - \theta\right) & = & -sin\left(-\alpha - \theta\right) \\ & = & sin\left(\alpha + \theta\right) \end{array} \tag{8.45}$$

So

$$\begin{array}{rcl} BD & = & \frac{bsin\theta}{sin\left(D\hat{B}C\right)} \\ & = & \frac{bsin\theta}{sin(\alpha+\theta)} \end{array} \tag{8.46}$$

Chapter 9

Statistics

9.1 Normal Distribution, Sampling, Function Fitting & Regression Analysis[1]

9.1.1 Introduction

In this chapter, you will use the mean, median, mode and standard deviation of a set of data to identify whether the data is normally distributed or whether it is skewed. You will learn more about populations and selecting different kinds of samples in order to avoid bias. You will work with lines of best fit, and learn how to find a regression equation and a correlation coefficient. You will analyse these measures in order to draw conclusions and make predictions.

9.1.2 A Normal Distribution

9.1.2.1 Investigation :

You are given a table of data below.

75	67	70	71	71	73	74	75
80	75	77	78	78	78	78	79
91	81	82	82	83	86	86	87

Table 9.1

1. Calculate the mean, median, mode and standard deviation of the data.
2. What percentage of the data is within one standard deviation of the mean?
3. Draw a histogram of the data using intervals $60 \leq x < 64$, $64 \leq x < 68$, etc.
4. Join the midpoints of the bars to form a frequency polygon.

If large numbers of data are collected from a population, the graph will often have a bell shape. If the data was, say, examination results, a few learners usually get very high marks, a few very low marks and most get a mark in the middle range. We say a distribution is *normal* if

- the mean, median and mode are equal.
- it is symmetric around the mean.

[1]This content is available online at <http://cnx.org/content/m39305/1.1/>.

- $\pm 68\%$ of the sample lies within one standard deviation of the mean, 95% within two standard deviations and 99% within three standard deviations of the mean.

Figure 9.1

What happens if the test was very easy or very difficult? Then the distribution may not be symmetrical. If extremely high or extremely low scores are added to a distribution, then the mean tends to shift towards these scores and the curve becomes skewed.

If the test was very difficult, the mean score is shifted to the left. In this case, we say the distribution is *positively skewed*, or *skewed right*. If it was very easy, then many learners would get high scores, and the mean of the distribution would be shifted to the right. We say the distribution is *negatively skewed*, or *skewed left*.

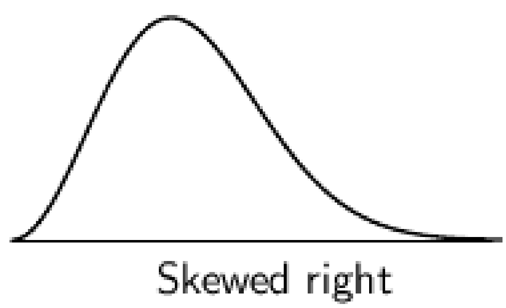

Figure 9.2

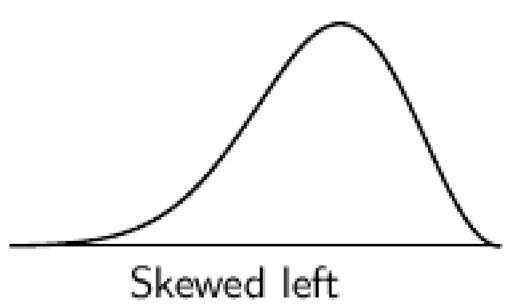

Figure 9.3

9.1.2.2 Normal Distribution

1. Given the pairs of normal curves below, sketch the graphs on the same set of axes and show any relation between them. An important point to remember is that the area beneath the curve corresponds to 100%.
   a. Mean = 8, standard deviation = 4 and Mean = 4, standard deviation = 8
   b. Mean = 8, standard deviation = 4 and Mean = 16, standard deviation = 4
   c. Mean = 8, standard deviation = 4 and Mean = 8, standard deviation = 8
2. After a class test, the following scores were recorded:

Test Score	**Frequency**
3	1
4	7
5	14
6	21
7	14
8	6
9	1
Total	64
Mean	6
Standard Deviation	1,2

Table 9.2

   a. Draw the histogram of the results.
   b. Join the midpoints of each bar and draw a frequency polygon.
   c. What mark must one obtain in order to be in the top 2% of the class?
   d. Approximately 84% of the pupils passed the test. What was the pass mark?
   e. Is the distribution normal or skewed?
3. In a road safety study, the speed of 175 cars was monitored along a specific stretch of highway in order to find out whether there existed any link between high speed and the large number of accidents along the route. A frequency table of the results is drawn up below.

Speed (km.h^{-1})	**Number of cars (Frequency)**
50	19
60	28
70	23
80	56
90	20
100	16
110	8
120	5

Table 9.3

The mean speed was determined to be around 82 km.h^{-1} while the median speed was worked out to be around 84,5 km.h^{-1}.

a. Draw a frequency polygon to visualise the data in the table above.
b. Is this distribution symmetrical or skewed left or right? Give a reason fro your answer.

9.1.3 Extracting a Sample Population

Suppose you are trying to find out what percentage of South Africa's population owns a car. One way of doing this might be to send questionnaires to peoples homes, asking them whether they own a car. However, you quickly run into a problem: you cannot hope to send every person in the country a questionnaire, it would be far to expensive. Also, not everyone would reply. The best you can do is send it to a few people, see what percentage of these own a car, and then use this to estimate what percentage of the entire country own cars. This smaller group of people is called the *sample population*.

The sample population must be carefully chosen, in order to avoid biased results. How do we do this?

First, it must be *representative*. If all of our sample population comes from a very rich area, then almost all will have cars. But we obviously cannot conclude from this that almost everyone in the country has a car! We need to send the questionnaire to rich as well as poor people.

Secondly, the *size* of the sample population must be large enough. It is no good having a sample population consisting of only two people, for example. Both may very well not have cars. But we obviously cannot conclude that no one in the country has a car! The larger the sample population size, the more likely it is that the statistics of our sample population corresponds to the statistics of the entire population.

So how does one ensure that ones sample is representative? There are a variety of methods available, which we will look at now.

1. **Random Sampling.** Every person in the country has an equal chance of being selected. It is unbiased and also independant, which means that the selection of one person has no effect on the selection on another. One way of doing this would be to give each person in the country a number, and then ask a computer to give us a list of random numbers. We could then send the questionnaire to the people corresponding to the random numbers.
2. **Systematic Sampling.** Again give every person in the country a number, and then, for example, select every hundredth person on the list. So person with number 1 would be selected, person with number 100 would be selected, person with number 200 would be selected, etc.
3. **Stratified Sampling.** We consider different subgroups of the population, and take random samples from these. For example, we can divide the population into male and female, different ages, or into different income ranges.
4. **Cluster Sampling.** Here the sample is concentrated in one area. For example, we consider all the people living in one urban area.

9.1.3.1 Sampling

1. Discuss the advantages, disadvantages and possible bias when using
   a. systematic sampling
   b. random sampling
   c. cluster sampling
2. Suggest a suitable sampling method that could be used to obtain information on:
   a. passengers views on availability of a local taxi service.
   b. views of learners on school meals.
   c. defects in an item made in a factory.

d. medical costs of employees in a large company.

3. 2% of a certain magazines' subscribers is randomly selected. The random number 16 out of 50, is selected. Then subscribers with numbers 16, 66, 116, 166, ... are chosen as a sample. What kind of sampling is this?

9.1.4 Function Fitting and Regression Analysis

In Grade 11 we recorded two sets of data (bivariate data) on a scatter plot and then we drew a line of best fit as close to as many of the data items as possible. Regression analysis is a method of finding out exactly which function best fits a given set of data. We can find out the equation of the regression line by drawing and estimating, or by using an algebraic method called "the least squares method", available on most scientific calculators. The linear regression equation is written $\hat{y} = a + bx$ (we say y-hat) or $y = A + Bx$. Of course these are both variations of a more familiar equation $y = mx + c$.

Suppose you are doing an experiment with washing dishes. You count how many dishes you begin with, and then find out how long it takes to finish washing them. So you plot the data on a graph of time taken versus number of dishes. This is plotted below.

Image not finished

Figure 9.4

If t is the time taken, and d the number of dishes, then it looks as though t is proportional to d, ie. $t = m \cdot d$, where m is the constant of proportionality. There are two questions that interest us now.

1. How do we find m? One way you have already learnt, is to draw a line of best-fit through the data points, and then measure the gradient of the line. But this is not terribly precise. Is there a better way of doing it?
2. How well does our line of best fit really fit our data? If the points on our plot don't all lie close to the line of best fit, but are scattered everywhere, then the fit is not 'good', and our assumption that $t = m \cdot d$ might be incorrect. Can we find a quantitative measure of how well our line really fits the data?

In this chapter, we answer both of these questions, using the techniques of *regression analysis*.

Phet simulation for curve fitting

This media object is a Flash object. Please view or download it at
<curve-fitting.swf>

Figure 9.5

Exercise 9.1.1: Fitting by hand ***(Solution on p. 141.)***

Use the data given to draw a scatter plot and line of best fit. Now write down the equation of the line that best seems to fit the data.

x	1,0	2,4	3,1	4,9	5,6	6,2
y	2,5	2,8	3,0	4,8	5,1	5,3

Table 9.4

9.2 Least Squares, Calculator Work, Correlation Coefficients[2]

9.2.1 The Method of Least Squares

We now come to a more accurate method of finding the line of best-fit. The method is very simple. Suppose we guess a line of best-fit. Then at at every data point, we find the distance between the data point and the line. If the line fitted the data perfectly, this distance should be zero for all the data points. The worse the fit, the larger the differences. We then square each of these distances, and add them all together.

Figure 9.6

The best-fit line is then the line that minimises the sum of the squared distances.

Suppose we have a data set of n points $\{(x_1;y_1),(x_2;y_2),...,(x_n,y_n)\}$. We also have a line $f(x)=mx+c$ that we are trying to fit to the data. The distance between the first data point and the line, for example, is

$$\text{distance} = y_1 - f(x) = y_1 - (mx+c) \tag{9.1}$$

We now square each of these distances and add them together. Lets call this sum $S(m,c)$. Then we have that

$$\begin{array}{rcl} S(m,c) & = & (y_1-f(x_1))^2+(y_2-f(x_2))^2+...+(y_n-f(x_n))^2 \\ & = & \sum_{i=1}^{n}(y_i-f(x_i))^2 \end{array} \tag{9.2}$$

Thus our problem is to find the value of m and c such that $S(m,c)$ is minimised. Let us call these minimising values m_0 and c_0. Then the line of best-fit is $f(x)=m_0x+c_0$. We can find m_0 and c_0 using calculus, but it is tricky, and we will just give you the result, which is that

$$\begin{array}{rcl} m_0 & = & \frac{n\sum_{i=1}^{n}x_iy_i-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}y_i}{n\sum_{i=1}^{n}(x_i)^2-\left(\sum_{i=1}^{n}x_i\right)^2} \\ c_0 & = & \frac{1}{n}\sum_{i=1}^{n}y_i-\frac{m_0}{n}\sum_{i=0}^{n}x_i=\overline{y}-m_0\overline{x} \end{array} \tag{9.3}$$

[2]This content is available online at <http://cnx.org/content/m39324/1.1/>.

Khan academy video on regression - 1

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/GAmzwIkGFgE&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 9.7

Exercise 9.2.1: Method of Least Squares ***(Solution on p. 141.)***
In the table below, we have the records of the maintenance costs in Rands, compared with the age of the appliance in months. We have data for 5 appliances.

appliance	1	2	3	4	5
age (x)	5	10	15	20	30
cost (y)	90	140	250	300	380

Table 9.5

Find the line of best fit using the method of least squares.

9.2.2 Using a calculator

Exercise 9.2.2: Using the Sharp EL-531VH calculator ***(Solution on p. 141.)***
Find a regression equation for the following data:

Days (x)	1	2	3	4	5
Growth in m (y)	1,00	2,50	2,75	3,00	3,50

Table 9.6

Exercise 9.2.3: Using the CASIO fx-82ES Natural Display calculator ***(Solution on p. 142.)***
Using a calculator determine the least squares line of best fit for the following data set of marks.

Learner	1	2	3	4	5
Chemistry (%)	52	55	86	71	45
Accounting (%)	48	64	95	79	50

Table 9.7

For a Chemistry mark of 65%, what mark does the least squares line predict for Accounting?

9.2.2.1

1. The table below lists the exam results for 5 students in the subjects of Science and Biology.

Learner	1	2	3	4	5
Science %	55	66	74	92	47
Biology %	48	59	68	84	53

Table 9.8

   a. Use the formulae to find the regression equation coefficients a and b.
   b. Draw a scatter plot of the data on graph paper.
   c. Now use algebra to find a more accurate equation.

2. Footlengths and heights of 7 students are given in the table below.

Height (cm)	170	163	131	181	146	134	166
Footlength (cm)	27	23	20	28	22	20	24

Table 9.9

   a. Draw a scatter plot of the data on graph paper.
   b. Indentify and describe any trends shown in the scatter plot.
   c. Find the equation of the least squares line by using algebraic methods and draw the line on your graph.
   d. Use your equation to predict the height of a student with footlength 21,6 cm.
   e. Use your equation to predict the footlength of a student 176 cm tall.

3. Repeat the data in question 2 and find the regression line using a calculator

9.2.3 Correlation coefficients

Once we have applied regression analysis to a set of data, we would like to have a number that tells us exactly how well the data fits the function. A correlation coefficient, r, is a tool that tells us to what degree there is a relationship between two sets of data. The correlation coefficient $r \in [-1;1]$ when $r=-1$, there is a perfect negative relationship, when $r=0$, there is no relationship and $r=1$ is a perfect positive correlation.

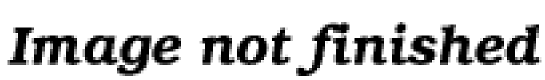

Figure 9.8

Positive, strong	Positive, strong
$r \approx 0,9$	$r \approx 0,7$

Table 9.10

We often use the correlation coefficient r^2 in order to examine the strength of the correlation only. In this case:

$r^2 = 0$	no correlation
$0 < r^2 < 0{,}25$	very weak
$0{,}25 < r^2 < 0{,}5$	weak
$0{,}5 < r^2 < 0{,}75$	moderate
$0{,}75 < r^2 < 0{,}9$	strong
$0{,}9 < r^2 < 1$	very strong
$r^2 = 1$	perfect correlation

Table 9.11

The correlation coefficient r can be calculated using the formula

$$r = \frac{1}{n-1} \sum \left(\frac{x - \overline{x}}{s_x} \right) \left(\frac{y - \overline{y}}{s_y} \right) \tag{9.4}$$

- where n is the number of data points,
- s_x is the standard deviation of the x-values and
- s_y is the standard deviation of the y-values.

This is known as the Pearson's product moment correlation coefficient. It is a long calculation and much easier to do on the calculator where you simply follow the procedure for the regression equation, and go on to find r.

9.2.4 Exercises

1. Below is a list of data concerning 12 countries and their respective carbon dioxide (CO_2) emmission levels per person and the gross domestic product (GDP - a measure of products produced and services delivered within a country in a year) per person.

	CO_2 emmissions per capita (x)	GDP per capita (y)
South Africa	8,1	3 938
Thailand	2,5	2 712
Italy	7,3	20 943
Australia	17,0	23 893
China	2,5	816
India	0,9	463
Canada	16,0	22 537
United Kingdom	9,0	21 785
United States	19,9	31 806
Saudi Arabia	11,0	6 853
Iran	3,8	1 493
Indonesia	1,2	986

Table 9.12

a. Draw a scatter plot of the data set and your estimate of a line of best fit.
b. Calculate equation of the line of regression using the method of least squares.
c. Draw the regression line equation onto the graph.
d. Calculate the correlation coefficient r.
e. What conclusion can you reach, regarding the relationship between CO_2 emission and GDP per capita for the countries in the data set?

2. A collection of data on the peculiar investigation into a foot size and height of students was recorded in the table below. Answer the questions to follow.

Length of right foot (cm)	Height (cm)
25,5	163,3
26,1	164,9
23,7	165,5
26,4	173,7
27,5	174,4
24	156
22,6	155,3
27,1	169,3

Table 9.13

a. Draw a scatter plot of the data set and your estimate of a line of best fit.
b. Calculate equation of the line of regression using the method of least squares or your calculator.
c. Draw the regression line equation onto the graph.
d. Calculate the correlation coefficient r.

e. What conclusion can you reach, regarding the relationship between the length of the right foot and height of the students in the data set?

3. A class wrote two tests, and the marks for each were recorded in the table below. Full marks in the first test was 50, and the second test was out of 30.
   a. Is there a strong association between the marks for the first and second test? Show why or why not.
   b. One of the learners (in row 18) did not write the second test. Given their mark for the first test, calculate an expected mark for the second test.

Learner	Test 1	Test 2
	(Full marks: 50)	(Full marks: 30)
1	42	25
2	32	19
3	31	20
4	42	26
5	35	23
6	23	14
7	43	24
8	23	12
9	24	14
10	15	10
11	19	11
12	13	10
13	36	22
14	29	17
15	29	17
16	25	16
17	29	18
18	17	
19	30	19
20	28	17

Table 9.14

4. A fast food company produces hamburgers. The number of hamburgers made, and the costs are recorded over a week.

Hamburgers made	Costs
495	R2382
550	R2442
515	R2484
500	R2400
480	R2370
530	R2448
585	R2805

Table 9.15

a. Find the linear regression function that best fits the data.
b. If the total cost in a day is R2500, estimate the number of hamburgers produced.
c. What is the cost of 490 hamburgers?

5. The profits of a new shop are recorded over the first 6 months. The owner wants to predict his future sales. The profits so far have been R90 000 , R93 000, R99 500, R102 000, R101 300, R109 000.
   a. For the profit data, calculate the linear regression function.
   b. Give an estimate of the profits for the next two months.
   c. The owner wants a profit of R130 000. Estimate how many months this will take.
6. A company produces sweets using a machine which runs for a few hours per day. The number of hours running the machine and the number of sweets produced are recorded.

Machine hours	Sweets produced
3,80	275
4,23	287
4,37	291
4,10	281
4,17	286

Table 9.16

Find the linear regression equation for the data, and estimate the machine hours needed to make 300 sweets.

Solutions to Exercises in Chapter 9

Solution to Exercise 9.1.1 (p. 133)

Step 1. The first step is to draw the graph. This is shown below.

Figure 9.13

Step 2. The equation of the line is

$$y = mx + c \tag{9.5}$$

From the graph we have drawn, we estimate the y-intercept to be 1,5. We estimate that $y = 3,5$ when $x = 3$. So we have that points $(3;3,5)$ and $(0;1,5)$ lie on the line. The gradient of the line, m, is given by

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3,5 - 1,5}{3 - 0} \\ &= \frac{2}{3} \end{aligned} \tag{9.6}$$

So we finally have that the equation of the line of best fit is

$$y = \frac{2}{3}x + 1,5 \tag{9.7}$$

Solution to Exercise 9.2.1 (p. 135)

Step 1.

appliance	x	y	xy	x^2
1	5	90	450	25
2	10	140	1400	100
3	15	250	3750	225
4	20	300	6000	400
5	30	380	11400	900
Total	80	1160	23000	1650

Table 9.17

Step 2.

$$\begin{aligned} m_0 &= \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} = \frac{5\times 23000 - 80\times 1160}{5\times 1650 - 80^2} = 12 \\ c_0 &= \overline{y} - b\overline{x} = \frac{1160}{5} - \frac{12\times 80}{5} = 40 \\ \therefore & \quad \underline{\hat{y} = 40 + 12x} \end{aligned} \tag{9.8}$$

Solution to Exercise 9.2.2 (p. 135)

Step 1. Using your calculator, change the mode from normal to "Stat xy". This mode enables you to type in bivariate data.

Step 2. Key in the data as follows:

1	(x, y)	1	DATA	$n = 1$
2	(x, y)	2,5	DATA	$n = 2$
3	(x, y)	2,75	DATA	$n = 3$
4	(x, y)	3,0	DATA	$n = 4$
5	(x, y)	3,5	DATA	$n = 5$

Table 9.18

Step 3. Ask for the values of the regression coefficients a and b.

RCL	a	gives	$a = 0{,}9$
RCL	b	gives	$b = 0{,}55$

Table 9.19

$$\therefore \hat{y} = 0,9 + 0,55x \tag{9.9}$$

Solution to Exercise 9.2.3 (p. 135)

Step 1. Switch on the calculator. Press [MODE] and then select STAT by pressing [2]. The following screen will appear:

1	1-VAR	2	A + BX
3	_ + CX^2	4	ln X
5	e ^ X	6	A . B ^ X
7	A . X ^ B	8	1/X

Table 9.20

Now press [2] for linear regression. Your screen should look something like this:

		x			y	
1						
2						
3						

Table 9.21

Step 2. Press [52] and then [=] to enter the first mark under x. Then enter the other values, in the same way, for the x-variable (the Chemistry marks) in the order in which they are given in the data set. Then move the cursor across and up and enter 48 under y opposite 52 in the x-column. Continue to enter the other y-values (the Accounting marks) in order so that they pair off correctly with the corresponding x-values.

		x			y	
1		52				
2		55				
3						

Table 9.22

Then press [AC]. The screen clears but the data remains stored.

1:	Type	2:	Data
3:	Edit	4:	Sum
5:	Var	6:	MinMax
7:	Reg		

Table 9.23

Now press [SHIFT][1] to get the stats computations screen shown below. Choose Regression by pressing [7].

1:	A	2:	B
3:	r	4:	$\hat{x}$
5:	$\hat{y}$		

Table 9.24

Step 3.
a. Press [1] and [=] to get the value of the y-intercept, $a = -5,065.. = -5,07$(to 2 d.p.) Finally, to get the slope, use the following key sequence: [SHIFT][1][7][2][=]. The calculator gives $b = 1,169.. = 1,17$(to 2 d.p.) The equation of the line of regression is thus: $\hat{y} = -5,07 + 1,17x$
b. Press [AC][65][SHIFT][1][7][5][=] This gives a (predicted) Accounting mark of $\hat{} = 70,94 = 71\%$

Chapter 10

Combinations and permutations

10.1 Introduction and Notation[1]

10.1.1 Introduction

Mathematics education began with counting. In the beginning, fingers, beans and buttons were used to help with counting, but these are only practical for small numbers. What happens when a large number of items must be counted?

This chapter focuses on how to use mathematical techniques to count combinations of items.

10.1.2 Counting

An important aspect of probability theory is the ability to determine the total number of possible outcomes when multiple events are considered.

For example, what is the total number of possible outcomes when a die is rolled and then a coin is tossed? The roll of a die has six possible outcomes (1, 2, 3, 4, 5 or 6) and the toss of a coin, 2 outcomes (head or tails). Counting the possible outcomes can be tedious.

10.1.2.1 Making a List

The simplest method of counting the total number of outcomes is by making a list:

1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T

or drawing up a table:

[1]This content is available online at <http://cnx.org/content/m39100/1.1/>.

die	coin
1	H
1	T
2	H
2	T
3	H
3	T
4	H
4	T
5	H
5	T
6	H
6	T

Table 10.1

Both these methods result in 12 possible outcomes, but both these methods have a lot of repetition. Maybe there is a smarter way to write down the result?

10.1.2.2 Tree Diagrams

One method of eliminating some of the repetition is to use *tree diagrams*. Tree diagrams are a graphical method of listing all possible combinations of events from a random experiment.

Image not finished

Figure 10.1: Example of a tree diagram. Each possible outcome is a branch of the tree.

10.1.3 Notation

10.1.3.1 The Factorial Notation

For an integer n, the notation $n!$ (read n factorial) represents:

$$n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1 \tag{10.1}$$

with the following definition: $0! = 1$.

The factorial notation will be used often in this chapter.

10.2 Fundamental Counting Principle and Combinations[2]

10.2.1 The Fundamental Counting Principle

The use of lists, tables and tree diagrams is only feasible for events with a few outcomes. When the number of outcomes grows, it is not practical to list the different possibilities and the fundamental counting principle is used.

The **fundamental counting principle** describes how to determine the total number of outcomes of a series of events.

Suppose that two experiments take place. The first experiment has n_1 possible outcomes, and the second has n_2 possible outcomes. Therefore, the first experiment, followed by the second experiment, will have a total of $n_1 \times n_2$ possible outcomes. This idea can be generalised to m experiments as the total number of outcomes for m experiments is:

$$n_1 \times n_2 \times n_3 \times ... \times n_m = \prod_{i=1}^{m} n_i \tag{10.2}$$

$\prod$ is the multiplication equivalent of $\sum$.

Note: the order in which the experiments are done does not affect the total number of possible outcomes.

Exercise 10.2.1: Lunch Special ***(Solution on p. 153.)***

A take-away has a 4-piece lunch special which consists of a sandwich, soup, dessert and drink for R25.00. They offer the following choices for :

Sandwich: chicken mayonnaise, cheese and tomato, tuna, and ham and lettuce

Soup: tomato, chicken noodle, vegetable

Dessert: ice-cream, piece of cake

Drink: tea, coffee, coke, Fanta and Sprite.

How many possible meals are there?

10.2.2 Combinations

The fundamental counting principle describes how to calculate the total number of outcomes when multiple independent events are performed together.

A more complex problem is determining how many combinations there are of selecting a group of objects from a set. Mathematically, a *combination* is defined as an un-ordered collection of unique elements, or more formally, a subset of a set. For example, suppose you have fifty-two playing cards, and select five cards. The five cards would form a combination and would be a subset of the set of 52 cards.

In a set, the order of the elements in the set does not matter. These are represented usually with curly braces. For example $\{2,4,6\}$ is a subset of the set $\{1,2,3,4,5,6\}$. Since the order of the elements does not matter, only the specific elements are of interest. Therefore,

$$\{2,4,6\} = \{6,4,2\} \tag{10.3}$$

and $\{1,1,1\}$ is the same as $\{1\}$ because in a set the elements don't usually appear more than once.

So in summary we can say the following: Given S, the set of all possible unique elements, a combination is a subset of the elements of S. The order of the elements in a combination is not important (two lists with the same elements in different orders are considered to be the same combination). Also, the elements cannot be repeated in a combination (every element appears once).

[2]This content is available online at <http://cnx.org/content/m39106/1.1/>.

10.2.2.1 Counting Combinations

Calculating the number of ways that certain patterns can be formed is the beginning of *combinatorics*, the study of combinations. Let S be a set with n objects. Combinations of r objects from this set S are subsets of S having r elements each (where the order of listing the elements does not distinguish two subsets).

10.2.2.1.1 Combination without Repetition

When the order does not matter, but each object can be chosen only once, the number of combinations is:

$$\frac{n!}{r!\,(n-r)!} = \binom{n}{r} \tag{10.4}$$

where n is the number of objects from which you can choose and r is the number to be chosen.

For example, if you have 10 numbers and wish to choose 5 you would have 10!/(5!(10 - 5)!) = 252 ways to choose.

For example how many possible 5 card hands are there in a deck of cards with 52 cards?

52! / (5!(52-5)!) = 2 598 960 combinations

10.2.2.1.2 Combination with Repetition

When the order does not matter and an object can be chosen more than once, then the number of combinations is:

$$\frac{(n+r-1)!}{r!\,(n-1)!} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1} \tag{10.5}$$

where n is the number of objects from which you can choose and r is the number to be chosen.

For example, if you have ten types of donuts to choose from and you want three donuts there are (10 + 3 - 1)! / 3!(10 - 1)! = 220 ways to choose.

Note that in this video permutations are mentioned, you will cover permutations in the next section.

Khan academy video on probability - 1

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/bCxMhncR7PU&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 10.2

10.2.2.2 Combinatorics and Probability

Combinatorics is quite useful in the computation of probabilities of events, as it can be used to determine exactly how many outcomes are possible in a given experiment.

Exercise 10.2.2: Probability ***(Solution on p. 153.)***

At a school, learners each play 2 sports. They can choose from netball, basketball, soccer, athletics, swimming, or tennis. What is the probability that a learner plays soccer and either netball, basketball or tennis?

10.3 Permutations and Applications[3]

10.3.1 Permutations

The concept of a combination did not consider the order of the elements of the subset to be important. A permutation is a combination with the order of a selection from a group being important. For example, for the set $\{1, 2, 3, 4, 5, 6\}$, the combination $\{1, 2, 3\}$ would be identical to the combination $\{3, 2, 1\}$, but these two combinations are different permutations, because the elements in the set are ordered differently.

More formally, a permutation is an ordered list without repetitions, perhaps missing some elements.

This means that $\{1, 2, 2, 3, 4, 5, 6\}$ and $\{1, 2, 4, 5, 5, 6\}$ are not permutations of the set $\{1, 2, 3, 4, 5, 6\}$.

Now suppose you have these objects:

1, 2, 3

Here is a list of all permutations of all three objects:

1 2 3; 1 3 2; 2 1 3; 2 3 1; 3 1 2; 3 2 1.

10.3.1.1 Counting Permutations

Let S be a set with n objects. Permutations of r objects from this set S refer to sequences of r different elements of S (where two sequences are considered different if they contain the same elements but in a different order). Formulas for the number of permutations and combinations are readily available and important throughout combinatorics.

It is easy to count the number of permutations of size r when chosen from a set of size n (with $r \leq n$).

1. Select the first member of the permutation out of n choices, because there are n distinct elements in the set.
2. Next, since one of the n elements has already been used, the second member of the permutation has $(n-1)$ elements to choose from the remaining set.
3. The third member of the permutation can be filled in $(n-2)$ ways since 2 have been used already.
4. This pattern continues until there are r members on the permutation. This means that the last member can be filled in $(n-(r-1)) = (n-r+1)$ ways.
5. Summarizing, we find that there is a total of

$$n\,(n-1)\,(n-2)\,...\,(n-r+1) \tag{10.6}$$

different permutations of r objects, taken from a pool of n objects. This number is denoted by $P\,(n, r)$ and can be written in factorial notation as:

$$P\,(n, r) = \frac{n!}{(n-r)!}. \tag{10.7}$$

For example, if we have a total of 5 elements, the integers $\{1, 2, 3, 4, 5\}$, how many ways are there for a permutation of three elements to be selected from this set? In this case, $n = 5$ and $r = 3$. Then, $P\,(5, 3) = 5!/7! = 60!$.

Khan academy video on probability - 2

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/XqQTXW7XfYA&rel=0&hl=en_US&feature=player_embedded&version=3>

Figure 10.3

[3]This content is available online at <http://cnx.org/content/m39103/1.1/>.

Exercise 10.3.1: Permutations *(Solution on p. 153.)*

Show that a collection of n objects has $n!$ permutations.

10.3.1.1.1 Permutation with Repetition

When order matters and an object can be chosen more than once then the number of permutations is:

$$n^r \tag{10.8}$$

where n is the number of objects from which you can choose and r is the number to be chosen.

For example, if you have the letters A, B, C, and D and you wish to discover the number of ways of arranging them in three letter patterns (trigrams) you find that there are 4^3 or 64 ways. This is because for the first slot you can choose any of the four values, for the second slot you can choose any of the four, and for the final slot you can choose any of the four letters. Multiplying them together gives the total.

10.3.2 Applications

10.3.2.1 The Binomial Theorem

In mathematics, the binomial theorem is an important formula giving the expansion of powers of sums. Its simplest version reads

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{10.9}$$

Whenever n is a positive integer, the numbers

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \tag{10.10}$$

are the binomial coefficients (the coefficients in front of the powers).

For example, here are the cases n = 2, n = 3 and n = 4:

$$\begin{aligned} (x+y)^2 &= x^2 + \mathbf{2}xy + y^2 \\ (x+y)^3 &= x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + y^3 \\ (x+y)^4 &= x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + y^4 \end{aligned} \tag{10.11}$$

The coefficients form a triangle, where each number is the sum of the two numbers above it:

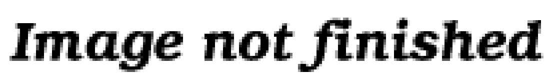

Figure 10.4

This formula, and the triangular arrangement of the binomial coefficients, are often attributed to Blaise Pascal who described them in the 17th century. It was, however, known to the Chinese mathematician Yang Hui in the 13th century, the earlier Persian mathematician Omar KhayyÃ¡m in the 11th century, and the even earlier Indian mathematician Pingala in the 3rd century BC.

Exercise 10.3.2: Number Plates *(Solution on p. 153.)*

The number plate on a car consists of any 3 letters of the alphabet (excluding the vowels and 'Q'), followed by any 3 digits (0 to 9). For a car chosen at random, what is the probability that the number plate starts with a 'Y' and ends with an odd digit?

Exercise 10.3.3: Factorial *(Solution on p. 153.)*

Show that

$$\frac{n!}{(n-1)!} = n \tag{10.12}$$

10.3.3 Exercises

1. Tshepo and Sally go to a restaurant, where the menu is:

Starter	Main Course	Dessert
Chicken wings	Beef burger	Chocolate ice cream
Mushroom soup	Chicken burger	Strawberry ice cream
Greek salad	Chicken curry	Apple crumble
	Lamb curry	Chocolate mousse
	Vegetable lasagne	

Table 10.2

   a. How many different combinations (of starter, main course, and dessert) can Tshepo have?
   b. Sally doesn't like chicken. How many different combinations can she have?

2. Four coins are thrown, and the outcomes recorded. How many different ways are there of getting three heads? First write out the possibilities, and then use the formula for combinations.
3. The answers in a multiple choice test can be A, B, C, D, or E. In a test of 12 questions, how many different ways are there of answering the test?
4. A girl has 4 dresses, 2 necklaces, and 3 handbags.
   a. How many different choices of outfit (dress, necklace and handbag) does she have?
   b. She now buys two pairs of shoes. How many choices of outfit (dress, necklace, handbag and shoes) does she now have?
5. In a soccer tournament of 9 teams, every team plays every other team.
   a. How many matches are there in the tournament?
   b. If there are 5 boys' teams and 4 girls' teams, what is the probability that the first match will be played between 2 girls' teams?
6. The letters of the word 'BLUE' are rearranged randomly. How many new words (a word is any combination of letters) can be made?
7. The letters of the word 'CHEMISTRY' are arranged randomly to form a new word. What is the probability that the word will start and end with a vowel?
8. There are 2 History classes, 5 Accounting classes, and 4 Mathematics classes at school. Luke wants to do all three subjects. How many possible combinations of classes are there?
9. A school netball team has 8 members. How many ways are there to choose a captain, vice-captain, and reserve?

10. A class has 15 boys and 10 girls. A debating team of 4 boys and 6 girls must be chosen. How many ways can this be done?
11. A secret pin number is 3 characters long, and can use any digit (0 to 9) or any letter of the alphabet. Repeated characters are allowed. How many possible combinations are there?

Solutions to Exercises in Chapter 10

Solution to Exercise 10.2.1 (p. 147)

Step 1. There are 4 parts: sandwich, soup, dessert and drink.

Step 2.

Meal component	Sandwich	Soup	Dessert	Drink
Number of choices	4	3	2	5

Table 10.3

Step 3.

$$4 \times 3 \times 2 \times 5 = 120 \quad (10.13)$$

So there are 120 possible meals.

Solution to Exercise 10.2.2 (p. 148)

Step 1. We count the events: soccer and netball, soccer and basketball, soccer and tennis. This gives three choices.

Step 2. There are 6 sports to choose from and we choose 2 sports. There are $\binom{6}{2} = 6!/\left(2!\left(6-2\right)!\right) = 15$ choices.

Step 3. The probability is the number of events we are counting, divided by the total number of choices. Probability $= \frac{3}{15} = \frac{1}{5} = 0{,}2$

Solution to Exercise 10.3.1 (p. 150)

Step 1. Proof: Constructing an ordered sequence of n objects is equivalent to choosing the position occupied by the first object, then choosing the position of the second object, and so on, until we have chosen the position of each of our n objects.

Step 2. There are n ways to choose a position for the first object. Once its position is fixed, we can choose from $(n-1)$ possible positions for the second object. With the first two placed, there are $(n-2)$ remaining possible positions for the third object; and so on. There are only two positions to choose from for the penultimate object, and the nth object will occupy the last remaining position.

Step 3. Therefore, according to the fundamental counting principle, there are

$$n\left(n-1\right)\left(n-2\right)...2 \times 1 = n! \quad (10.14)$$

ways of constructing an ordered sequence of n objects.

Solution to Exercise 10.3.2 (p. 150)

Step 1. The number plate starts with a 'Y', so there is only 1 choice for the first letter, and ends with an even digit, so there are 5 choices for the last digit (1, 3, 5, 7, 9).

Step 2. Use the counting principle. For each of the other letters, there are 20 possible choices (26 in the alphabet, minus 5 vowels and 'Q') and 10 possible choices for each of the other digits.
Number of events $= 1 \times 20 \times 20 \times 10 \times 10 \times 5 = 200\ 000$

Step 3. Use the counting principle. This time, the first letter and last digit can be anything.
Total number of choices $= 20 \times 20 \times 20 \times 10 \times 10 \times 10 = 8\ 000\ 000$

Step 4. The probability is the number of events we are counting, divided by the total number of choices. Probability $= \frac{200\ 000}{8\ 000\ 000} = \frac{1}{40} = 0,025$

Solution to Exercise 10.3.3 (p. 151)

Step 2.

Step 1. **Method 1:** Expand the factorial notation.

$$\frac{n!}{(n-1)!} = \frac{n \times (n-1) \times (n-2) \times ... \times 2 \times 1}{(n-1) \times (n-2) \times ... \times 2 \times 1} \tag{10.15}$$

Cancelling the common factor of $(n-1) \times (n-2) \times ... \times 2 \times 1$ on the top and bottom leaves n.
So $\frac{n!}{(n-1)!} = n$

Method 2: We know that $P(n,r) = \frac{n!}{(n-r)!}$ is the number of permutations of r objects, taken from a pool of n objects. In this case, $r = 1$. To choose 1 object from n objects, there are n choices.
So $\frac{n!}{(n-1)!} = n$

Glossary

A **Arithmetic Sequence**

An *arithmetic* (or *linear*) *sequence* is a sequence of numbers in which each new term is calculated by **adding** a constant value to the previous term

Arithmetic Sequence

An *arithmetic* (or *linear*) *sequence* is a sequence of numbers in which each new term is calculated by adding a constant value to the previous term:

$$a_n = a_{n-1} + d \qquad (1.3)$$

where

- a_n represents the new term, the n^{th}-term, that is calculated;
- a_{n-1} represents the previous term, the $(n-1)^{th}$-term;
- d represents some constant.

D **Derivative**

The derivative of a function $f(x)$ is written as $f'(x)$ and is defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (5.6)$$

F **Factor Theorem**

For any polynomial, $f(x)$, for all values of a which satisfy $f(a) = 0$, $(x - a)$ is a factor of $f(x)$. Or, more concisely:

$$f(x) = (x - a)\, q(x) \qquad (3.1)$$

is a polynomial.

In other words: If the remainder when dividing $f(x)$ by $(x - a)$ is zero, then $(x - a)$ is a factor of $f(x)$.

So if $f\left(-\frac{b}{a}\right) = 0$, then $(ax + b)$ is a factor of $f(x)$.

G **Geometric Sequences**

A geometric sequence is a sequence in which every number in the sequence is equal to the previous number in the sequence, **multiplied** by a constant number.

L **Logarithms**

If $a^n = x$, then: $log_a(x) = n$, where $a > 0$; $a \neq 1$ and $x > 0$.

Index of Keywords and Terms

Keywords are listed by the section with that keyword (page numbers are in parentheses). Keywords do not necessarily appear in the text of the page. They are merely associated with that section. *Ex.* apples, § 1.1 (1) **Terms** are referenced by the page they appear on. *Ex.* apples, 1

Attributions

Collection: *Siyavula textbooks: Grade 12 Maths*
Edited by: Free High School Science Texts Project
URL: http://cnx.org/content/col11242/1.2/

Module: "Logarithms"
By: Rory Adams, Free High School Science Texts Project, Mark Horner, Heather Williams
URL: http://cnx.org/content/m31883/1.4/
Pages: 1-10

Module: "Sequences and series: Arithmetic & Geometric Sequences, Recursive Formulae (Grade 12)"
Used here as: "Arithmetic & Geometric Sequences, Recursive Formulae"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39302/1.1/
Pages: 11-16

Module: "Sequences and series: Sigma notation, Finite & Infinite Series (Grade 12)"
Used here as: "Sigma notation, Finite & Infinite Series"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39301/1.1/
Pages: 17-28

Module: "Finance: Introduction, Sequences & Series, Future Value of Payments (Grade 12)"
Used here as: "Introduction, Sequences & Series, Future Value of Payments"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39196/1.1/
Pages: 29-35

Module: "Finance: Investments & Loans, Loan Schedules, Capital Outstanding, Formulae (Grade 12)"
Used here as: "Investments & Loans, Loan Schedules, Capital Outstanding, Formulae"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39194/1.1/
Pages: 35-40

Module: "Factorising Cubic Polynomials: Introduction, Factor Theorem, Factorising Cubic Polynomials"
Used here as: "Introduction, Factor Theorem, Factorising Cubic Polynomials"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39279/1.1/
Pages: 45-47

Module: "Factorising Cubic Polynomials: Factor Theorem Exercises, Solving Cubic Equations"
Used here as: "Factor Theorem Exercises, Solving Cubic Equations"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39276/1.1/
Pages: 47-48
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

Module: "Functions and Graphs: Introduction, Definition, Notation (Grade 12)"
Used here as: "Introduction, Definition, Notation"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39286/1.1/
Pages: 53-54
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

Module: "Functions and Graphs: Graphs of Inverse Functions (Grade 12)"
Used here as: "Graphs of Inverse Functions"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39282/1.1/
Pages: 54-60
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

Module: "Differential Calculus: Introduction, Limits, Average Gradient (Grade 12)"
Used here as: "Introduction, Limits, Average Gradient"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39270/1.1/
Pages: 61-67
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

Module: "Differential Calculus: Differentiation (First Principles, Rules) and Sketching Graphs (Grade 12)"
Used here as: "Differentiation (First Principles, Rules) and Sketching Graphs"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39313/1.1/
Pages: 67-72
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

Module: "Differential Calculus: Solving Problems (Grade 12)"
Used here as: "Solving Problems"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39273/1.1/
Pages: 72-76
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

Module: "Linear Programming: Introduction, Terminology, The Feasible Region (Grade 12)"
Used here as: "Introduction, Terminology, The Feasible Region"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39293/1.1/
Pages: 85-88

Module: "Linear Programming: Method, Exercises (Grade 12)"
Used here as: "Method, Exercises"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39297/1.1/
Pages: 88-90

Module: "Geometry: Introduction, Circle Geometry (Grade 12)"
Used here as: "Introduction, Circle Geometry"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39327/1.1/
Pages: 93-103

Module: "Geometry: Coordinate Geometry, Equation of Tangent, Transformations (Grade 12)"
Used here as: "Coordinate Geometry, Equation of Tangent, Transformations"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39288/1.1/
Pages: 103-110

Module: "Trigonometry: Compound Identities, Problem Solving Strategies (Grade 12)"
Used here as: "Compound Identities, Problem Solving Strategies"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39319/1.1/
Pages: 115-119

Module: "Trigonometry: Applications of Trig Functions (2D & 3D), Other Geometries (Grade 12)"
Used here as: "Applications of Trig Functions (2D & 3D), Other Geometries"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39310/1.1/
Pages: 119-124

Module: "Statistics: Normal Distribution, Sampling, Function Fitting & Regression Analysis (Grade 12)"
Used here as: "Normal Distribution, Sampling, Function Fitting & Regression Analysis"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39305/1.1/
Pages: 129-134
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

Module: "Statistics: Least Squares, Calculator Work, Correlation Coefficients"
Used here as: "Least Squares, Calculator Work, Correlation Coefficients"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39324/1.1/
Pages: 134-140
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

Module: "Combinations and permutations: Introduction and Notation"
Used here as: "Introduction and Notation"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39100/1.1/
Pages: 145-146
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

Module: "Combinations and Permutations: Fundamental Counting Principle and Combinations"
Used here as: "Fundamental Counting Principle and Combinations"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39106/1.1/
Pages: 147-148
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

Module: "Combinations and Permutations: Permutations and Applications"
Used here as: "Permutations and Applications"
By: Free High School Science Texts Project
URL: http://cnx.org/content/m39103/1.1/
Pages: 149-152
Copyright: Free High School Science Texts Project
License: http://creativecommons.org/licenses/by/3.0/

About Connexions
Since 1999, Connexions has been pioneering a global system where anyone can create course materials and make them fully accessible and easily reusable free of charge. We are a Web-based authoring, teaching and learning environment open to anyone interested in education, including students, teachers, professors and lifelong learners. We connect ideas and facilitate educational communities.

Connexions's modular, interactive courses are in use worldwide by universities, community colleges, K-12 schools, distance learners, and lifelong learners. Connexions materials are in many languages, including English, Spanish, Chinese, Japanese, Italian, Vietnamese, French, Portuguese, and Thai. Connexions is part of an exciting new information distribution system that allows for **Print on Demand Books**. Connexions has partnered with innovative on-demand publisher QOOP to accelerate the delivery of printed course materials and textbooks into classrooms worldwide at lower prices than traditional academic publishers.